THE FUNDAMENTAL LEMMA FOR LIE ALGEBRAS

by

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Contents

Introduction

In this article, we propose a proof of certain conjectures of Langlands, Shelstad, and Waldspurger, better known as the fundamental lemma for Lie algebras and the nonstandard fundamental lemma. For further details in the following statements, we refer to Sections 1.11.1 and 1.12.7.

Theorem 1. — *Let k be a finite field with q elements, $\mathcal{O}$ a complete discrete valuation ring with residue field k, and F its fraction field. Let G be a reductive group scheme over $\mathcal{O}$ whose Weyl group order is not divisible by the characteristic of k. Let (κ, ρ_κ) be an endoscopic datum for G over $\mathcal{O}$, and let H be the associated endoscopic group scheme.*

Then one has the equality between the κ-orbital integral and the stable orbital integral

$$\Delta_G(a)\mathbf{O}_a^\kappa(1_\mathfrak{g}, \mathrm{d}t) = \Delta_H(a_H)\mathbf{SO}_{a_H}(1_\mathfrak{h}, \mathrm{d}t)$$

attached to the corresponding regular semisimple stable conjugacy classes a in $\mathfrak{g}(F)$ and a_H in $\mathfrak{h}(F)$, to the characteristic functions $1_\mathfrak{g}$ and $1_\mathfrak{h}$ of the compact subsets $\mathfrak{g}(\mathcal{O})$ and $\mathfrak{h}(\mathcal{O})$ in $\mathfrak{g}(F)$ and $\mathfrak{h}(F)$ respectively, and where we have set

$$\Delta_G(a) = q^{-\mathrm{val}(\mathfrak{D}_G(a))/2} \quad \text{and} \quad \Delta_H(a_H) = q^{-\mathrm{val}(\mathfrak{D}_H(a_H))/2},$$

with $\mathfrak{D}_G$ and $\mathfrak{D}_H$ denoting the discriminant functions of G and H.

Theorem 2. — *Let G_1 and G_2 be two reductive group schemes over $\mathcal{O}$ having isogenous root data and whose Weyl group orders are not divisible by the characteristic of k. Then, one has the following equality between the stable orbital integrals*

$$\mathbf{SO}_{a_1}(1_{\mathfrak{g}_1}, \mathrm{d}t) = \mathbf{SO}_{a_2}(1_{\mathfrak{g}_2}, \mathrm{d}t)$$

attached to the corresponding regular semisimple stable conjugacy classes a_1 in $\mathfrak{g}_1(F)$ and a_2 in $\mathfrak{g}_2(F)$, and to the characteristic functions $1_{\mathfrak{g}_1}$ and $1_{\mathfrak{g}_2}$ of the compact sets $\mathfrak{g}_1(\mathcal{O})$ and $\mathfrak{g}_2(\mathcal{O})$ in $\mathfrak{g}_1(F)$ and $\mathfrak{g}_2(F)$ respectively.

We prove these theorems in the equal-characteristic case. According to Waldspurger, the case of mixed characteristic follows (see [**82**]).

The main applications of the fundamental lemma are found in the realization of certain particular cases of Langlands' functoriality principle via the comparison of trace formulas and in the construction of Galois representations attached to automorphic forms through the computation of the cohomology of Shimura varieties. For applications to the comparison of trace formulas, we refer to the work of Arthur [**2**], and for those to Shimura varieties, to Kottwitz [**45**] and to Harris's forthcoming book.

Known Cases and Reductions. — The fundamental lemma has been established in a great number of particular cases. Its archimedean analogue was completely solved by Shelstad in [**71**]. This case inspired Langlands and Shelstad to formulate their conjecture for a nonarchimedean field. The case of the group SL(2) was treated by Labesse and Langlands in [**48**]. The case of the three-variable unitary group was resolved by Rogawski in [**65**]. Cases related to the twisted Sp(4) and GL(4) have been solved by Hales, Schröder, and Weissauer via explicit computations (see [**31**], [**68**] and [**84**]). More recently, Whitehouse continued these computations to prove the weighted twisted fundamental lemma in this case (see [**85**]).

The fundamental lemma for stable base change was established by Clozel [**14**] and Labesse [**47**], starting from the case of the unit element of the Hecke algebra proved by Kottwitz [**43**]. Previously, the case of GL(2) was established by Saito and Shintani *cf.* [**67**] and by Langlands *cf.* [**49**], and the case of GL(3) by Kottwitz *cf.* [**39**].

Another important case is that of SL(n), which was resolved by Waldspurger in [**79**]. The case of SL(3) with an elliptic torus was established earlier by Kottwitz (see [**40**]) and the case of SL(n) with an elliptic torus by Kazhdan (see [**35**]).

Recently, together with Laumon, we proved the fundamental lemma for the Lie algebras of unitary groups in the equal-characteristic case. Our method is geometric and applies only to local fields of equal characteristic. As already mentioned, the mixed-characteristic case then follows from Waldspurger's work [**82**]. The change of characteristic was also established by Cluckers and Loeser [**15**] in a more general framework but with less precision regarding the bound on the residual characteristic.

In a series of works, notably [**80**] and [**83**], Waldspurger showed that the fundamental lemma for groups as well as the transfer is deduced from the ordinary fundamental lemma for Lie algebras. Similarly, the twisted fundamental lemma is deduced from the combination of the ordinary lemma for Lie algebras and what he calls the nonstandard lemma. In the remainder of this article, we restrict ourselves to the ordinary fundamental lemma for Lie algebras and its nonstandard variant over a local field of equal characteristic.

Local Geometric Approach. — Kazhdan and Lusztig introduced in [**36**] the affine Springer fibers, which are geometric incarnations of orbital integrals. That work provides basic information on the geometry of affine Springer fibers, which we recall in Chapter 3 of the present article.

In the appendix to [**36**], Bernstein and Kazhdan constructed an affine Springer fiber for the group $\mathrm{Sp}(6)$ whose point count is not a polynomial in q. In fact, the associated motive of this affine Springer fiber contains the motive of a hyperelliptic curve. This example suggests that it is unlikely that one can obtain an explicit formula for the orbital integrals.

The interpretation of the κ-orbital integrals in terms of the quotients of affine Springer fibers was established in the work of Goresky, Kottwitz, and MacPherson [**26**]. They also introduced in [**26**] the use of equivariant cohomology in the study of affine Springer fibers. This strategy is very well adapted to the particular case of elements belonging to an unramified torus since in that case one has an action of a large torus on the affine Springer fiber with only isolated fixed points. They also discovered a remarkable relation between the equivariant cohomology of an affine Springer fiber for G and the corresponding fiber for the endoscopic group H in this unramified setting. This relation also depends on a purity conjecture for the cohomology of these affine Springer fibers. This conjecture was verified for elements with equal root valuations in [**27**].

In [**51**] and [**52**], Laumon introduced a method of deforming Springer fibers in the case of unitary groups based on the deformation theory of plane curves. His strategy is to introduce a plane curve of geometric genus zero having a singularity prescribed by the local situation and then to deform this plane curve. He also observed that in the unitary case, there exists a one-dimensional torus acting on the affine Springer fibers of $U(n)$ associated with a stable class coming from $U(n_1) \times U(n_2)$, whose fixed point set is the corresponding affine Springer fiber for $U(n_1) \times U(n_2)$. By computing the equivariant cohomology in family, he was able to prove the fundamental lemma in the unitary case, even for possibly ramified elements. His proof again relies on the purity conjecture for affine Springer fibers formulated by Goresky, Kottwitz, and MacPherson.

This purity conjecture seemed to me to be the main obstacle in the local geometric approach. It plays an essential role in degenerating certain spectral sequences and allows one to apply the Atiyah-Borel-Segal localization theorem.

In fact, there is another obstacle, at least as serious, to using equivariant cohomology for general affine Springer fibers. For very ramified elements of groups other than the unitary groups, there is no torus action on the corresponding affine Springer fiber. In this case, even if one has the purity conjecture, it is not clear that the strategies of [**26**] and [**52**] can be applied.

Global Geometric Approach. — In [**34**], Hitchin showed that the cotangent bundle of the moduli space of stable bundles over a compact Riemann surface is a completely integrable Hamiltonian system. To do this, he interpreted this cotangent bundle as the moduli space of principal bundles equipped with a Higgs field. The Hamiltonians are then given by the coefficients of the characteristic polynomial of the Higgs field. In this way, he defined the famous Hitchin fibration $f : \mathscr{M} \to \mathscr{A}$, where $\mathscr{M}$ is the aforementioned cotangent bundle, $\mathscr{A}$ is the affine space parametrizing the characteristic polynomials with coefficients in the global sections of suitable powers of the canonical bundle of X, and where f is a morphism whose generic fiber is essentially an abelian variety.

From our point of view, the Hitchin bundles are global analogues of affine Springer fibers. Moreover, it is important in our approach to allow the Higgs bundles to have values in an arbitrary line bundle of sufficiently large degree instead of the canonical bundle. In this generality, the moduli space of Higgs bundles $\mathscr{M}$ is no longer endowed with a symplectic form but still admits a Hitchin fibration $f : \mathscr{M} \to \mathscr{A}$.

We observed in [**57**] that a formal point count of $\mathscr{M}$ over a finite field yields an expression nearly identical to the geometric side of the trace formula for the Lie algebra. In [**57**], we proposed to interpret the stabilization process of the Langlands-Kottwitz trace formula in terms of the ℓ-adic cohomology of the Hitchin fibration. This interpretation leads to a formulation of a global variant of the fundamental lemma in terms of the ℓ-adic cohomology of the Hitchin fibration (see [**16**] and [**58**]). The geometric interpretation of the stabilization process with the Hitchin fibration, as well as the formulation of this global variant of the fundamental lemma, is more complex than the local analogue with affine Springer fibers. On the other hand, one enjoys a much richer geometry.

The geometric interpretation of the stabilization process is based on the action of a Picard stack $\mathscr{P} \to \mathscr{A}$ on $\mathscr{M}$, which is in some sense the stack of natural symmetries of the Hitchin fibration. The point count with the help of $\mathscr{P}$, carried out in Section 9 of [**57**], is revisited in a more systematic way in the final Chapter 8 of the present article. The construction of $\mathscr{P}$ is based on that of the regular centralizer, which we recall in Chapter 2. An important observation in the point count is that for every $a \in \mathscr{M}(k)$, the quotient $[\mathscr{M}_a/\mathscr{P}_a]$ can be expressed as a product of local quotients of the affine Springer fibers $\mathscr{M}_v(a)$ by their natural symmetry groups $\mathscr{P}_v(J_a)$ (see 4.15.1 and [**57**, 4.6]). This product formula appears implicitly throughout the article.

The algebraic stack $\mathscr{M}$ is not of finite type. However, there exists an anisotropic open subset $\mathscr{A}^{\mathrm{ani}}$ of $\mathscr{A}$, constructed in Chapter 6, over which the restricted fibration $f^{\mathrm{ani}} : \mathscr{M}^{\mathrm{ani}} \to \mathscr{A}^{\mathrm{ani}}$ is a proper morphism. It is known that $\mathscr{M}^{\mathrm{ani}}$ is smooth over the base field k. According to Deligne [**18**], the derived direct image $f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell$ is a pure complex - that is, the perverse cohomology sheaves

$$^p\mathrm{H}^n(f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)$$

are pure perverse sheaves. By the decomposition theorem, these become semisimple after base change to $\mathscr{A}^{\mathrm{ani}} \otimes_k \bar{k}$. In order to prove the fundamental lemma, it is essential to understand the geometrically simple factors in this decomposition.

The action of $\mathscr{P}$ on $\mathscr{M}$ induces an action of $\mathscr{P}^{\mathrm{ani}}$ on the perverse sheaves $^p\mathrm{H}^n(f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)$ and decomposes them into isotypic components

$$^p\mathrm{H}^n(f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell) = \bigoplus_{[\kappa]} {}^p\mathrm{H}^n(f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_{[\kappa]}.$$

Here the index $[\kappa]$ runs over the set of semisimple conjugacy classes in the dual group $\hat{G}$, only finitely many of which have a non-trivial contribution. We denote by $^p\mathrm{H}^n(f^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ the direct factor corresponding to $\kappa = 1$. The appearance of the dual group here is a consequence of the computation of the sheaf of connected components of the fibers of $\mathscr{P}$ via the Tate-Nakayama isomorphism, which was initiated in [**57**, 6] and completed in Sections 4.10 and 5.5 of the present article. In [**57**] and [**58**], we showed that this decomposition exactly corresponds to the endoscopic decomposition on the geometric side of the trace formula established by Langlands and Kottwitz (cf. [**50**] and [**44**]).

In this work we change our point of view slightly. Instead of working over $\mathscr{A}^{\mathrm{ani}}$, we pass to an étale cover $\tilde{\mathscr{A}}^{\mathrm{ani}}$ depending on the choice of a point $\infty \in X$, just as in [**54**]. Over this étale open set we have a finer decomposition

$$^{p}\mathrm{H}^{n}(\tilde{f}^{\mathrm{ani}}_{*}\overline{\mathbb{Q}}_{\ell}) = \bigoplus_{\kappa} {}^{p}\mathrm{H}^{n}(\tilde{f}^{\mathrm{ani}}_{*}\overline{\mathbb{Q}}_{\ell})_{\kappa},$$

where κ runs over a finite subset of the maximal torus $\hat{\mathbb{T}}$ of the dual group $\hat{G}$. We no longer stick exactly to the trace formula, but in return we rid ourselves of a number of incidental difficulties. Our geometric stabilization theorem 6.4.1 consists in expressing the piece ${}^{p}\mathrm{H}^{n}(\tilde{f}^{\mathrm{ani}}_{*}\overline{\mathbb{Q}}_{\ell})_{\kappa}$ in terms of endoscopic groups. In fact one expresses it as a direct sum of the stable pieces — that is, those corresponding to $\kappa = 1$ — but for the groups H associated with pointed endoscopic data *cf.* 1.8.2. There is an arithmetic variant 6.4.2 of 6.4.1, of which the fundamental lemma of Langlands-Shelstad is a consequence.

In [**57**] we showed that ${}^{p}\mathrm{H}^{n}(\tilde{f}^{\mathrm{ani}}_{*}\overline{\mathbb{Q}}_{\ell})_{\kappa}$ is supported on the union of the images of the morphisms $\tilde{\mathscr{A}}^{\mathrm{ani}}_{H} \to \tilde{\mathscr{A}}^{\mathrm{ani}}$, or rather a variant of this. We take up those arguments again in 6.3. As we observed in [**57**] and [**54**], this statement can replace the purity conjecture of Goresky, Kottwitz and MacPherson in the context of the Hitchin fibration. Moreover, it is very tempting to conjecture that all the simple factors of ${}^{p}\mathrm{H}^{n}(\tilde{f}^{\mathrm{ani}}_{*}\overline{\mathbb{Q}}_{\ell})_{\kappa}$ have as support the image of one of the closed immersions $\tilde{\mathscr{A}}^{\mathrm{ani}}_{H} \to \tilde{\mathscr{A}}^{\mathrm{ani}}$. Note that the geometric stabilization theorem follows from this support conjecture, since in fact it is not difficult to establish it over a dense open subset of $\tilde{\mathscr{A}}^{\mathrm{ani}}_{H}$.

In the unitary case, Laumon and I used the Atiyah-Borel-Segal exact sequence in equivariant cohomology to prove an ad hoc variant of the support statement above. My later attempts to generalize this equivariant method to other groups were hampered by the absence of a torus action on certain Hitchin fibers associated to highly ramified elements.

In this work we prove, for all groups, a weak form of this support conjecture. In Chapter 7 we show that one can determine all the supports in the case of a δ-regular abelian fibration *cf.* 7.2.1. Since it is not possible at present to prove that the Hitchin fibration is δ-regular — see however [**60**, p. 4] — we use certain intermediate statements 7.2.2 and 7.2.3, combined with what is available in positive characteristic in the direction of δ-regularity 5.7.2 and an intensive point count in Chapter 8, to conclude the proof of 6.4.2.

The delta inequality 7.2.2, whose proof occupies Chapter 7, allows one to bound from below the codimension of the supports in the case of an abelian fibration. It is in fact an elaborate variant of the inequality on the codimension of supports due to Goresky and MacPherson, which we recall in 7.3.

Let us now review the organization of the article. The first chapter contains the statement of the fundamental lemma for Lie algebras and its nonstandard variant. The heart of the article is the next-to-last chapter 7, where we prove the inequalities 7.2.2 and 7.2.3, from which a support theorem 7.8.5 will be deduced. The final chapter 8 contains the point counting argument that allows one to deduce the fundamental lemma from the support theorem 7.8.5. The other chapters are of a preparatory nature. Chapter 3 contains a study of the geometry of affine Springer fibers. The following chapter 4 presents a parallel study of the geometry of the Hitchin fibration. Our favorite tool in these studies is the action of the natural symmetries based on the construction of the regular centralizer recalled in Chapter 2. In Chapter 5, various stratifications of the Hitchin base are defined. In the next chapter 6, the anisotropic open subset of the Hitchin base is defined over which the fibration has all the good properties so that one may formulate the geometric stabilization theorem 6.4.2, which will be proved in the final two chapters 7 and 8. At the beginning of each chapter, the reader will find a summary of its content and the notations used.

1. The Langlands-Shelstad and Waldspurger Conjectures

In this chapter, we recall the statement of the fundamental lemma for Lie algebras conjectured by Langlands and Shelstad (see 1.11.1), as well as the nonstandard variant conjectured by Waldspurger (see 1.12.7). In the final paragraph of this chapter, we recall where the Langlands-Shelstad fundamental lemma enters in the stabilization process on the geometric side of the trace formula. Our study of the Hitchin fibration, which will be carried out later in this article, is directly inspired by this stabilization process. We have tried to keep the presentation in this chapter as elementary as possible.

1.1. Chevalley Morphism. — Let k be a field and $\mathbb{G}$ a split reductive group over k. Our definition is that of [**21**]: reductive implies affine, smooth and connected. We fix a split maximal torus $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$. Let $N_G(\mathbb{T})$ be the normalizer of $\mathbb{T}$ and $\mathbb{W} = N_G(\mathbb{T})/\mathbb{T}$ be the Weyl group. We denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$ and $k[\mathfrak{g}]$ the algebra of regular functions on $\mathfrak{g}$. Similarly, we denote by $\mathfrak{t} = \mathrm{Spec}(k[\mathfrak{t}])$ the Lie algebra of $\mathbb{T}$. Let r be the rank of $\mathbb{G}$, which by definition is the dimension of the maximal torus $\mathbb{T}$.

The action of $\mathbb{T}$ on the Lie algebra $\mathfrak{g}$ defines a set of roots Φ, on which the Weyl group $\mathbb{W}$ acts. The choice of the Borel subgroup $\mathbb{B}$ defines a set Δ of simple roots. Every root can be written uniquely in the form $\sum_{\alpha\in\Delta} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}$; the integer $\sum_{\alpha\in\Delta} n_\alpha$ is then called the *height* of that root. We write $h-1$ for the maximal height of a root; if the root system is simple, h is its Coxeter number. *Throughout this work we shall assume that the characteristic p is either zero or greater than* $2h$. Since the order of the Weyl group is a product of natural numbers not exceeding h, this hypothesis implies in particular that p does not divide the order of the Weyl group. It also implies that p is a good prime in the sense of [**12**, 1.14], and hence that the Killing form is non-degenerate in the semisimple case.

The group $\mathbb{G}$ acts on its Lie algebra via the adjoint action. The algebra of $\mathbb{G}$-invariant functions on $\mathfrak{g}$ is identified by the restriction from $\mathfrak{g}$ to $\mathfrak{t}$ with the algebra of $\mathbb{W}$-invariant functions on $\mathfrak{t}$.

Here is the statement of the Chevalley restriction theorem, a proof of which may be found, in the semisimple adjoint case, in [**75**, 3.17]. The general case is deduced from it under the hypothesis $p > 2$ *cf.* [**55**, 0.8].

Theorem 1.1.1. — *By restriction from $\mathfrak{g}$ to $\mathfrak{t}$, we have an algebra isomorphism $k[\mathfrak{g}]^{\mathbb{G}} = k[\mathfrak{t}]^{\mathbb{W}}$.*

We denote $\mathfrak{c} = \mathrm{Spec}(k[\mathfrak{t}]^{\mathbb{W}}) = \mathrm{Spec}(k[\mathfrak{g}]^{\mathbb{G}})$. The action of $\mathbb{G}_m$ by homothety on $\mathfrak{g}$ commutes with the adjoint action of $\mathbb{G}$, and therefore induces an action on $\mathfrak{c}$.

The inclusion $k[\mathfrak{t}]^{\mathbb{W}} \subset k[\mathfrak{t}]$ defines a morphism $\pi : \mathfrak{t} \to \mathfrak{c}$. It is a finite and flat morphism that realizes $\mathfrak{c}$ as the quotient in the sense of invariants of $\mathfrak{t}$ by the action of $\mathbb{W}$. Moreover, there exists a non-empty open subset $\mathfrak{c}^{\mathrm{rs}}$ of $\mathfrak{c}$ above which π is a finite étale Galois morphism with Galois group $\mathbb{W}$. This morphism is compatible with the action of $\mathbb{G}_m$ by homothety on $\mathfrak{t}$ and the action of $\mathbb{G}_m$ on $\mathfrak{c}$ by exponents.

We denote $\chi : \mathfrak{g} \to \mathfrak{c}$ the characteristic Chevalley morphism which is deduced from the inclusion of algebras $k[\mathfrak{g}]^{\mathbb{G}} \subset k[\mathfrak{g}]$. This morphism is compatible with the action of $\mathbb{G}_m$ by homothety on $\mathfrak{g}$ and the action of $\mathbb{G}_m$ on $\mathfrak{c}$ by exponents. By analogy with the case of matrices, we will call $\chi(x)$ the characteristic polynomial of x and $\mathfrak{c}$ the space of characteristic polynomials.

In addition to the $\mathbb{G}_m$-equivariance property, χ is $\mathbb{G}$-invariant by construction. This gives rise to morphisms between algebraic stacks

$$[\chi] : [\mathfrak{g}/\mathbb{G}] \to \mathfrak{c} \text{ and } [\chi/\mathbb{G}_m] : [\mathfrak{g}/\mathbb{G}\times\mathbb{G}_m] \to [\mathfrak{c}/\mathbb{G}_m].$$

1.2. Kostant Section. — The morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ admits sections. In [**38**], Kostant constructed such a section in a uniform way. Although he assumes that the base field has characteristic zero, his construction generalizes to positive characteristic $p > 2h$. In [**78**], Veldkamp sought to construct the Kostant section under the weaker hypothesis that p does not divide the order of the Weyl group; but, as Deligne pointed out to me, Veldkamp's argument appears to be incomplete.

Fix a pinning of $\mathbb{G}$ which consists of the choice of a maximal split torus $\mathbb{T}$, a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$ with unipotent radical $\mathbb{U}$ and a vector $\mathbb{x}_+ \in \mathrm{Lie}(\mathbb{U})$ of the form $\mathbb{x}_+ = \sum_{\alpha\in\Delta} \mathbb{x}_\alpha$ where Δ is the set of simple roots and where $\mathbb{x}_\alpha$ is a non-zero vector of the eigenspace $\mathrm{Lie}(\mathbb{U})_\alpha$ of $\mathrm{Lie}(\mathbb{U})$ corresponding to the eigenvalue α for the action of $\mathbb{T}$.

There then exists a unique $\mathfrak{sl}_2$-triple $(h, \mathbb{x}_+, \mathbb{x}_-)$ in $\mathbb{g}$ with the semi-simple element $h \in \mathbb{t} \cap \mathrm{Lie}(\mathbb{G}^{\mathrm{der}})$. Indeed, arguing on the height of the roots, one shows that $\mathrm{ad}(\mathbb{x}_+)^{2h-1} = 0$, as in [**12**, 5.5.2]. One then has a theorem of Jacobson-Morozov type [**12**, 5.3.2] under the hypothesis $p > 2h$. One even knows that the principal representation of $\mathfrak{sl}_2$ on $\mathbb{g}$ is completely reducible under the hypothesis $p > 2h$, by [**12**, 5.4.8].

The following statement was proved by Kostant in characteristic zero [**38**, Theorem 0.10].

Lemma 1.2.1. — *Let $\mathbb{g}^{\mathbb{x}_+}$ be the centralizer of $\mathbb{x}_+$ in $\mathbb{g}$. The restriction of the characteristic morphism $\chi : \mathbb{g} \to \mathbb{c}$ to the affine subspace $\mathbb{x}_- + \mathbb{g}^{\mathbb{x}_+}$ of the vector space $\mathbb{g}$ is an isomorphism.*

He first observes that $\mathrm{Lie}(\mathbb{B}) = [\mathbb{x}_-, \mathrm{Lie}(\mathbb{U})] \oplus \mathbb{g}^{\mathbb{x}_+}$. This equality follows from the complete reducibility of the principal representation of $\mathfrak{sl}_2$ in $\mathbb{g}$, which is valid under the hypothesis $p > 2h$. His argument [**38**, Prop. 19] for deducing the lemma from this observation is in fact independent of the characteristic. It has also been presented on the last page of [**6**].

The inverse of this isomorphism

$$\epsilon : \mathbb{c} \to \mathbb{x}_- + \mathbb{g}^{\mathbb{x}_+} \hookrightarrow \mathbb{g} \tag{1.2.2}$$

defines a section of the Chevalley morphism $\chi : \mathbb{g} \to \mathbb{c}$ called the Kostant section. In fact, Kostant constructed an entire family of sections of the Chevalley morphism. For classical groups, it is also possible to construct explicit sections $\mathbb{c} \to \mathbb{g}$ using linear algebra without using the pinning *cf.* [**59**]. These sections, which generalize the companion matrix in the linear case, are probably more adapted to explicit calculations of affine Springer fibers and Hitchin fibers.

Let us return to the section constructed by Kostant. It follows from 1.2.1 that $\mathbb{c}$ is isomorphic to an affine space. The action of $\mathbb{G}_m$ on $\mathbb{G}$ induces an action on $\mathbb{c}$. Using the complete reducibility of the principal $\mathfrak{sl}_2$-representation in $\mathbb{g}$, one may choose coordinates $a_1, \ldots, a_r$ of $\mathbb{c}$ so that this action is written

$$t(a_1, \ldots, a_r) = (t^{e_1} a_1, \ldots, t^{e_r} a_r).$$

The integers $e_1 - 1, \ldots, e_r - 1$ are the exponents of the root system, see [**11**].

Let $\mathbb{g}^{\mathrm{reg}}$ be the open subset of $\mathbb{g}$ consisting of the points $x \in \mathbb{g}$ whose centralizer I_x has dimension r. It was shown in [**38**, Lemma 10] that the image of the Kostant section is contained in the open subset $\mathbb{g}^{\mathrm{reg}}$. Kostant proves that if $x, x' \in \mathbb{g}^{\mathrm{reg}}(\bar{k})$ have the same characteristic polynomial $\chi(x) = \chi(x')$, then they are conjugate *cf.* [**38**, Th. 2]. Combining this with the existence of the section, he obtains the following statement.

Lemma 1.2.3. — *The restriction of χ to $\mathbb{g}^{\mathrm{reg}}$ is a smooth morphism. Its geometric fibers are homogeneous spaces under the action of G.*

1.3. Outer Twisting. — For arithmetic applications, it is necessary to consider the quasi-split forms of the group $\mathbb{G}$. Fix an pinning $(\mathbb{T}, \mathbb{B}, \mathbb{x}_+)$ of $\mathbb{G}$, as in 1.2. Let $\mathrm{Out}(\mathbb{G})$ denote the group of automorphisms of $\mathbb{G}$ that fix this pinning. This is a discrete group which may possibly be infinite. It acts on the set of roots Φ, leaving the subset of simple roots stable. It also acts on the Weyl group $\mathbb{W}$ in a compatible way-that is, there is an action of the semidirect product $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ on $\mathbb{T}$ and on the set Φ of roots.

Definition 1.3.1. — *A quasi-split form of $\mathbb{G}$ over a k-scheme X is the data of an* $\mathrm{Out}(\mathbb{G})$*-torsor ρ_G over X (with respect to the étale topology).*

1.3.2. — The datum of ρ_G allows one to twist $\mathbb{G}$ to obtain a smooth reductive X-group $G = \rho_G \wedge^{\mathrm{Out}(\mathbb{G})} \mathbb{G}$ which is equipped with a pinning defined over X, that is, a triple $(T, B, \mathbf{x}_+)$, where B is a closed smooth subgroup scheme of G over X, T is a subtorus of B, and $\mathbf{x}_+$ is a global section of $\mathrm{Lie}(B)$ such that, fiber by fiber, $(T, B, \mathbf{x}_+)$ is isomorphic to the pinning $(\mathbb{T}, \mathbb{B}, \mathbb{x}_+)$. Conversely, any smooth reductive X-group equipped with a pinning $(T, B, \mathbf{x}_+)$ and locally isomorphic to $\mathbb{G}$ for the étale topology defines a torsor under the group $\mathrm{Out}(\mathbb{G})$. We will abuse language by saying that G is a quasi-split form, omitting mention of the attached pinning.

1.3.3. — Let ρ_G be an $\mathrm{Out}(\mathbb{G})$-torsor over X, and let G be the corresponding quasi-split form with pinning $(T, B, \mathbf{x}_+)$. The structures discussed in the previous two paragraphs carry over to the quasi-split form. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t} = \mathrm{Lie}(T)$. The action of $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ on $\mathbb{t}$ induces an action of $\mathrm{Out}(\mathbb{G})$ on $\mathbb{c} = \mathbb{t}/\mathbb{W}$. Thus, we define the space of characteristic polynomials of $\mathfrak{g} = \mathrm{Lie}(G)$ as the X-scheme

$$\mathfrak{c} = \rho_G \wedge^{\mathrm{Out}(\mathbb{G})} \mathbb{c}.$$

It is equipped with a morphism

$$\chi : \mathfrak{g} \to \mathfrak{c}$$

which is deduced from the Chevalley morphism $\chi : \mathbb{g} \to \mathbb{c}$. Since $\mathrm{Out}(\mathbb{G})$ fixes the $\mathfrak{sl}_2$-triple $(h, \mathbb{x}_+, \mathbb{x}_-)$, the Kostant section $\epsilon : \mathbb{c} \to \mathbb{g}$ is $\mathrm{Out}(\mathbb{G})$-equivariant. By twisting, we obtain an X-morphism

$$\epsilon : \mathfrak{c} \to \mathfrak{g} \tag{1.3.4}$$

which is a section of the Chevalley morphism $\chi : \mathfrak{g} \to \mathfrak{c}$, and which we will also call the Kostant section. Moreover, there is a finite flat morphism

$$\pi : \mathfrak{t} \to \mathfrak{c}$$

obtained from $\pi : \mathbb{t} \to \mathbb{c}$. The finite étale X-group scheme

$$W = \rho_G \wedge^{\mathrm{Out}(\mathbb{G})} \mathbb{W}$$

acts on $\mathfrak{t}$. Since $\mathbb{c}$ is the invariant quotient of $\mathbb{t}$ by $\mathbb{W}$, $\mathfrak{c}$ is also the invariant quotient of $\mathfrak{t}$ by W. Over the open subset

$$\mathfrak{c}^{\mathrm{rs}} = \rho_G \wedge^{\mathrm{Out}(\mathbb{G})} \mathbb{c},$$

the morphism $\pi : \mathfrak{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ is a W-torsor under the finite étale group scheme.

1.3.5. — To study endoscopic groups, it is necessary to consider reductions of the $\mathrm{Out}(\mathbb{G})$-torsor ρ_G. A reduction of ρ_G is a torsor $\rho : X_\rho \to X$ under a discrete group Θ_ρ, together with a homomorphism $\mathbf{o}_{\mathbb{G}} : \Theta_\rho \to \mathrm{Out}(\mathbb{G})$ such that

$$\rho \wedge^{\Theta_\rho} \mathrm{Out}(\mathbb{G}) = \rho_G.$$

Then one has $\mathfrak{t} = X_\rho \wedge^{\Theta_\rho} \mathbb{t}$ and $\mathfrak{c} = X_\rho \wedge^{\Theta_\rho} \mathbb{c}$, and a cartesian diagram

$$\begin{array}{ccc} X_\rho \times \mathbb{t} & \xrightarrow{\pi} & X_\rho \times \mathbb{c} \\ \downarrow & & \downarrow \\ \mathfrak{t} & \xrightarrow[\pi]{} & \mathfrak{c} \end{array}$$

in which both vertical arrows are Θ_ρ-torsors. The semidirect product $\mathbb{W} \rtimes \Theta_\rho$ acts on $X_\rho \times \mathbb{t}$ and acts freely on the open subset $X_\rho \times \mathbb{t}^{\mathrm{rs}}$. The diagonal arrow in the above diagram,

$$\pi_\rho : X_\rho \times \mathbb{t} \to \mathfrak{c},$$

is then a finite flat $\mathbb{W} \rtimes \Theta_\rho$-invariant morphism that realizes $\mathfrak{c}$ as the invariant quotient of $X_\rho \times \mathbb{t}$ by $\mathbb{W} \rtimes \Theta_\rho$. Over the open subset $\mathfrak{c}^{\mathrm{rs}}$, π_ρ is a finite étale Galois morphism with Galois group $\mathbb{W} \rtimes \Theta_\rho$. It is often convenient in calculations to replace the W-torsor $\mathfrak{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ by the torsor $X_\rho \times \mathbb{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ under the constant group $\mathbb{W} \rtimes \Theta_\rho$.

1.3.6. — In what follows we shall assume that X is connected and normal. It is often convenient to pass from the language of torsors to the more concrete language of representations of the profinite fundamental group. Let x be a geometric point of X and $\pi_1(X,x)$ the pointed fundamental group of X. Let x_{Out} be a geometric point of ρ_G above x. The choice of this point defines a homomorphism

$$\rho_G^\bullet : \pi_1(X,x) \to \mathrm{Out}(\mathbb{G}).$$

A pointed reduction $\rho \wedge^{\Theta_\rho} \mathrm{Out}(\mathbb{G}) = \rho_G$ (see 1.3.5) is given by a geometric point x_ρ of X_ρ lying above x_{Out}. This is equivalent to giving a homomorphism $\rho^\bullet : \pi_1(X,x) \to \Theta_\rho$ in the commutative diagram

$$\begin{tikzcd} \pi_1(X,x) \arrow[r, "\rho^\bullet"] \arrow[dr, "\rho_G^\bullet"'] & \Theta_\rho \arrow[d, "\mathbf{o}_G"] \\ & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

where $\mathbf{o}_{\mathbb{G}} : \Theta_\rho \to \mathrm{Out}(\mathbb{G})$ is the reduction homomorphism.

1.4. Regular Semisimple Centralizer. — Let $\mathfrak{g}^{\mathrm{rs}}$ denote the inverse image of the open subset $\mathfrak{c}^{\mathrm{rs}}$ under the morphism $\chi : \mathfrak{g} \to \mathfrak{c}$. For any $a \in \mathfrak{c}^{\mathrm{rs}}(\bar{k})$, it is well known that the fiber $\chi^{-1}(a)$ consists of a single orbit under the adjoint action of G. For any $\gamma \in \mathfrak{g}^{\mathrm{rs}}(\bar{k})$, the centralizer I_γ of γ is a maximal torus of $G \otimes_k \bar{k}$.

1.4.1. — Let $a \in \mathfrak{c}^{\mathrm{rs}}(\bar{k})$ and let $\gamma, \gamma' \in \chi^{-1}(a)$. Then there exists an element $g \in G(\bar{k})$ which transports γ to γ', i.e. such that $\mathrm{ad}(g)\gamma = \gamma'$. The inner automorphism $\mathrm{ad}(g)$ then defines an isomorphism $\mathrm{ad}(g) : I_\gamma \xrightarrow{\sim} I_{\gamma'}$. Moreover, if g and g' are two elements of $G(\bar{k})$ transporting γ to γ', then g and g' differ by an element of I_γ. Since I_γ is a torus, hence commutative, the two isomorphisms

$$\mathrm{ad}(g) \quad \text{and} \quad \mathrm{ad}(g') : I_\gamma \xrightarrow{\sim} I_{\gamma'}$$

are identical. This shows that the tori I_γ and $I_{\gamma'}$ are canonically isomorphic. This therefore defines a torus that depends only on a and is canonically isomorphic to I_γ for every $\gamma \in \chi^{-1}(a)$. This torus can be described directly in terms of a with arbitrary coefficients as follows.

Let S be an X-scheme and $a \in \mathfrak{c}^{\mathrm{rs}}(S)$ an S-point of $\mathfrak{c}^{\mathrm{rs}}$. We call the *cameral cover* associated to a the W-torsor $\pi_a : \tilde{S}_a \to S$ obtained by forming the cartesian diagram

$$\begin{tikzcd} \tilde{S}_a \arrow[r] \arrow[d, "\pi_a"'] & \mathfrak{t}^{\mathrm{rs}} \arrow[d, "\pi"] \\ S \arrow[r, "a"'] & \mathfrak{c}^{\mathrm{rs}} \end{tikzcd}$$

Set

$$J_a = \pi_a \wedge^W T. \tag{1.4.2}$$

The following lemma is a particular case of a result of Donagi and Gaitsgory that was probably well known. We shall recall the general statement in paragraph 2.4.

Lemma 1.4.3. — *Let S be an X-scheme and $a \in \mathfrak{c}^{\mathrm{rs}}(S)$ an S-point of $\mathfrak{c}^{\mathrm{rs}}$. Let x be an S-point of $\mathfrak{g}^{\mathrm{rs}}(S)$ such that $\chi(x) = a$, and let I_x be the pullback of the centralizer over $\mathfrak{g}$. Then there is a canonical isomorphism $J_a = I_x$.*

1.4.4. — Let us consider a pointed variant of the construction above. Choose a geometric point s of S lying over x, and $\tilde{s}$ of $\tilde{S}_a$ lying over s. As in 1.3.6, we have a commutative diagram

$$\begin{tikzcd} & \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) \arrow[d] \\ \pi_1(S,s) \arrow[ur, "\pi_a^\bullet"] \arrow[r] & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

The torus J_a is then obtained by twisting the constant torus $\mathbb{T}$ by the action of $\pi_1(S,s)$ on $\mathbb{T}$ deduced from the homomorphism

$$\pi_a^\bullet : \pi_1(S,s) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}),$$

that is to say

$$J_a = S_\bullet \wedge^{\pi_1(S,s),\pi_a^\bullet} \mathbb{T}, \tag{1.4.5}$$

where $(S_\bullet, s_\bullet)$ is the universal pro-finite étale Galois cover of (S,s).

1.5. Conjugacy Classes within a Stable Class. — In this paragraph, G will be a quasi-split form of $\mathbb{G}$ over a field F containing k. By a stable semisimple regular conjugacy class in $\mathfrak{g}$ over F, we mean an element $a \in \mathfrak{c}^{\mathrm{rs}}(F)$. Langlands's original definition of stable conjugacy classes is more complicated, but once restricted to semisimple regular elements in the Lie algebra, it coincides with ours. Since we restrict ourselves to stable semisimple regular classes, henceforth by a stable conjugacy class we mean semisimple regular, unless explicitly mentioned otherwise.

1.5.1. — Let $a \in \mathfrak{c}^{\mathrm{rs}}(F)$ be a stable conjugacy class. Let $\gamma_0 = \epsilon(a) \in \mathfrak{g}(F)$ be the image of a under the Kostant section. The centralizer I_{γ_0} of γ_0 is a torus defined over F, canonically isomorphic to the torus J_a defined in the previous paragraph (see 1.4.3). Let γ be another F-point in $\chi^{-1}(a)$. Since, as elements of $\mathfrak{g}(\overline{F})$, γ_0 and γ are conjugate (i.e. there exists $g \in G(\overline{F})$ such that $\gamma = \mathrm{ad}(g)\gamma_0$), it follows that for every $\sigma \in \mathrm{Gal}(\overline{F}/F)$, one has $g^{-1}\sigma(g) \in I_{\gamma_0}(\overline{F})$. The map $\sigma \mapsto g^{-1}\sigma(g)$ then defines an element

$$\mathrm{inv}(\gamma_0,\gamma) \in \mathrm{H}^1(F, I_{\gamma_0})$$

which depends only on the $G(F)$-conjugacy class of γ and not on the choice of the transporting element g. Its image in $\mathrm{H}^1(F,G)$ is trivial by construction. According to Langlands, the map $\gamma \mapsto \mathrm{inv}(\gamma_0,\gamma)$ establishes a bijection between the set of $G(F)$-conjugacy classes in the set of F-points of $\chi^{-1}(a)$ and the fiber of the map

$$\mathrm{H}^1(F, I_{\gamma_0}) \to \mathrm{H}^1(F,G)$$

over the neutral element (see [**50**]). This fiber will be denoted

$$\ker(\mathrm{H}^1(F, I_{\gamma_0}) \to \mathrm{H}^1(F,G)).$$

1.5.2. — Instead of $\mathfrak{g}(F)$, it is often necessary to consider the groupoid $[\mathfrak{g}/G](F)$ of pairs (E,ϕ) consisting of a G-torsor E over F and an F-point ϕ of $\mathrm{ad}(E) = E \wedge^G \mathfrak{g}$. The Chevalley morphism defines a functor $[\chi]$ from $[\mathfrak{g}/G](F)$ to the set $\mathfrak{c}(F)$. Let $a \in \mathfrak{c}^{\mathrm{rs}}(F)$. Consider the groupoid of F-points of $[\chi]^{-1}(a)$. The objects of $[\chi]^{-1}(a)(F)$ are locally isomorphic for the étale topology. Moreover, there is a base point (E_0,γ_0) of the groupoid, where E_0 is the trivial G-torsor and $\gamma_0 \in \mathfrak{g}(F)$ is the element $\gamma_0 = \epsilon(a)$ defined by the Kostant section. For any F-point (E,ϕ) of $[\chi]^{-1}(a)$, one obtains an invariant

$$\mathrm{inv}((E_0,\gamma_0),(E,\phi)) \in \mathrm{H}^1(F, I_{\gamma_0}).$$

The map $(E,\phi) \mapsto \mathrm{inv}((E_0,\gamma_0),(E,\phi))$ defines a bijection from the set of isomorphism classes of $[\chi]^{-1}(a)(F)$ onto $\mathrm{H}^1(F, I_{\gamma_0})$.

For a given F-point (E,ϕ) of $[\chi]^{-1}(a)$ and its invariant $\mathrm{inv}((E_0,\gamma_0),(E,\phi))$, the isomorphism class of E corresponds to the image of this invariant in $\mathrm{H}^1(F,G)$ under the map

$$\mathrm{H}^1(F,I_{\gamma_0}) \to \mathrm{H}^1(F,G).$$

Thus, we recover the bijection mentioned above between the $G(F)$-conjugacy classes in $\chi^{-1}(a)(F)$ and the subset of $\mathrm{H}^1(F,I_{\gamma_0})$ consisting of the elements whose image in $\mathrm{H}^1(F,G)$ is trivial.

1.6. The Tate-Nakayama Duality. — The discussion in the previous paragraph takes a very explicit form in the case of a non-archimedean local field thanks to the Tate-Nakayama duality. Let F_v be a non-archimedean local field, $\mathcal{O}_v$ its ring of integers, and v the valuation. Let F_v^{sep} be a separable closure of F_v. Denote by Γ_v the Galois group $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$. Write $X=\mathrm{Spec}(F_v)$ and let x be the chosen geometric point.

***1.6.1.* —** Let G be the quasi-split form over F_v of $\mathbb{G}$ associated to a homomorphism $\rho_G^\bullet : \Gamma_v \to \mathrm{Out}(\mathbb{G})$. Let $\hat{\mathbb{G}}$ be the dual complex of $\mathbb{G}$. By definition, it is equipped with a pinning $(\hat{\mathbb{T}},\hat{\mathbb{B}},\hat{\mathbb{x}})$ whose associated root datum is obtained from that of $\mathbb{G}$ by interchanging roots and coroots. In particular, $\mathrm{Out}(\mathbb{G})=\mathrm{Out}(\hat{\mathbb{G}})$. Thus, we have an action $\rho_G^\bullet$ of Γ_v on $\hat{\mathbb{G}}$ that fixes the pinning.

According to Kottwitz (see [**41**] and [**44**]), $\mathrm{H}^1(F,G)$ is finite and carries a natural abelian group structure. Moreover, its Pontryagin dual is given by

$$\mathrm{H}^1(F,G)^* = \pi_0((Z_{\hat{\mathbb{G}}})^{\rho_G^\bullet(\Gamma_v)})$$

where $(Z_{\hat{\mathbb{G}}})^{\rho_G^\bullet(\Gamma_v)}$ is the subgroup of fixed points in the center $Z_{\hat{\mathbb{G}}}$ of $\hat{\mathbb{G}}$ under the action of $\rho_G^\bullet(\Gamma_v)$. In particular, $\mathrm{H}^1(F_v,G)^*$ is a finitely generated abelian group.

***1.6.2.* —** Let us now assume the situation of 1.3.6. In particular, we have a finite étale Galois cover $\rho : X_\rho \to X$ with Galois group Θ_ρ, and a point $x_\rho \in X_\rho$ above x. For every $a \in \mathfrak{c}^{\mathrm{rs}}(F_v)$, we have a $\mathbb{W}\rtimes\Theta_\rho$-torsor $\pi_{\rho,a} : \widetilde{X}_{\rho,a} \to X$ coming from 1.3.5. Choose a point $x_{\rho,a}$ of $\widetilde{X}_{\rho,a}$ above x_ρ.

We then have a homomorphism $\pi^\bullet_{\rho,a} : \Gamma_v \to \mathbb{W}\rtimes\Theta_\rho$ making the diagram

$$\begin{tikzcd} \Gamma_v \arrow[r, "\pi^\bullet_{\rho,a}"] \arrow[dr, "\rho^\bullet_G"'] & \mathbb{W}\rtimes\Theta_\rho \arrow[d] \\ & \Theta_\rho \end{tikzcd}$$

commute.

Let $\gamma_0=\epsilon(a)$ be the element obtained by applying the Kostant section to a. According to Lemma 1.4.3, we have

$$I_{\gamma_0} = J_a = \pi_{\rho,a} \wedge^{\mathbb{W}\rtimes\Theta_\rho} \mathbb{T}.$$

By the local Tate-Nakayama duality (see [**44**, 1.1]), one then obtains

$$\mathrm{H}^1(F_v,J_a)^* = \pi_0(\hat{\mathbb{T}}^{\pi^\bullet_{\rho,a}(\Gamma_v)}). \tag{1.6.3}$$

In other words, the group of complex characters of $\mathrm{H}^1(F_v,J_a)$ coincides with the group of connected components of the fixed point group $\hat{\mathbb{T}}^{\pi^\bullet_{\rho,a}(\Gamma_v)}$. Here $\hat{\mathbb{T}}$ denotes the complex dual torus of $\mathbb{T}$, defined by interchanging the character group with the cocharacter group.

***1.6.4.* —** The inclusion $\iota : \hat{\mathbb{T}} \hookrightarrow \hat{\mathbb{G}}$ is Γ-equivariant up to conjugation; that is, for any $t\in\hat{\mathbb{T}}$ and any $\sigma\in\Gamma_v$, the elements $\rho^\bullet(\sigma)(\iota(t))$ and $\iota(\pi_a^\bullet(\sigma)(t))$ are conjugate in $\hat{\mathbb{G}}$. This yields the inclusion

$$(Z_{\hat{\mathbb{G}}})^{\rho_G^\bullet(\Gamma_v)} \subset \hat{\mathbb{T}}^{\pi_a^\bullet(\Gamma_v)}$$

which induces a homomorphism between the groups of connected components

$$\pi_0((Z_{\hat{\mathbb{G}}})^{\rho_G^\bullet(\Gamma_v)}) \to \pi_0(\hat{\mathbb{T}}^{\pi_a^\bullet(\Gamma_v)}).$$

By duality, one recovers the map $\mathrm{H}^1(F_v, I_{\gamma_0}) \to \mathrm{H}^1(F_v, G)$ defined in the previous paragraph.

1.7. κ-Orbital Integrals. — Retain the notations of the previous paragraph. In addition, assume that G is given as a quasi-split form of $\mathbb{G}$ over $\mathcal{O}_v$. The locally compact group $G(F_v)$ is then equipped with a maximal compact open subgroup $G(\mathcal{O}_v)$. Let $\mathrm{d}g_v$ denote the Haar measure on $G(F_v)$ normalized so that $G(\mathcal{O}_v)$ has volume one.

For $a \in \mathfrak{c}^{\mathrm{rs}}(F_v)$, fix a Haar measure $\mathrm{d}t_v$ on the torus $J_a(F_v)$. For any $\gamma \in \mathfrak{g}(F_v)$ with $\chi(\gamma) = a$, the canonical isomorphism $J_a = I_\gamma$ allows one to transport the Haar measure $\mathrm{d}t_v$ from $J_a(F_v)$ to a Haar measure on $I_\gamma(F_v)$.

For any such γ, and for any locally constant compactly supported function f on $\mathfrak{g}(F_v)$, one may then define the orbital integral

$$\mathbf{O}_\gamma(f, \mathrm{d}t_v) = \int_{I_\gamma(F_v)\backslash G(F_v)} f(\mathrm{ad}(g_v)^{-1}\gamma)\frac{\mathrm{d}g_v}{\mathrm{d}t_v}.$$

***Definition 1.7.1.* —** *Let κ be an element of $\hat{\mathbb{T}}^{\pi_a^\bullet(\Gamma_v)}$. For any locally constant compactly supported function f on $\mathfrak{g}(F_v)$, we define the κ-orbital integral of f on the stable conjugacy class $a \in \mathfrak{c}(F_v)$ by the formula*

$$\mathbf{O}_a^\kappa(f, \mathrm{d}t_v) = \sum_\gamma \langle \mathrm{inv}(\gamma_0, \gamma), \kappa\rangle\, \mathbf{O}_\gamma(f, \mathrm{d}t_v)$$

where γ runs over the set of $G(F_v)$-conjugacy classes within the stable class of characteristic a, where the base point $\gamma_0 = \epsilon(a)$ is defined by the Kostant section, and where $\mathrm{d}t_v$ is a Haar measure on the torus $J_a(F_v)$.

Note for future reference that the definition of the κ-orbital integral depends on the choice of a geometric point $x_{\rho,a}$ in $\tilde{X}_{\rho,a}$; without such a point one cannot relate the cohomology group $\mathrm{H}^1(F_v, J_a)$ with the dual torus $\hat{\mathbb{T}}$, and therefore express the Tate-Nakayama duality in the form 1.6.3.

1.8. Endoscopic Groups. — By construction, the dual group $\hat{\mathbb{G}}$ is equipped with a pinning $(\hat{\mathbb{T}}, \hat{\mathbb{B}}, \hat{\mathbb{x}}_+)$. Let κ be an element of the maximal torus $\hat{\mathbb{T}}$ in this pinning. The neutral component of the centralizer of κ in $\hat{\mathbb{G}}$ is a reductive subgroup, which we denote by $\hat{\mathbb{H}}$. The pinning of $\hat{\mathbb{G}}$ induces on $\hat{\mathbb{H}}$ a pinning with the same maximal torus $\hat{\mathbb{T}}$. Let $\mathbb{H}$ be the split reductive group over k equipped with a pinning whose dual complex is $\hat{\mathbb{H}}$ (with its pinning). Then one has $\mathrm{Out}(\mathbb{H}) = \mathrm{Out}(\hat{\mathbb{H}})$.

Consider the centralizer $(\hat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ of κ in the semidirect product $\hat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G})$. There is an exact sequence

$$1 \to \hat{\mathbb{H}} \to (\hat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa \to \pi_0(\kappa) \to 1,$$

where $\pi_0(\kappa)$ is the group of connected components of $(\hat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$. One then has canonical homomorphisms

$$\begin{tikzcd} & \pi_0(\kappa) \arrow[dl, "\mathbf{o}_{\mathbb{H}}(\kappa)"'] \arrow[dr, "\mathbf{o}_{\mathbb{G}}(\kappa)"] & \\ \mathrm{Out}(\mathbb{H}) & & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

***Definition 1.8.1.* —** *Let G be a quasi-split form of $\mathbb{G}$ over X, given by an* $\mathrm{Out}(\mathbb{G})$*-torsor ρ_G. An endoscopic datum for G over X is a pair (κ, ρ_κ) with κ as above and where ρ_κ is a $\pi_0(\kappa)$-torsor which induces ρ_G via the change of structure groups $\mathbf{o}_{\mathbb{G}}(\kappa)$.*

The endoscopic group associated with the endoscopic datum (κ, ρ_κ) is the quasi-split form H over X of $\mathbb{H}$ given by the $\mathrm{Out}(\mathbb{H})$*-torsor ρ_H obtained from ρ_κ by the change of structure group $\mathbf{o}_{\mathbb{H}}(\kappa)$.*

There is a pointed variant of the notion of endoscopic datum that is useful. Let X be a scheme with a geometric point x. Let G be a connected reductive X-group, a quasi-split form of the constant group $\mathbb{G}$, defined by a homomorphism

$$\rho_G^\bullet : \pi_1(X,x) \to \mathrm{Out}(\mathbb{G}).$$

Definition 1.8.2. — *A pointed endoscopic datum for G over X is a pair $(\kappa, \rho_\kappa^\bullet)$ where $\kappa \in \hat{\mathbb{T}}$ and*

$$\rho_\kappa^\bullet : \pi_1(X,x) \to \pi_0(\kappa)$$

is a homomorphism lying over $\rho_G^\bullet$.

Given a pointed endoscopic datum, one can form a commutative diagram

$$\begin{tikzcd}
 & \pi_1(X,x) \arrow[d, "\rho_\kappa^\bullet"] \arrow[ddl, "\rho_H^\bullet"'] \arrow[ddr, "\rho_G^\bullet"] & \\
 & \pi_0(\kappa) \arrow[dl, "\mathbf{o}_{\mathbb{H}}(\kappa)"] \arrow[dr, "\mathbf{o}_{\mathbb{G}}(\kappa)"'] & \\
\mathrm{Out}(\mathbb{H}) & & \mathrm{Out}(\mathbb{G})
\end{tikzcd}$$

The endoscopic group H associated with the pointed endoscopic datum $(\kappa, \rho_\kappa^\bullet)$ can then be formed using the homomorphism $\rho_H^\bullet$.

1.9. Transfer of Stable Conjugacy Classes. — Let (κ, ρ_κ) be an endoscopic datum as in 1.8.1, and let H be the associated endoscopic group. We now define the transfer of stable conjugacy classes from H to G by constructing a morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$.

By construction, one has a simultaneous reduction of the torsors ρ_G and ρ_H to the torsor

$$\rho_\kappa : X_{\rho_\kappa} \to X$$

under the group $\pi_0(\kappa)$. Then one may realize $\mathfrak{c}$ as the invariant quotient of $X_{\rho_\kappa} \times \mathfrak{t}$ under the action of $\mathbb{W} \rtimes \pi_0(\kappa)$ (see 1.3.5). Similarly, one may realize $\mathfrak{c}_H$ as the invariant quotient of $X_{\rho_\kappa} \times \mathfrak{t}$ under the action of $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$. To define the morphism $\mathfrak{c}_H \to \mathfrak{c}$, it suffices to define a homomorphism

$$\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$$

compatible with the actions on $X_{\rho_\kappa} \times \mathfrak{t}$, which leads us to the following lemma.

Lemma 1.9.1. — *Let $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$ be the semidirect product defined by $\mathbf{o}_{\mathbb{H}}(\kappa) : \pi_0(\kappa) \to \mathrm{Out}(\mathbb{H})$. Let $\mathbb{W} \rtimes \pi_0(\kappa)$ be the semidirect product defined by $\mathbf{o}_{\mathbb{G}}(\kappa) : \pi_0(\kappa) \to \mathrm{Out}(\mathbb{G})$. There exists a canonical homomorphism*

$$\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$$

which induces the obvious homomorphism on the normal subgroups $\mathbb{W}_{\mathbb{H}} \subset \mathbb{W}$, induces the identity on the quotient $\pi_0(\kappa)$, and is compatible with the actions of $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$ and $\mathbb{W} \rtimes \pi_0(\kappa)$ on $\mathfrak{t}$.

Proof. — Recall that the centralizer $(\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ of κ in $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ is canonically isomorphic to the semidirect product $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$ (see [**57**, Lemma 10.1]). One deduces from this an homomorphism $\theta : \pi_0(\kappa) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ such that one has a group homomorphism

$$\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes^\theta \pi_0(\kappa),$$

where the latter semidirect product is formed using the action of $\pi_0(\kappa)$ on $\mathbb{W}$ defined by the homomorphism $\theta : \pi_0(\kappa) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ and the natural action of $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ on $\mathbb{W}$. This homomorphism is visibly compatible with the actions on $\mathfrak{t}$.

It remains to construct an isomorphism between the semidirect products

$$\mathbb{W} \rtimes^\theta \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa),$$

where the latter is formed using the homomorphism $\mathbf{o}_{\mathbb{G}}(\kappa) : \pi_0(\kappa) \to \mathrm{Out}(\mathbb{G})$. For every $\alpha \in \pi_0(\kappa)$, the element $\theta(\alpha) \in \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ can be written uniquely as $\theta(\alpha) = w(\alpha)\mathbf{o}_{\mathbb{G}}(\alpha)$ with $w(\alpha) \in \mathbb{W}$. This allows us to define a homomorphism

$$\pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$$

by the formula $\alpha \mapsto w(\alpha)\alpha$. This homomorphism induces an isomorphism $\mathbb{W} \rtimes^{\theta} \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$ which makes the diagram

$$\begin{tikzcd}
\mathbb{W} \rtimes^{\theta} \pi_0(\kappa) \arrow[rr] \arrow[dr] & & \mathbb{W} \rtimes \pi_0(\kappa) \arrow[dl] \\
& \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) &
\end{tikzcd}$$

commute. In particular, this isomorphism is compatible with the actions on $\mathbb{t}$. □

Over the regular semisimple open $\mathfrak{c}^{\mathrm{rs}}$ of $\mathfrak{c}$, there is a finite étale morphism

$$\nu^{\mathrm{rs}} : \mathfrak{c}_H^{G-\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$$

where we denote by $\mathfrak{c}_H^{G-\mathrm{rs}}$ the inverse image of $\mathfrak{c}^{\mathrm{rs}}$ in $\mathfrak{c}_H$. This morphism realizes the transfer of stable conjugacy classes of H that are semisimple and G-regular to the stable conjugacy classes of G that are semisimple regular.

Lemma 1.9.2. — *Let $a_H \in \mathfrak{c}_H^{G-\mathrm{rs}}(S)$ be an S-point for a scheme S, and let $a \in \mathfrak{c}^{\mathrm{rs}}(S)$ be its image. Then there is a canonical isomorphism between the torus J_a defined by formula 1.4.2 and the torus J_{H,a_H} defined by the same formula applied to H.*

Proof. — Taking the inverse image of 1.3.5 by a, one obtains a $\mathbb{W} \rtimes \pi_0(\kappa)$-torsor $\pi_{\rho_\kappa,a}$ over S. Similarly, one has a $\mathbb{W}_H \rtimes \pi_0(\kappa)$-torsor π_{ρ_κ,a_H} over S. By the very construction of the morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$, there is a morphism

$$\pi_{\rho_\kappa,a_H} \to \pi_{\rho_\kappa,a}$$

compatible with the actions of $\mathbb{W}_H \rtimes \pi_0(\kappa)$ and $\mathbb{W} \rtimes \pi_0(\kappa)$. Consequently, the second torsor is obtained from the first by the change of structure groups $\mathbb{W}_H \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$ (see 1.9.1). It now suffices to apply formula 1.4.5. □

Remark 1.9.3. — Now, let $S = \mathrm{Spec}(F_v)$ where F_v is a local field as in 1.6. By choosing an F_v^{sep}-point $x_{\rho_\kappa,a}$ in the torsor π_{ρ_κ,a_H}, one obtains a homomorphism $\pi^{\bullet}_{\rho_\kappa,a_H} : \Gamma_v \to \mathbb{W}_H \rtimes \pi_0(\kappa)$. Then, by Tate-Nakayama duality 1.6.3,

$$\mathrm{H}^1(F_v, J_a)^* = \mathrm{H}^1(F_v, J_{H,a_H})^* = \hat{\mathbb{T}}^{\pi^{\bullet}_{\rho_\kappa,a_H}(\Gamma_v)}.$$

By construction, $\kappa \in \hat{\mathbb{T}}^{\mathbb{W}_H \rtimes \pi_0(\kappa)}$, so that one may define the κ-orbital integral $\mathbf{O}_a^\kappa(f, \mathrm{d}t_v)$ as in 1.7.1.

1.10. Discriminant and Resultant. — Let Φ be the root system associated with the split group $\mathbb{G}$. For each root $\alpha \in \Phi$, let $\mathrm{d}\alpha \in k[\mathbb{t}]$ denote the derivative of the character $\alpha : \mathbb{T} \to \mathbb{G}_m$. Form the discriminant

$$\underline{\mathfrak{D}}_{\mathbb{G}} = \prod_{\alpha \in \Phi} \mathrm{d}\alpha \in k[\mathbb{t}],$$

which is clearly an invariant element under $\mathbb{W}$ in this polynomial algebra. It thus defines a function on the space of characteristic polynomials $\mathbb{c} = \mathrm{Spec}(k[\mathbb{t}]^{\mathbb{W}})$. Let $\mathfrak{D}_{\mathbb{G}}$ be the divisor on $\mathbb{c}$ defined by this function. We recall the well-known statement.

Lemma 1.10.1. — *The divisor $\mathfrak{D}_{\mathbb{G}}$ is a reduced divisor on $\mathbb{c}$ which is stable under the action of* $\mathrm{Out}(\mathbb{G})$. *The open complement of this divisor is the regular semisimple open $\mathbb{c}^{\mathrm{rs}}$.*

Proof. — The pullback of $\mathfrak{D}_{\mathbb{G}}$ is the divisor of the hyperplanes in $\mathbb{t}$ defined by the roots, each hyperplane occurring with multiplicity 2 since α and $-\alpha$ define the same hyperplane. The group $\mathbb{W}$ acts freely on the complement of this divisor in $\mathbb{t}$, so that the morphism $\pi_{\mathbb{t}} : \mathbb{t} \to \mathbb{c}$ is étale over this open set and clearly ramified along this divisor. Moreover, $\mathrm{Out}(\mathbb{G})$ acts on the set of roots, hence it leaves this divisor stable.

It remains to prove that $\mathfrak{D}_{\mathbb{G}}$ is a reduced divisor. Since it is a complete intersection, it suffices to show that a dense open subset of $\mathfrak{D}_{\mathbb{G}}$ is reduced. One may therefore remove from $\mathfrak{D}_{\mathbb{G}}$ the image of the points of $\mathbb{t}$ belonging to more than two root hyperplanes. One then reduces to the case of a group of semisimple rank one, where the assertion can be verified by hand. □

Let X be a k-scheme and let G be a quasi-split form of $\mathbb{G}$ over X given by an $\mathrm{Out}(\mathbb{G})$-torsor ρ_G. By twisting $\mathfrak{D}_{\mathbb{G}}$ by ρ_G, one obtains a reduced divisor $\mathfrak{D}_G$ on $\mathfrak{c}$ whose open complement is $\mathfrak{c}^{\mathrm{rs}}$. Let (κ, ρ_κ) be an endoscopic datum (see 1.8.1) for G. One has a divisor $\mathfrak{D}_{\mathbb{H}}$ on $\mathbb{c}_{\mathbb{H}}$ and by twisting, a divisor $\mathfrak{D}_H$ on $\mathfrak{c}_H$.

The following statement was pointed out to me by two of the anonymous referees.

Lemma 1.10.2. — *Choose a subset $\Psi \subset \Phi - \Phi_{\mathbb{H}}$ such that for every pair of opposite roots $\pm\alpha \in \Phi - \Phi_{\mathbb{H}}$ the cardinality of $\{\pm\alpha\} \cap \Psi$ equals one. Then the function $\prod_{\alpha\in\Psi} \mathrm{d}\alpha \in k[\mathbb{t}]$ is invariant under the action of $\mathbb{W}_{\mathbb{H}}$.*

Proof. — An element $w \in \mathbb{W}_{\mathbb{H}}$ acts on $\Phi - \Phi_{\mathbb{H}}$ by sending a pair of opposite roots $\{\pm\alpha\} \subset \Phi - \Phi_{\mathbb{H}}$ to a generally different pair of opposite roots in the same set. It follows that there is a sign $\epsilon(w)$ such that $w(\prod_{\alpha\in\Psi} \mathrm{d}\alpha) = \epsilon(w) \prod_{\alpha\in\Psi} \mathrm{d}\alpha$, and that this sign does not depend on the choice of the set Ψ. It remains to show that this sign equals one.

Choose a subset $\Psi_{\mathbb{G}} \subset \Phi$ such that for every pair of opposite roots $\pm\alpha \in \Phi$ the cardinality of $\{\pm\alpha\} \cap \Psi_{\mathbb{G}}$ equals one, and set $\Psi_{\mathbb{H}} = \Psi_{\mathbb{G}} \cap \Phi_{\mathbb{H}}$. We may also assume that $\Psi_{\mathbb{G}} = \Psi_{\mathbb{H}} \cup \Psi$. For every $w \in \mathbb{W}$ there is a sign $\epsilon_{\mathbb{G}}(w)$ such that $w(\prod_{\alpha\in\Psi_{\mathbb{G}}} \mathrm{d}\alpha) = \epsilon_{\mathbb{G}}(w) \prod_{\alpha\in\Psi_{\mathbb{G}}} \mathrm{d}\alpha$, and this sign does not depend on the choice of $\Psi_{\mathbb{G}}$. Taking for $\Psi_{\mathbb{G}}$ the set of positive roots, one sees that $\epsilon_{\mathbb{G}}(w) = (-1)^{\ell_{\mathbb{G}}(w)}$, where $\ell_{\mathbb{G}}(w)$ is the usual length of w in the Coxeter group $\mathbb{W}_{\mathbb{G}}$. If $w \in \mathbb{W}_{\mathbb{H}}$, one likewise has $w(\prod_{\alpha\in\Psi_{\mathbb{H}}} \mathrm{d}\alpha) = \epsilon_{\mathbb{H}}(w) \prod_{\alpha\in\Psi_{\mathbb{H}}} \mathrm{d}\alpha$ with $\epsilon_{\mathbb{H}}(w) = (-1)^{\ell_{\mathbb{H}}(w)}$.

To show that $\epsilon(w) = 1$, it remains to prove the equality

$$(-1)^{\ell_{\mathbb{G}}(w)} = (-1)^{\ell_{\mathbb{H}}(w)}.$$

Note that both these signs may be interpreted as the determinant of the reflection representation. The equality of signs follows from the fact that the reflection representation of $\mathbb{W}_{\mathbb{H}}$ is obtained from that of $\mathbb{W}_{\mathbb{G}}$ by restriction. □

Let $\mathfrak{R}^{\mathbb{G}}_{\mathbb{H}}$ be the effective divisor of $\mathbb{c}_{\mathbb{H}}$ defined by the invariant function $\prod_{\alpha\in\Psi} \mathrm{d}\alpha \in k[\mathbb{t}]^{\mathbb{W}_{\mathbb{H}}}$ of the preceding lemma. One then has the equality of divisors

$$\nu^* \mathfrak{D}_{\mathbb{G}} = \mathfrak{D}_{\mathbb{H}} + 2\mathfrak{R}^{\mathbb{G}}_{\mathbb{H}}.$$

1.10.3. — By twisting $\mathfrak{R}^{\mathbb{G}}_{\mathbb{H}}$ by ρ_κ, one obtains an effective divisor $\mathfrak{R}^G_H$ on $\mathfrak{c}_H$ satisfying the relation

$$\nu^* \mathfrak{D}_G = \mathfrak{D}_H + 2\mathfrak{R}^G_H.$$

1.11. The Fundamental Lemma for Lie Algebras.

— We are now in a position to state the fundamental lemma for Lie algebras, conjectured by Langlands and Shelstad and by Waldspurger.

Let F_v be a non-archimedean local field, $\mathcal{O}_v$ its ring of integers, and let $v : F_v^\times \to \mathbb{Z}$ be the discrete valuation. Let $\mathbb{F}_q$ denote the residue field of $\mathcal{O}_v$. Let $X_v = \mathrm{Spec}(\mathcal{O}_v)$. Let F_v^{sep} be a separable closure of F_v. This defines a geometric point x of X_v.

Let G be a quasi-split form of $\mathbb{G}$ over $\mathcal{O}_v$ defined by a homomorphism $\rho_G^\bullet : \pi_1(X_v, x) \to \mathrm{Out}(\mathbb{G})$. Consider a pointed endoscopic datum $(\kappa, \rho_\kappa^\bullet)$ consisting of an element $\kappa \in \hat{\mathbb{T}}$ and a homomorphism $\rho_\kappa^\bullet : \pi_1(X_v, x) \to \pi_0(\kappa)$ lying over $\rho_G^\bullet$ (see 1.8.2). Then there is a reductive group scheme H over X_v and a morphism of X_v-schemes $\mathfrak{c}_H \to \mathfrak{c}$.

Let $a_H \in \mathfrak{c}_H(\mathcal{O}_v)$ with image $a \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$. Choose a Haar measure $\mathrm{d}t_v$ on the torus $J_a(F_v)$. By Lemma 1.9.2, we have an isomorphism between the tori J_a and J_{H,a_H}, so that one may transport the Haar measure $\mathrm{d}t_v$ to $J_{H,a_H}(F_v)$.

By choosing an F_v^{sep}-point x_a as in 1.9.3, one can define the κ-orbital integral $\mathbf{O}_a^\kappa$ of any locally constant compactly supported function on $\mathfrak{g}(F_v)$. Let $1_{\mathfrak{g}_v}$ be the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$ in $\mathfrak{g}(F_v)$ and $1_{\mathfrak{h}_v}$ the characteristic function of $\mathfrak{h}(\mathcal{O}_v)$ in $\mathfrak{h}(F_v)$.

Theorem 1.11.1. — *With the above notations, one has the equality*

$$\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v}, \mathrm{d}t_v) = q^{r^G_{H,v}(a_H)}\, \mathbf{SO}_{a_H}(1_{\mathfrak{h}_v}, \mathrm{d}t_v)$$

where $r^G_{H,v}(a_H) = \deg_v(a_H^*\mathfrak{R}^G_H)$.

This theorem is the Lie algebra variant of the original conjecture of Langlands and Shelstad, which concerns Lie groups over a non-archimedean local field. This variant was formulated by Waldspurger, who showed that it implies the original conjecture for groups. He also proved that the case where F is a local field of positive characteristic not dividing the order of $\mathbb{W}$ implies the case where F is a local field of characteristic zero with residue characteristic not dividing the order of $\mathbb{W}$ (see [**82**]). We restrict ourselves here to the first case.

The original Langlands-Shelstad statement is somewhat more complicated, notably due to the presence of a sign in the transfer factor. In [**46**], Kottwitz established the link between the Langlands-Shelstad transfer factor and the Kostant section, which allows us to state the Langlands-Shelstad conjecture in this simpler form. In the case of classical groups, Waldspurger gave a more explicit form of this conjecture in [**81**]. Hales also wrote an expository paper (see [**32**]) on the statement of the Langlands-Shelstad conjecture.

1.11.2. — Note that the statement above extends trivially to the case where one starts with an element $a_H \in \mathfrak{c}^{G-\mathrm{rs}}(F_v)$ that is not in $\mathfrak{c}_H(\mathcal{O}_v)$. In this case, we also have $a \notin \mathfrak{c}(\mathcal{O}_v)$. This follows from the valuative criterion of properness applied to the finite morphism $\nu_H : \mathfrak{c}_H \to \mathfrak{c}$. In such cases, it is evident that the orbital integrals $\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v}, \mathrm{d}t_v)$ and $\mathbf{SO}_{a_H}(1_{\mathfrak{h}_v}, \mathrm{d}t_v)$ vanish.

1.11.3. — Finally, note that the equality of divisors

$$a_*\mathfrak{D}_G = a_H^*\mathfrak{D}_H + 2a_H^*\mathfrak{R}^H_G$$

which follows from 1.10.3 implies

$$\Delta_H(a_H)\Delta_G(a)^{-1} = q^{\deg_v(a_H^*\mathfrak{R}^H_G)}$$

with $\Delta_H(a_H) = q^{-\deg(a_H^*\mathfrak{D}_H)/2}$ and $\Delta_G(a) = q^{-\deg(a^*\mathfrak{D}_G)/2}$. This allows one to rewrite formula 1.11.1 in the more usual form

$$\Delta_G(a)\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v}, \mathrm{d}t_v) = \Delta_H(a_H)\mathbf{SO}_{a_H}(1_{\mathfrak{h}_v}, \mathrm{d}t_v).$$

1.12. The Nonstandard Fundamental Lemma. — In [**83**], Waldspurger formulated a variant of the Langlands-Shelstad conjecture which he calls the nonstandard fundamental lemma. In this paragraph, we recall this conjecture.

Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two split reductive groups over k with pinnings. For each $i \in \{1, 2\}$, we have in particular a maximal torus $\mathbb{T}_i$ of $\mathbb{G}_i$, the set of roots $\Phi_i \subset \mathbb{X}^*(\mathbb{T}_i)$, the set of simple roots $\Delta_i \subset \Phi_i$, and the set of coroots $\Phi_i^\vee \subset \mathbb{X}_*(\mathbb{T}_i)$. The quintuple

$$(\mathbb{X}^*(\mathbb{T}_i), \mathbb{X}_*(\mathbb{T}_i), \Phi_i, \Phi_i^\vee, \Delta_i)$$

is the root datum associated with the pinned group $\mathbb{G}_i$, which determines it up to unique isomorphism.

Definition 1.12.1. — *An isogeny of root data between* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *consists of a pair of isomorphisms of* $\mathbb{Q}$*-vector spaces*

$$\psi^* : \mathbb{X}^*(\mathbb{T}_2) \otimes \mathbb{Q} \longrightarrow \mathbb{X}^*(\mathbb{T}_1) \otimes \mathbb{Q}$$

and

$$\psi_* : \mathbb{X}_*(\mathbb{T}_1) \otimes \mathbb{Q} \longrightarrow \mathbb{X}_*(\mathbb{T}_2) \otimes \mathbb{Q},$$

which are dual to each other and such that ψ^* sends bijectively the $\mathbb{Q}$-lines of the form $\mathbb{Q}\alpha_2$, for $\alpha_2 \in \Phi_2$, onto the $\mathbb{Q}$-lines of the form $\mathbb{Q}\alpha_1$, for $\alpha_1 \in \Phi_1$, sending the lines of the simple roots onto the lines of the simple roots, and similarly for ψ_* with the $\mathbb{Q}$-lines generated by the coroots.

Example 1.12.2. — Two semisimple groups having the same adjoint group have isogenous root data. Indeed, in this case there is a canonical isomorphism $\mathbb{X}^*(\mathbb{T}_1) \otimes \mathbb{Q} \to \mathbb{X}^*(\mathbb{T}_2) \otimes \mathbb{Q}$ which respects the set of roots and the set of simple roots, and whose dual isomorphism respects the coroots.

Example 1.12.3. — The most interesting example is that of two reductive groups dual to each other in the sense of Langlands. By definition, the root datum of the dual group $\hat{\mathbb{G}}$ is obtained from that of $\mathbb{G}$ by interchanging the character group with the cocharacter group, and the set of roots with the set of coroots. Consider the direct sum decomposition of $\mathbb{Q}$-vector spaces

$$\begin{aligned} \mathbb{X}^*(\mathbb{T}_i) \otimes \mathbb{Q} &= \mathbb{Q}\Phi \oplus \mathbb{X}^*(Z_{\mathbb{G}}) \otimes \mathbb{Q}, \\ \mathbb{X}_*(\mathbb{T}_i) \otimes \mathbb{Q} &= \mathbb{Q}\Phi^\vee \oplus \mathbb{X}_*(Z_{\mathbb{G}}) \otimes \mathbb{Q}, \end{aligned}$$

where $\mathbb{Q}\Phi$ is the subspace of $\mathbb{X}^*(\mathbb{T}_i) \otimes \mathbb{Q}$ spanned by the roots and $\mathbb{X}^*(Z_{\mathbb{G}})$ is the character group of the center of $\mathbb{G}$. The linear map $\alpha \mapsto \alpha^\vee$ induces an isomorphism of vector spaces $\mathbb{Q}\Phi \to \mathbb{Q}\Phi^\vee$ that sends the lines of the form $\mathbb{Q}\beta$ for a certain root β onto the lines of the form $\mathbb{Q}\beta'^\vee$ for a certain coroot β'. The dual linear map has the same property. The interesting cases are those of $B_n \leftrightarrow C_n$, F_4, and G_2, where there exist roots of different lengths. For a more detailed discussion, see [**83**, page 14].

1.12.4. — Since the reflection associated to a root α depends only on the $\mathbb{Q}$-line spanned by α, the isomorphisms ψ^* and ψ_* induce an isomorphism between the Weyl groups $\mathbb{W}_1 \xrightarrow{\sim} \mathbb{W}_2$ of the two reductive groups $\mathbb{G}_1$ and $\mathbb{G}_2$ that are paired.

1.12.5. — Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two split groups whose root data are isogenous. Let Out_{12} denote the group of automorphisms of $\mathbb{X}_*(\mathbb{T}_1) \otimes \mathbb{Q}$ that stabilize Φ_1, Δ_1 but also $\mathbb{X}^*(\mathbb{T}_2)$, Φ_2, Δ_2 (viewed as subsets of $\mathbb{X}_*(\mathbb{T}_1) \otimes \mathbb{Q}$), and similarly for the dual automorphisms of $\mathbb{X}_*(T_1) \otimes \mathbb{Q}$. For any k-scheme X, for any Out_{12}-torsor ρ_{12}, one can twist $\mathbb{G}_1$ and $\mathbb{G}_2$, with their pinnings, by ρ_{12} to obtain quasi-split forms G_1 and G_2. The pairs of quasi-split forms obtained by this procedure are called *paired*.

Since we have the isomorphism ψ^* between the $\mathbb{Q}$-vector spaces

$$\mathbb{X}^*(\mathbb{T}_1) \otimes \mathbb{Q} \simeq \mathbb{X}^*(\mathbb{T}_2) \otimes \mathbb{Q},$$

one may compare the positions of the two lattices $\mathbb{X}^*(\mathbb{T}_1)$ and $\mathbb{X}^*(\mathbb{T}_2)$ viewed as lattices in a common $\mathbb{Q}$-vector space. A prime number p is said to be *good* relative to ψ^* if p does not divide the integers

$$|\mathbb{X}^*(\mathbb{T}_1)/(\mathbb{X}^*(\mathbb{T}_1) \cap \mathbb{X}^*(\mathbb{T}_2))| \quad \text{and} \quad |\mathbb{X}^*(\mathbb{T}_2)/(\mathbb{X}^*(\mathbb{T}_2) \cap \mathbb{X}^*(\mathbb{T}_1))|.$$

If k is a field of characteristic good relative to ψ^*, then ψ^* induces an isomorphism $\mathbb{X}^*(\mathbb{T}_1) \otimes k \xrightarrow{\sim} \mathbb{X}^*(\mathbb{T}_2) \otimes k$.

Lemma 1.12.6. — *Let G_1 and G_2 be paired groups over a base X with good residual characteristics. Let T_1 and T_2 be the maximal tori of the pinnings of G_1 and G_2, and let $\mathfrak{t}_1, \mathfrak{t}_2$ be their Lie algebras. Then there exists a canonical isomorphism $\mathfrak{t}_1 \to \mathfrak{t}_2$ and a compatible isomorphism*

$$\nu : \mathfrak{c}_{G_1} \xrightarrow{\sim} \mathfrak{c}_{G_2}.$$

Proof. — We have

$$\mathfrak{t}_i = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{X}^*(\mathbb{T}_i) \otimes \mathcal{O}_X)),$$

where $\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{X}^*(\mathbb{T}_i)\otimes\mathcal{O}_X)$ is the symmetric $\mathcal{O}_X$-algebra associated with the free $\mathcal{O}_X$-module $\mathbb{X}^*(\mathbb{T}_i)\otimes\mathcal{O}_X$. If the residue characteristics of X are good, ψ^* induces an isomorphism of free $\mathcal{O}_X$-modules

$$\mathbb{X}^*(\mathbb{T}_2)\otimes\mathcal{O}_X \longrightarrow \mathbb{X}^*(\mathbb{T}_1)\otimes\mathcal{O}_X,$$

and thus an isomorphism $\mathbb{t}_1 \xrightarrow{\sim} \mathbb{t}_2$. We already saw that (ψ^*,ψ_*) induces an isomorphism $\mathbb{W}_1 \xrightarrow{\sim} \mathbb{W}_2$ which is clearly compatible with their actions on $\mathbb{t}_1 \xrightarrow{\sim} \mathbb{t}_2$. This yields an isomorphism between

$$\mathbb{c}_1 = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{X}^*(\mathbb{T}_i)\otimes\mathcal{O}_X))^{\mathbb{W}_1}$$

and

$$\mathbb{c}_2 = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{X}^*(\mathbb{T}_i)\otimes\mathcal{O}_X))^{\mathbb{W}_2}.$$

By applying the outer twisting by ρ_{12}, one obtains the isomorphism $\mathfrak{c}_1 = \mathfrak{c}_2$, as desired. □

Now assume the situation of the previous lemma. Let X be the disk $\mathrm{Spec}(\mathcal{O}_v)$ where $\mathcal{O}_v = k[[\epsilon_v]]$ with k a finite field of characteristic good relative to ψ^*. Let $a_1 \in \mathfrak{c}_{G_1}(\mathcal{O}_v)$ and $a_2 \in \mathfrak{c}_{G_2}(\mathcal{O}_v)$ be such that $\nu(a_1) = a_2$. The isogeny $\mathbb{T}_1 \to \mathbb{T}_2$ given by ψ^* induces, by twisting, an isogeny of the tori $J_{a_1} \to J_{a_2}$ by Lemma 1.4.3. Moreover, the isogeny induces an isomorphism between the Lie algebras of these tori under the assumption that the characteristic is good relative to ψ. One can thus transport Haar measures between $J_{a_1}(F_v)$ and $J_{a_2}(F_v)$ (and vice versa). We will denote by the same symbol $\mathrm{d}t_v$ these compatible Haar measures.

Theorem 1.12.7. — *Suppose that the residual characteristic is greater than twice the Coxeter number of G_1 and of G_2. Then we have the following equality between stable orbital integrals:*

$$\mathbf{SO}_{a_1}(1_{G_1}, \mathrm{d}t_v) = \mathbf{SO}_{a_2}(1_{G_2}, \mathrm{d}t_v),$$

where 1_{G_i} is the characteristic function of the compact subset $\mathfrak{g}_i(\mathcal{O}_v)$ in $\mathfrak{g}_i(F_v)$.

This equality was conjectured by Waldspurger, who calls it the nonstandard fundamental lemma. In [**83**], he showed that the combination of the standard fundamental lemma 1.11.1 and the nonstandard lemma above implies the twisted fundamental lemma.

1.13. Global Stabilization Formula. — We now return to the Langlands-Shelstad conjecture. The fundamental lemma consists of an identity of local orbital integrals. However, it is necessary to place it in its original global context, namely the stabilization of the trace formula. We now review the structure of the anisotropic part over a global field of positive characteristic. This review will later serve as a guide to study the structure of the cohomology of the Hitchin fibration.

Let $k = \mathbb{F}_q$ and let F be the field of rational functions on a smooth, projective, geometrically connected curve X over k. For every closed point $v \in |X|$, denote by F_v the completion of F with respect to the valuation defined by v, and by $\mathcal{O}_v$ its ring of integers. For simplicity, in this discussion we assume that G is a split semisimple group.

For each class $\xi \in \mathrm{H}^1(F,G)$, denote by G^ξ the inner form of G defined by the image of ξ in $\mathrm{H}^1(F,G^{\mathrm{ad}})$. We will call such a form G^ξ a strongly inner form. Instead of considering the trace formula for G, one considers the sum of the trace formulas over the strongly inner forms that are locally trivial. The stabilization of the sum becomes simpler and, moreover, admits a direct geometric interpretation. This stabilization process, due to Langlands and Kottwitz (see [**50**] and [**44**]), is presented here following the exposition in [**58**].

Consider the sum

$$\sum_{\xi\in\ker^1(F,G)}\ \sum_{\gamma\in\mathfrak{g}^{\xi,\mathrm{ani}}(F)/\sim} \mathbf{O}_\gamma(1_D), \tag{1.13.1}$$

where

– $\ker^1(F,G)$ is the set of isomorphism classes of G-torsors over F whose image in $\mathrm{H}^1(F_v,G)$ is trivial for all $v \in |X|$.

– $\mathfrak{g}^\xi$ is the form of $\mathfrak{g}$ over F defined by ξ.

– γ runs over the set of conjugacy classes of semisimple regular elements in $\mathfrak{g}^\xi(F)$ whose centralizer in $\mathfrak{g}^\xi(F\otimes_k \bar k)$ is an anisotropic torus.

– $\mathbf{O}_\gamma(1_D)$ is the global orbital integral

$$\mathbf{O}_\gamma(1_D) = \int_{G^\xi_\gamma(F)\backslash G(\mathbb{A})} 1_D(\mathrm{ad}(g)^{-1}\gamma)dg,$$

of the function

$$1_D = \bigotimes_{v\in|X|} 1_{D_v} : \mathfrak{g}(\mathbb{A}) \longrightarrow \mathbb{C},$$

where 1_{D_v} is the characteristic function of the compact open subset $\varepsilon^{-d_v}\mathfrak{g}(O_v)$ of $\mathfrak{g}(F_v)$ (the integers d_v being even and zero except for finitely many places). The integral converges for the anisotropic classes γ.

– dg is the Haar measure on $G(\mathbb{A})$ normalized so that $G(O_{\mathbb{A}})$ has volume one.

Consider the Chevalley characteristic morphism $\chi : \mathfrak{g} \to \mathfrak{c}$, as well as its twisted forms

$$\chi^\xi : \mathfrak{g}^\xi \longrightarrow \mathfrak{c}$$

for $\xi \in \mathrm{H}^1(F,G)$. Note that the group of automorphisms of G acts on $\mathfrak{c}$ via the outer automorphism group, so that twisting by ξ does not affect $\mathfrak{c}$. Thus every conjugacy class γ in $\mathfrak{g}^\xi(F)$ defines an element $a \in \mathfrak{c}(F)$. Since the centralizer of a semisimple regular γ depends only on a, there exists a subset $\mathfrak{c}^{\mathrm{ani}}(F)$ of $\mathfrak{c}(F)$ consisting of those elements a arising from semisimple regular anisotropic classes in $\mathfrak{g}^\xi(F\otimes_k \bar k)$. The sum (1.13.1) can be rewritten as a sum over $a \in \mathfrak{c}^{\mathrm{ani}}(F)$:

$$\sum_{a\in\mathfrak{c}^{\mathrm{ani}}(F)} \sum_{\xi\in\ker^1(F,G)} \sum_{\gamma\in\mathfrak{g}^\xi(F)/\sim,\,\chi(\gamma)=a} \mathbf{O}_\gamma(1_D). \tag{1.13.2}$$

For each element $a \in \mathfrak{c}^{\mathrm{ani}}(F)$, the Kostant section (see 1.2.1) produces an element $\gamma_0 = \epsilon(a) \in \mathfrak{g}(F)$ with $\chi(\gamma_0) = a$. Denote by J_a the centralizer I_{γ_0} of γ_0. Since we have restricted to the semisimple regular anisotropic part, J_a is an anisotropic torus. The dual torus $\hat J_a$, defined over $\overline{\mathbb{Q}}_\ell$, is equipped with a finite action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$ so that the group $\hat J_a^\Gamma$ of fixed points is finite.

For every $\xi \in \ker^1(F,G)$, the conjugacy classes $\gamma \in \mathfrak{g}^\xi(F)$ with $\chi(\gamma) = a$ are in bijection with the cohomology classes

$$\alpha = \mathrm{inv}(\gamma_0,\gamma) \in \mathrm{H}^1(F,J_a)$$

whose image in $\mathrm{H}^1(F,G)$ is the element ξ. Thus the set of pairs (ξ,γ) in the sum (1.13.2), with $\xi \in \ker^1(F,G)$ and γ a conjugacy class in $\mathfrak{g}^\xi(F)$ mapping to $a \in \mathfrak{c}^{\mathrm{ani}}(F)$, is in bijection with

$$\ker[\mathrm{H}^1(F,J_a) \to \bigoplus_{v\in|X|} \mathrm{H}^1(F_v,G)].$$

In order for a collection of conjugacy classes $(\gamma_v)_{v\in|X|}$ of $\mathfrak{g}(F_v)$ with $\chi(\gamma_v) = a$ to come from a pair (ξ,γ) as in (1.13.2), it is necessary and sufficient that $\gamma_v = \gamma_0$ for almost all v, and that

$$\sum_{v\in|X|} \alpha_v\big|_{\hat J_a^\Gamma} = 0, \tag{1.13.3}$$

where $\alpha_v = \mathrm{inv}_v(\gamma_0,\gamma_v)$ (see [**41**]). If this holds, the number of pairs (ξ,γ) mapping to the collection $(\gamma_v)_{v\in|X|}$ equals the cardinality of the group

$$\ker^1(F,J_a) = \ker[\mathrm{H}^1(F,J_a) \to \bigoplus_{v\in|X|} \mathrm{H}^1(F_v,J_a)].$$

We now bring in the local orbital integrals. To define these, one needs a Haar measure on the centralizer. For each a, choose an invariant volume form on the generic fiber of J_a. Such a form exists and is unique up to a scalar. It induces at each place v a Haar measure $\mathrm{d}t_v$ on $J_a(F_v)$ whose tensor product $\bigotimes_{v\in|X|}\mathrm{d}t_v$ is independent of the choice of the volume form. This is the Tamagawa measure.

Thus, the sum (1.13.2) can be rewritten as

$$\sum_{a\in\mathfrak{c}^{\mathrm{ani}}(F)} \left|\ker^1(F,J_a)\right|\, \tau(J_a) \sum_{(\gamma_v)_{v\in|X|}} \prod_v \mathbf{O}_{\gamma_v}(1_{D_v},\mathrm{d}t_v), \tag{1.13.4}$$

where the γ_v are conjugacy classes in $\mathfrak{g}(F_v)$ satisfying condition (1.13.3) and

$$\tau(J_a)=\mathrm{vol}(J_a(F)\backslash J_a(\mathbb{A}),\bigotimes_{v\in|X|}\mathrm{d}t_v)$$

is a Tamagawa number. By factoring out $\tau(J_a)$, we obtain a sum of products of local orbital integrals $\prod_v \mathbf{O}_{\gamma_v}(1_{D_v},\mathrm{d}t_v)$ instead of the global orbital integrals $\mathbf{O}_\gamma(1_D)$.

Using Ono's formula (see [**62**])

$$\left|\ker^1(F,J_a)\right|\,\tau(J_a)=\left|\pi_0(\hat{J}_a^\Gamma)\right|, \tag{1.13.5}$$

the sum (1.13.2) becomes

$$\sum_{a\in\mathfrak{c}^{\mathrm{ani}}(F)} \left|\pi_0(\hat{J}_a^\Gamma)\right| \sum_{(\gamma_v)_{v\in|X|}} \prod_v \mathbf{O}_{\gamma_v}(1_{D_v},\mathrm{d}t_v), \tag{1.13.6}$$

where the (γ_v) satisfy condition (1.13.3). Note that under the assumption that J_a is anisotropic, the group $\hat{J}_a^\Gamma$ is finite so that $\pi_0(\hat{J}_a^\Gamma)=\hat{J}_a^\Gamma$.

Using Fourier transform on the finite group $\hat{J}_a^\Gamma$, the sum (1.13.2) becomes

$$\sum_{a\in\mathfrak{c}^{\mathrm{ani}}(F)} \sum_{\kappa\in\hat{J}_a^\Gamma} \mathbf{O}_a^\kappa\Big(1_D,\bigotimes_{v\in|X|}\mathrm{d}t_v\Big) \tag{1.13.7}$$

with

$$\mathbf{O}_a^\kappa\Big(1_D,\bigotimes_{v\in|X|}\mathrm{d}t_v\Big)=\prod_{v\in|X|}\sum_{\substack{\gamma_v\in\mathfrak{g}(F_v)/\sim\\ \chi(\gamma_v)=a}}\langle \mathrm{inv}_v(\gamma_0,\gamma_v),\kappa\rangle\,\mathbf{O}_{\gamma_v}(1_{D_v},\mathrm{d}t_v). \tag{1.13.8}$$

The next step is to exchange the sums over a and over κ. By choosing an embedding of $\hat{J}_a$ into $\hat{G}$, the element κ defines a semisimple conjugacy class $[\kappa]$. The intersection $\hat{J}_a^\Gamma\cap[\kappa]$ of $\hat{J}_a^\Gamma$ with the conjugacy class $[\kappa]$ does not depend on the choice of embedding of $\hat{J}_a$ into $\hat{G}$. The sum (1.13.7) then becomes

$$\sum_{[\kappa]\in\hat{G}/\sim}\ \sum_{a\in\mathfrak{c}^{\mathrm{ani}}(F)}\ \sum_{\kappa\in\hat{J}_a^\Gamma\cap[\kappa]} \mathbf{O}_a^\kappa\Big(1_D,\bigotimes_{v\in|X|}\mathrm{d}t_v\Big). \tag{1.13.9}$$

Recall that we assumed G is a semisimple adjoint group. For each conjugacy class $[\kappa]$, choose a representative $\kappa\in\hat{G}$. Since $\hat{G}$ is semisimple and simply connected, the centralizer $\hat{G}_\kappa$ is a connected reductive group. Let $\hat{H}=\hat{G}_\kappa$ and let H be the reductive group dual to $\hat{H}$. Since we are only interested in the anisotropic part, we discard all H which are not semisimple. Suppose then that H is semisimple and consider the morphism

$$\nu_H:\mathfrak{c}_H^{\mathrm{ani}}(F)\longrightarrow\mathfrak{c}^{\mathrm{ani}}(F).$$

An element a lies in the image of ν_H if and only if $\hat{J}_a^\Gamma\cap[\kappa]$ is nonempty. In general, there is a canonical bijection between the set $\hat{J}_a^\Gamma\cap[\kappa]$ and the set of $a_H\in\mathfrak{c}_H^{\mathrm{ani}}(F)$ lying over a.

Assuming the fundamental lemma, the sum (1.13.2) becomes

$$\sum_H\ \sum_{a_H\in\mathfrak{c}_H^{\mathrm{ani}}(F)} \mathbf{SO}_a\Big(1_D,\bigotimes_{v\in|X|}\mathrm{d}t_v\Big), \tag{1.13.10}$$

where the first sum runs over the set of equivalence classes of elliptic endoscopic groups of G.

As we noted in [**57**, 1], counting the k-points (with k finite) of the moduli space of Higgs bundles essentially yields the expression (1.13.2). We have even proposed a geometric interpretation of the stabilization process (1.13.2) = (1.13.7) as a decomposition of the cohomology of the Hitchin fibration according to the action of its natural symmetries. This is a decomposition into a direct sum of a pure complex over the base of the Hitchin fibration.

The equality (1.13.2) = (1.13.10) may then be interpreted as an equality in a Grothendieck group between two pure complexes. Theorem 6.4.2 is a precise variant of this interpretation from which we will deduce the Langlands-Shelstad fundamental lemma 1.11.1.

2. Regular Centralizer and Kostant Section

In this chapter we recall the construction of the regular centralizer and the morphism from the regular centralizer to the centralizer of [**57**]. We also recall the Galois description of the regular centralizer due to Donagi and Gaitsgory [**23**].

We retain the notations from 1.3. In particular, $\mathbb{G}$ is a split reductive group over a field k and G is a quasi-split form of $\mathbb{G}$ over a k–scheme X. We assume that the characteristic of k does not divide the order of $\mathbb{W}$.

2.1. Regular Centralizer. — Let I be the group scheme of centralizers over $\mathfrak{g}$. The fiber of I over a point $x \in \mathfrak{g}$ is the subgroup of G which centralizes x, i.e.,

$$I_x = \{g \in G \mid \mathrm{ad}(g)x = x\}.$$

In general, the dimension of I_x depends on x, so that I is not flat over $\mathfrak{g}$. However, the restriction I^{reg} of I to the open subset $\mathfrak{g}^{\mathrm{reg}}$ is a smooth group scheme of relative dimension r. Since its generic fiber is a torus, it is a smooth commutative group scheme.

The following lemma [**57**, 3.2] is the starting point for our study of the Hitchin fibration. For the reader's convenience, we briefly recall its proof.

Lemma 2.1.1. **—** *There exists a unique smooth commutative group scheme J over $\mathfrak{c}$ endowed with a G–equivariant isomorphism*

$$(\chi^* J)|_{\mathfrak{g}^{\mathrm{reg}}} \cong I|_{\mathfrak{g}^{\mathrm{reg}}}.$$

Moreover, this isomorphism extends to a homomorphism $\chi^* J \to I$.

Proof. — Let $x_1, x_2 \in \mathfrak{g}^{\mathrm{reg}}(\bar{k})$ be such that $\chi(x_1) = \chi(x_2) = a$. Denote by I_{x_1} and I_{x_2} the fibers of I at x_1 and x_2, respectively. There exists $g \in G(\bar{k})$ such that $\mathrm{ad}(g)x_1 = x_2$. Conjugation by g induces an isomorphism $I_{x_1} \to I_{x_2}$, and since I_{x_1} is commutative, this isomorphism is independent of the choice of g. This allows one to define the fiber J_a of J at a.

The definition of J as an affine group scheme over $\mathfrak{c}$ is obtained by faithfully flat descent. Denote by I_1^{reg} and I_2^{reg} the pullbacks of $I^{\mathrm{reg}} = I|_{\mathfrak{g}^{\mathrm{reg}}}$ to $\mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{c}} \mathfrak{g}^{\mathrm{reg}}$ via the first and second projections, respectively. The descent datum for I^{reg} along the morphism $\chi^{\mathrm{reg}} : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c}$ consists of an isomorphism $\sigma_{12} : I_2^{\mathrm{reg}} \to I_1^{\mathrm{reg}}$ satisfying a cocycle condition. We now construct σ_{12}, leaving the verification of the cocycle condition to the reader.

The construction of σ_{12} also proceeds by descent. Consider the morphism

$$G \times \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{c}} \mathfrak{g}^{\mathrm{reg}}, \quad (g, x) \mapsto (x, \mathrm{ad}(g)x).$$

This morphism is smooth and surjective (hence faithfully flat). Over $G \times \mathfrak{g}^{\mathrm{reg}}$, there is a canonical isomorphism between I_1^{reg} and I_2^{reg} provided by the G–equivariance of I. To descend this isomorphism to $\mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{c}} \mathfrak{g}^{\mathrm{reg}}$, one must verify an identity over the fiber product of $G \times \mathfrak{g}^{\mathrm{reg}}$ with itself over $\mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{c}} \mathfrak{g}^{\mathrm{reg}}$. After identifying this fiber product with $G \times I_1^{\mathrm{reg}}$, the required identity follows from the commutativity of the group scheme I_1^{reg}.

Thus we have constructed a smooth commutative group scheme J over $\mathfrak{c}$ equipped with a G–equivariant isomorphism

$$(\chi^*J)|_{\mathfrak{g}^{\mathrm{reg}}} \to I|_{\mathfrak{g}^{\mathrm{reg}}}.$$

This isomorphism extends to a homomorphism of group schemes $\chi^*J \to I$ because χ^*J is a smooth k–scheme, I is an affine k–scheme, and the complement $\chi^*J - \chi^*J|_{\mathfrak{g}^{\mathrm{reg}}}$ has codimension three in χ^*J. □

We call J the *regular centralizer*. In fact, one may take as a definition $J := \epsilon^* I$, where ϵ is the Kostant section from 1.2. Note that J is equipped with a $\mathbb{G}_m$–equivariant structure for the $\mathbb{G}_m$–action on $\mathfrak{c}$ defined by the exponents. By descent, one obtains a group scheme over $[\mathfrak{c}/\mathbb{G}_m]$, which we also denote by J (see [**57**, 3.3]).

2.2. On the Quotient $[\mathfrak{g}/G]$**. —** Since the Chevalley morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ is G–invariant, it factors through the quotient stack $[\mathfrak{g}/G]$, yielding the morphism

$$[\chi] : [\mathfrak{g}/G] \to \mathfrak{c}.$$

Recall that $[\mathfrak{g}/G]$ assigns to every k–scheme S the groupoid of pairs (E, ϕ) where E is a G–torsor on S and ϕ is a section of the adjoint bundle $\mathrm{ad}(E)$ associated with the adjoint representation of G.

Over $\mathfrak{c}$, we have defined a smooth commutative group scheme J. Let $\mathbf{B}J$ denote the classifying stack of J, which assigns to every $\mathfrak{c}$–scheme S the Picard groupoid of J–torsors on S. Lemma 2.1.1 shows that there is an action of $\mathbf{B}J$ on $[\mathfrak{g}/G]$ over $\mathfrak{c}$. Indeed, one may twist a pair $(E, \phi) \in [\mathfrak{g}/G](S)$ by a J–torsor using the homomorphism $\chi^*J \to I$ from Lemma 2.1.1.

Proposition 2.2.1. — *The morphism*

$$[\chi^{\mathrm{reg}}] : [\mathfrak{g}^{\mathrm{reg}}/G] \to \mathfrak{c}$$

is a gerbe banded by the regular centralizer J. Moreover, this gerbe is neutral.

Proof. — That $[\chi^{\mathrm{reg}}]$ is a gerbe follows from the fact that the restriction of the homomorphism $\chi^*J \to I$ to $\mathfrak{g}^{\mathrm{reg}}$ is a G–equivariant isomorphism by the construction of J. The Kostant section $\epsilon : \mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}}$, composed with the quotient map $\mathfrak{g}^{\mathrm{reg}} \to [\mathfrak{g}^{\mathrm{reg}}/G]$, neutralizes the gerbe. We denote this section by $[\epsilon] : \mathfrak{c} \to [\mathfrak{g}^{\mathrm{reg}}/G]$. □

***2.2.2.* —** It is thus natural to view $[\mathfrak{g}/G]$ as a kind of compactification of the Picard stack $\mathbf{B}J$. This mould can be used to construct more concrete geometric situations such as the affine Springer fiber and the Hitchin fibration by evaluating it on different schemes S. For affine Springer fibers, one evaluates it on the ring of formal power series (see 3). For the Hitchin fibration, one evaluates it on a smooth projective curve (see 4). For the latter, one must also take into account the $\mathbb{G}_m$–action which acts by homothety on $\mathfrak{g}$.

***2.2.3.* —** The regular centralizer J is equipped with a $\mathbb{G}_m$–action that lifts the $\mathbb{G}_m$–action on $\mathfrak{c}$. The classifying stack $\mathbf{B}J$ over $[\mathfrak{c}/\mathbb{G}_m]$ acts on $[\mathfrak{g}/G \times \mathbb{G}_m]$. This latter stack contains the open substack $[\mathfrak{g}^{\mathrm{reg}}/G \times \mathbb{G}_m]$. The morphism

$$[\chi^{\mathrm{reg}}/\mathbb{G}_m] : [\mathfrak{g}^{\mathrm{reg}}/G \times \mathbb{G}_m] \to [\mathfrak{c}/\mathbb{G}_m] \tag{2.2.4}$$

is again a gerbe banded by J. In general, this gerbe is not neutral. However, it becomes neutral after extracting a square root of the universal line bundle on $\mathbf{B}\mathbb{G}_m$. Consider the homomorphism $[2] : \mathbb{G}_m \to \mathbb{G}_m$ defined by $t \mapsto t^2$. It induces a morphism $\mathbf{B}[2] : \mathbf{B}\mathbb{G}_m \to \mathbf{B}\mathbb{G}_m$, which amounts to taking a square root of the universal line bundle on $\mathbf{B}\mathbb{G}_m$. We denote by a superscript $[2]$ the base change by this morphism. In particular, we have a morphism

$$[\chi/\mathbb{G}_m]^{[2]} : [\mathfrak{g}/G \times \mathbb{G}_m]^{[2]} \to [\mathfrak{c}/\mathbb{G}_m]^{[2]}.$$

Indeed, $[\mathfrak{c}/\mathbb{G}_m]^{[2]}$ is the quotient of $\mathfrak{c}$ by the $\mathbb{G}_m$–action defined as the square of the usual action by the exponents, and $[\mathfrak{g}/G\times\mathbb{G}_m]^{[2]}$ is the quotient of $\mathfrak{g}$ by the adjoint action of G and the square of the homothety. Consider the composition of the two homomorphisms

$$\mathbb{G}_m \to T\times\mathbb{G}_m \to G\times\mathbb{G}_m,$$

where the first is defined by $t\mapsto(2\rho(t),t)$ with 2ρ being the sum of the positive coroots. The Kostant section $\epsilon:\mathfrak{c}\to\mathfrak{g}$ is equivariant with respect to this homomorphism and hence induces a section of $[\chi/\mathbb{G}_m]^{[2]}$. We now rephrase the above in a more convenient form.

Lemma 2.2.5. — *Let S be a k–scheme equipped with a line bundle D and let*

$$h_D : S\to \mathbf{B}\mathbb{G}_m$$

be the morphism associated to the universal line bundle on $\mathbf{B}\mathbb{G}_m$. Let

$$a : S\to[\mathfrak{c}/\mathbb{G}_m]$$

be a morphism over h_D. Then the Kostant section together with the choice of a square root D' of D defines a section

$$[\epsilon]^{D'}(a): S\to[\mathfrak{g}^{\mathrm{reg}}/G\times\mathbb{G}_m].$$

2.3. The Center of G and the Connected Components of J. — The content of this paragraph is well known.

Proposition 2.3.1. — *If G is a group with connected center, then the regular centralizer J has connected fibers.*

Proof. — In the case of a regular nilpotent element, this is a theorem of Springer (see [**75**, III, 3.7 and 1.14] and [**73**, Theorem 4.11]). The general case reduces to the nilpotent case by the Jordan decomposition. Let $x\in\mathfrak{g}(\bar k)$ be a regular element with Jordan decomposition $x=s+n$, where $s\in\mathfrak{g}(\bar k)$ is semisimple and $n\in\mathfrak{g}(\bar k)$ is nilpotent with $[s,n]=0$. According to [**38**, 3, Lemmas 5 and 8], the centralizer G_s of s is a connected reductive subgroup of G, and moreover its center is connected. One then applies Springer's theorem to the regular nilpotent element n in $\mathrm{Lie}(G_s)$. □

Corollary 2.3.2. — *For every $x\in\mathfrak{g}^{\mathrm{reg}}(\bar k)$, the canonical homomorphism*

$$Z_G\to I_x$$

induces a surjective homomorphism

$$\pi_0(Z_G)\to\pi_0(I_x).$$

Proof. — Let G^{ad} be the adjoint group of G and I_x^{ad} the centralizer of x in G^{ad}. We have the exact sequence

$$1\to Z_G\to I_x\to I_x^{\mathrm{ad}}\to 1.$$

Since I_x^{ad} is connected by the previous proposition, the canonical map $\pi_0(Z_G)\to\pi_0(I_x)$ is surjective. □

2.4. Galois Description of J. — Following Donagi and Gaitsgory [**23**], one can describe the group scheme J over $\mathfrak{c}$ via the finite flat cover $\pi:\mathfrak{t}\to\mathfrak{c}$. Our presentation will be slightly different from theirs.

Consider the Weil restriction of the torus $T\times\mathfrak{t}$ from $\mathfrak{t}$ to $\mathfrak{c}$,

$$\Pi := \prod_{\mathfrak{t}/\mathfrak{c}}(T\times\mathfrak{t}) = \pi_*(T\times\mathfrak{t}).$$

As an abelian group scheme over $\mathfrak{c}$, we have

$$\Pi(S)=\mathrm{Hom}_{\mathfrak{t}}(S\times_{\mathfrak{c}}\mathfrak{t},T\times\mathfrak{t})$$

for every $\mathfrak{c}$–scheme S. This functor is representable by a commutative group scheme over $\mathfrak{c}$ because the morphism $\mathfrak{t} \to \mathfrak{c}$ is finite and flat. Since the Weil restriction preserves smoothness, Π is a smooth commutative group scheme over $\mathfrak{c}$ of relative dimension $r \cdot \sharp\,\mathbb{W}$. Over the open subset $\mathfrak{c}^{\mathrm{rs}}$, the cover $\mathfrak{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ is finite étale so that the restriction of Π to this open is a torus.

The finite étale group scheme W acts simultaneously on T and on $\mathfrak{t}$. The diagonal action of W on $T \times \mathfrak{t}$ induces an action of W on Π. The fixed points under W define a closed subfunctor J^1 of Π.

Lemma 2.4.1. — *The closed subfunctor J^1 of Π is a smooth commutative group scheme over $\mathfrak{c}$.*

Proof. — Since the order of W is prime to the characteristic, the W–fixed points in the smooth group scheme Π form a closed subfunctor J^1, which is smooth over k. Let x be a geometric point of Π fixed by W, and let a be its image in $\mathfrak{c}$. Let $\mathbf{T}_x$ be the tangent space of Π at x, and $\mathbf{T}_a$ the tangent space of $\mathfrak{c}$ at a. Since Π is smooth over $\mathfrak{c}$, the map $\mathbf{T}_x \to \mathbf{T}_a$ is surjective. The tangent space $\mathbf{T}_x J^1$ is the subspace of $\mathbf{T}_x$ fixed by W. Because the characteristic of k does not divide $\sharp\,\mathbb{W}$, the restriction of $\mathbf{T}_x \to \mathbf{T}_a$ to the W–fixed part remains surjective. It follows that J^1 is smooth over $\mathfrak{c}$. □

The following proposition is a strengthening of 1.4.2.

Proposition 2.4.2. — *There exists a canonical homomorphism $J \to J^1$ which is an isomorphism over the open subset $\mathfrak{c}^{\mathrm{rs}}$ of $\mathfrak{c}$.*

Proof. — We begin by constructing a homomorphism from J into the Weil restriction $\pi_*(T \times \mathfrak{t})$. By adjunction, this is equivalent to constructing a homomorphism

$$\pi^* J \to T \times \mathfrak{t}$$

of group schemes over $\mathfrak{t}$.

To do so, we require the simultaneous Grothendieck-Springer resolution. Let $\tilde{\mathfrak{g}}$ be the scheme of pairs (x, gB) with $x \in \mathfrak{g}$ and $gB \in G/B$ such that $\mathrm{ad}(g)^{-1}(x) \in \mathrm{Lie}(B)$ (here B denotes the Borel subgroup of the fixed pinning of G). Denote by $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ the projection on x. The projection $\mathrm{Lie}(B) \to \mathfrak{t}$ defines a morphism $\tilde{\chi} : \tilde{\mathfrak{g}} \to \mathfrak{t}$ which completes the commutative square

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\tilde{\chi}"] \arrow[d, "\pi_{\mathfrak{g}}"'] & \mathfrak{t} \arrow[d, "\pi"] \\ \mathfrak{g} \arrow[r, "\chi"'] & \mathfrak{c} \end{tikzcd}$$

Moreover, when restricted to the open subset $\mathfrak{g}^{\mathrm{reg}}$, one obtains a cartesian diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}}^{\mathrm{reg}} \arrow[r, "\tilde{\chi}^{\mathrm{reg}}"] \arrow[d, "\pi_{\mathfrak{g}}^{\mathrm{reg}}"'] & \mathfrak{t} \arrow[d, "\pi"] \\ \mathfrak{g}^{\mathrm{reg}} \arrow[r, "\chi^{\mathrm{reg}}"'] & \mathfrak{c} \end{tikzcd}$$

By 2.1.1, we have $(\chi^{\mathrm{reg}})^* J = I|_{\mathfrak{g}^{\mathrm{reg}}}$, so that to construct a homomorphism of $\mathfrak{t}$–group schemes $\pi^* J \to (T \times \mathfrak{t})$, it suffices to construct a homomorphism of group schemes over $\tilde{\mathfrak{g}}^{\mathrm{reg}}$

$$(\pi_{\mathfrak{g}}^{\mathrm{reg}})^*(I|_{\mathfrak{g}^{\mathrm{reg}}}) \to T \times \tilde{\mathfrak{g}}^{\mathrm{reg}},$$

which is G–equivariant. We now state a lemma.

Lemma 2.4.3. — *For every $(x, gB) \in \tilde{\mathfrak{g}}^{\mathrm{reg}}(\bar{k})$, we have $I_x \subset \mathrm{ad}(g)B$.*

Proof. — The assertion is clear for regular semisimple elements x, since for such x the centralizer I_x is a torus that acts on the fiber $\pi_{\mathfrak{g}}^{-1}(x)$, which is discrete; hence the torus must act trivially.

Now consider the group scheme H over $\tilde{\mathfrak{g}}^{\mathrm{reg}}$ whose fiber over (x, gB) is the subgroup of I_x consisting of elements h with $h \in \mathrm{ad}(g)B$. By construction, this is a closed subgroup scheme of

$(\pi_{\mathfrak{g}}^{\mathrm{reg}})^* I$ which agrees with $(\pi_{\mathfrak{g}}^{\mathrm{reg}})^* I|_{\mathfrak{g}^{\mathrm{reg}}}$ over the dense open set of pairs (x, gB) with x regular semisimple. Since $(\pi_{\mathfrak{g}}^{\mathrm{reg}})^* I|_{\mathfrak{g}^{\mathrm{reg}}}$ is flat over $\tilde{\mathfrak{g}}^{\mathrm{reg}}$, it follows that this closed subgroup scheme equals $(\pi_{\mathfrak{g}}^{\mathrm{reg}})^* I|_{\mathfrak{g}^{\mathrm{reg}}}$. □

Consider the group scheme $\underline{B}$ over G/B whose fiber over gB is the subgroup $\mathrm{ad}(g)B \subset G$. Denote by $\underline{B}|_{\tilde{\mathfrak{g}}^{\mathrm{reg}}}$ its base change to $\tilde{\mathfrak{g}}^{\mathrm{reg}}$. The above lemma shows that over $\tilde{\mathfrak{g}}^{\mathrm{reg}}$, we have a homomorphism

$$I|_{\tilde{\mathfrak{g}}^{\mathrm{reg}}} \to \underline{B}|_{\tilde{\mathfrak{g}}^{\mathrm{reg}}}.$$

Moreover, there is a homomorphism from $\underline{B}$ into the G/B–torus $T \times G/B$. Composing, we obtain a G–equivariant homomorphism

$$I|_{\tilde{\mathfrak{g}}^{\mathrm{reg}}} \to T \times \tilde{\mathfrak{g}}^{\mathrm{reg}},$$

which is an isomorphism over the regular semisimple locus. Over the regular semisimple locus, the W–action on $I|_{\tilde{\mathfrak{g}}^{\mathrm{reg}}}$ transfers to the diagonal action on $T \times \mathfrak{g}^{\mathrm{rs}}$. By adjunction, we obtain a homomorphism

$$I|_{\mathfrak{g}^{\mathrm{reg}}} \to (\pi_{\mathfrak{g}})_*(T \times \tilde{\mathfrak{g}}^{\mathrm{reg}}),$$

which factors through the fixed-point subfunctor under the diagonal W–action on $(\pi_{\mathfrak{g}})_*(T \times \tilde{\mathfrak{g}}^{\mathrm{reg}})$.

By descent, one obtains a homomorphism $J \to J^1$ which is an isomorphism over $\mathfrak{c}^{\mathrm{rs}}$. □

The previous proposition admits the following variant. Consider a reduction

$$\rho \wedge^{\Theta_\rho} \mathrm{Out}(\mathbb{G}) = \rho_G$$

of the torsor ρ_G (see 1.3.5). Then one has a finite flat cover

$$\pi_\rho : X_\rho \times \mathbb{t} \to \mathfrak{c}$$

which is generically étale Galois with Galois group $\mathbb{W} \rtimes \Theta_\rho$ (see 1.3.5). This gives a slightly different description of J^1.

Lemma 2.4.4. — *J^1 is canonically isomorphic to the subfunctor of fixed points in the Weil restriction*

$$\prod_{(X_\rho \times \mathbb{t})/\mathfrak{c}} (\mathbb{T} \times X_\rho \times \mathbb{t})$$

for the diagonal action of $\mathbb{W} \rtimes \Theta_\rho$ on $\mathbb{T} \times X_\rho \times \mathbb{t}$.

Proof. — Since the assertion is local for the étale topology on X, we may assume that ρ is trivial. In that case, the assertion is immediate. □

Following [**23**], we now define a subfunctor J' of J^1 which we will show coincides with the image of $J \to J^1$. This definition will require some preparations. The following construction is local for the étale topology on the base X, so we may assume G is split. For every root $\alpha \in \Phi$, let h_α be the hyperplane in $\mathfrak{t}$ which is the kernel of the differential $\mathrm{d}\alpha : \mathfrak{t} \to \mathbb{G}_a$. Let $s_\alpha \in W$ be the reflection with respect to the hyperplane h_α. Let T^{s_α} be the subgroup of T of elements fixed by s_α. Then one has the inclusion

$$\alpha(T^{s_\alpha}) \subset \{\pm 1\}.$$

Let x be a geometric point of $\mathfrak{t}$ such that $s_\alpha(x) = x$ and let a be its image in $\mathfrak{c}$. Since J^1 is the W–invariant part of the functor $\prod_{\mathfrak{t}/\mathfrak{c}}(T \times \mathfrak{t})$, there is a canonical homomorphism from J^1_a into the fiber $T \times \{x\}$. Its image is contained in $T^{s_\alpha} \times \{x\}$. Composing with the root $\alpha : T \to \mathbb{G}_m$, one obtains a homomorphism

$$\alpha_x : J^1_a \to \mathbb{G}_m$$

whose image is contained in $\{\pm 1\}$.

Let J^0 be the open subgroup scheme of J^1 consisting of the neutral components of the fibers. By construction, J^0_a is the neutral component of J^1_a, so that J^0_a is contained in the kernel of α_x.

Definition 2.4.5. — *Define J' to be the subfunctor of J^1 as follows. For every $\mathfrak{c}$–scheme S, the group $J'(S)$ is the subgroup of $J^1(S)$ consisting of morphisms*

$$f : S \times_{\mathfrak{c}} \mathfrak{t} \to T$$

such that for every geometric point x of $S \times_{\mathfrak{c}} \mathfrak{t}$ with $s_\alpha(x) = x$ for some root α, one has

$$\alpha(f(x)) = 1.$$

We have inclusions of functors $J^0 \subset J' \subset J^1$.

Lemma 2.4.6. — *The functor J' from the category of $\mathfrak{c}$–schemes to the category of abelian groups is representable by an open subgroup scheme of J^1.*

Proof. — It suffices to show that J' is representable by an open subscheme of J^1, since being a subgroup functor, the representing scheme will automatically be a group scheme. Consider the open subset U of $T \times \mathfrak{t}$ defined as the complement of the union, over all roots $\alpha \in \Phi$, of the closed subschemes $\alpha^{-1}(-1) \times h_\alpha$. The Weil restriction $\mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(U)$ is then an open subscheme of $\mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(T \times \mathfrak{t})$. Hence,

$$J' = \mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(U) \cap J^1,$$

so that J' is an open subgroup scheme of J^1. □

The following proposition is a variant of a theorem of Donagi and Gaitsgory [**23**, Theorem 11.6]. We give here a slightly different proof.

Proposition 2.4.7. — *The homomorphism $J \to J^1$ of Proposition 2.4.2 factors through the open subgroup scheme J' defined in Definition 2.4.5 and induces an isomorphism $J \to J'$.*

Proof. — To show that $J \to J^1$ factors through J', it suffices to prove that for every $a \in \mathfrak{c}(\bar{k})$, the homomorphism $\pi_0(J_a) \to \pi_0(J^1_a)$ factors through $\pi_0(J'_a)$. It is enough to check, fiberwise, that the homomorphism $\pi_0(J_a) \to \pi_0(J^1_a)$ factors through $\pi_0(J'_a)$. Recall that the homomorphism $Z_G \to J_a$ induces a surjective homomorphism $\pi_0(Z_G) \to \pi_0(J_a)$ (see 2.3.2). Thus, it suffices to verify that the homomorphism $Z_G \to J^1_a$ factors through J'_a. But this is evident since the restriction of any root to Z_G is trivial.

Since J and J' are smooth affine group schemes over $\mathfrak{c}$, to show that the homomorphism $J \to J'$ is an isomorphism, it suffices to do so over an open subset of $\mathfrak{c}$ whose complement has codimension two. The inverse image of $\mathfrak{D}_G$ in $\mathfrak{t}$ is the union of the hyperplanes h_α for the roots. Let $\mathfrak{D}_G^{\mathrm{sing}}$ be the closed subset of $\mathfrak{D}_G$ whose inverse image consists of points belonging to at least two hyperplanes h_α. It is clear that $\mathfrak{D}_G^{\mathrm{sing}}$ is a closed subset of codimension two in $\mathfrak{c}$. It suffices to prove that the isomorphism between J and J' on $\mathfrak{c} - \mathfrak{D}_G$ extends to an isomorphism on $\mathfrak{c} - \mathfrak{D}_G^{\mathrm{sing}}$.

Let $a \in (\mathfrak{c} - \mathfrak{D}_G^{\mathrm{sing}})(\bar{k})$. It suffices to show that the homomorphism $J \to J'$ is an isomorphism in an étale neighborhood of a. If $a \notin \mathfrak{D}_G$, there is nothing to prove because we have seen that $J = J' = J^1$ over the open $\mathfrak{c} - \mathfrak{D}_G$. Now suppose $a \in \mathfrak{D}_G - \mathfrak{D}_G^{\mathrm{sing}}$. Then one can find $s \in \mathfrak{t}(\bar{k})$ mapping to a such that s is annihilated by a unique root α. Let T_α be the kernel of $\alpha : T \to \mathbb{G}_m$ and H_α the centralizer of T_α. Let n be a regular nilpotent element of the Lie algebra $\mathfrak{h}_\alpha$ of H_α (which can be identified with a Lie subalgebra of $\mathfrak{g}$). Then the element $x = s + n$ is a regular element of $\mathfrak{g}$ with image $a \in \mathfrak{c}$. It is also a regular element of $\mathfrak{h}_\alpha$ with image a_{H_α} in the space of characteristic polynomials $\mathfrak{c}_{H_\alpha}$ of H_α. One verifies that the morphism $\mathfrak{c}_{H_\alpha} \to \mathfrak{c}$ sends a_{H_α} to a and is étale at that point. One can also verify that in a neighborhood of a_{H_α}, the inverse images of J, J', and J^1 to $\mathfrak{c}_{H_\alpha}$ all coincide with the same groups (now associated to H_α). Thus the problem reduces to the special case of groups of semisimple rank one.

A group of semisimple rank one is isomorphic to a product of SL_2, PGL_2, or GL_2 with a torus. Therefore, it suffices to restrict to these three groups. By a direct calculation, one verifies that the centralizer J has a nonconnected fiber in the case of SL_2, whereas in the other two cases its fibers are all connected. The same holds for J'. □

Corollary 2.4.8. — *The homomorphism $J \to J^1$ from 2.4.2 induces an isomorphism on the open subgroup schemes of the neutral components of the fibers.*

2.5. The Case of Endoscopic Groups. — Consider an endoscopic datum (κ, ρ_κ) for G and the associated endoscopic group H (see 1.8.1). There is a finite flat morphism

$$\nu : \mathfrak{c}_H \to \mathfrak{c}$$

between the X–schemes of characteristic polynomials of H and G (see 1.9). The regular centralizer J on $\mathfrak{c}$ and the regular centralizer J_H of H on $\mathfrak{c}_H$ are related as follows.
Proposition 2.5.1. — *There exists a canonical homomorphism*

$$\mu : \nu^* J \to J_H$$

which is an isomorphism over the open subset $\mathfrak{c}_H^{G-\mathrm{rs}} = \nu^{-1}(\mathfrak{c}^{\mathrm{rs}})$.
Proof. — We use the notations of 1.9. In particular, there is a finite flat cover

$$X_{\rho_\kappa} \times \mathbb{t} \to \mathfrak{c}$$

which realizes $\mathfrak{c}$ as the invariant quotient of $X_{\rho_\kappa} \times \mathbb{t}$ by $\mathbb{W} \rtimes \pi_0(\kappa)$. Similarly, $\mathfrak{c}_H$ is the invariant quotient of $X_{\rho_\kappa} \times \mathbb{t}$ by $\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa)$. Recall that the morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$ was constructed in 1.9 by producing a homomorphism

$$\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$$

compatible with the actions of these two groups on $X_{\rho_\kappa} \times \mathbb{t}$.

Consider the group scheme

$$J^1 = \prod_{(X_\rho \times \mathbb{t})/\mathfrak{c}} (T \times \mathbb{t})^{\mathbb{W} \rtimes \pi_0(\kappa)}$$

and its analogue for H

$$J_H^1 = \prod_{(X_\rho \times \mathbb{t})/\mathfrak{c}_H} (T \times \mathbb{t})^{\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa)}.$$

The obvious morphism

$$X_{\rho_\kappa} \times \mathbb{t} \to (X_{\rho_\kappa} \times \mathbb{t}) \times_\mathfrak{c} \mathfrak{c}_H$$

which is $\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa)$–equivariant, induces a homomorphism $\nu^* J^1 \to J_H^1$. One verifies that this homomorphism is an isomorphism over the open subset $\mathfrak{c}_H^{G-\mathrm{rs}}$ (see 1.9.2).

According to Corollary 2.4.8, we have a homomorphism $J \to J^1$ which induces an isomorphism on the open subgroup schemes of the neutral components of the fibers. To show that the homomorphism $\nu^* J^1 \to J_H^1$ restricts to a homomorphism $\nu^* J \to J_H$, it suffices to verify that for every $a_H \in \mathfrak{c}_H(\bar{k})$ with image $a \in \mathfrak{c}(\bar{k})$, the induced homomorphism

$$\pi_0(J_a^1) \to \pi_0(J_{H,a_H}^1)$$

sends the subgroup $\pi_0(J_a) \subset \pi_0(J_a^1)$ into the subgroup $\pi_0(J_{H,a_H}) \subset \pi_0(J_{H,a_H}^1)$. Since the set of roots of H is a subset of the set of roots for G, the conditions that define the subgroup $\pi_0(J_{H,a_H})$ in $\pi_0(J_{H,a_H}^1)$ are satisfied by the elements of $\pi_0(J_a)$ (see 2.4.5). The proposition follows. □

3. Affine Springer Fibers

By analogy with the Springer fibers in the simultaneous Grothendieck–Springer resolution, Kazhdan and Lusztig introduced the affine Springer fibers and studied their geometric properties. Goresky, Kottwitz, and MacPherson established the link between the affine Springer fibers and stable orbital integrals via the point counting on a certain quotient of the affine Springer fibers. In this chapter, we review the geometric properties of the affine Springer fibers following Kazhdan and Lusztig and provide some complementary results. The point counting will be revisited in Chapter 8.

Below are the notations that will be used in this chapter. Let k be a finite field with q elements and $\bar{k}$ a separable closure of k. Let F_v be a local field of equal characteristic, $\mathcal{O}_v$ its ring of integers whose residue field k_v is a finite extension of k. We denote by

$$X_v = \mathrm{Spec}(\mathcal{O}_v)$$

the associated formal disc and by

$$X_v^\bullet$$

the punctured formal disc. Let v be the closed point of X_v and η_v its generic point.

Set

$$\bar{\mathcal{O}}_v = \mathcal{O}_v \hat{\otimes}_k \bar{k} \quad \text{and} \quad \bar{X}_v = \mathrm{Spec}(\bar{\mathcal{O}}_v).$$

The set of connected components of $\bar{X}_v$ is in bijection with the set of embeddings of the extension k_v of k into $\bar{k}$:

$$\bar{X}_v = \bigsqcup_{\bar{v}:k_v \to \bar{k}} \bar{X}_{\bar{v}}.$$

By choosing a geometric point $\bar{\eta}_v$ in the generic fiber of $\bar{X}_{\bar{v}}$, one obtains the usual exact sequence

$$1 \to I_v \to \Gamma_v \to \mathrm{Gal}(\bar{k}/\bar{k}_v) \to 1,$$

where $\Gamma_v = \pi_1(\eta_v, \bar{\eta}_v)$ is the Galois group of F_v and

$$I_v = \pi_1(\bar{X}_{\bar{v}}, \bar{\eta}_v)$$

its inertia subgroup.

Let G be a quasi-split form of $\mathbb{G}$ over X_v associated to an $\mathrm{Out}(\mathbb{G})$–torsor ρ_G. By choosing a geometric point of $\bar{\eta}_{\mathrm{Out},v}$ of ρ_G lying over $\bar{\eta}_v$, one obtains a homomorphism

$$\rho_G^\bullet : \Gamma_v \to \mathrm{Out}(\mathbb{G})$$

which factors through $\mathrm{Gal}(\bar{k}/k_v)$. Over $\bar{X}_v$, the torsor ρ_G is trivial. The point in $\bar{\eta}_{\mathrm{Out},v}$ provides a trivialization of ρ_G over $\bar{X}_{\bar{v}}$.

3.1. Review of the Affine Grassmannian. — For every k–scheme S, denote by $X_v \hat{\times} S$ the v–adic completion of $X_v \times S$, and by $X_v^\bullet \hat{\times} S$ the open subset obtained by removing $\{v\} \times S$ from $X_v \hat{\times} S$. The affine Grassmannian is the functor $\mathcal{G}_v$ which assigns to every k–scheme the groupoid of G–torsors E_v on $X_v \hat{\times} S$ equipped with a trivialization on $X_v^\bullet \hat{\times} S$. Note that an automorphism of E_v that is trivial on $X_v^\bullet \hat{\times} S$ is necessarily trivial; hence this groupoid is a discrete category. One may also replace it by the set of isomorphism classes.

According to [**4**] and [**86**], $\mathcal{G}_v$ is strictly representable by an ind–scheme over k. More precisely, there exists an inductive system of projective k–schemes with closed immersions whose inductive limit represents the functor $\mathcal{G}_v$. Since G is a smooth group scheme with connected fibers, the set of k–points of $\mathcal{G}_v$ can be expressed as the quotient

$$\mathcal{G}_v(k) = G(F_v)/G(\mathcal{O}_v).$$

When $G = \mathrm{GL}_r$, this set is naturally identified with the set of $\mathcal{O}_v$–lattices in the F_v–vector space $F_v^{\oplus r}$.

The same definition holds when one replaces k by $\bar{k}$ and X_v by $\bar{X}_{\bar{v}}$ for every embedding $\bar{v} : k_v \to \bar{k}$. Then one obtains the affine Grassmannian $\mathcal{G}_{\bar{v}}$ defined over $\bar{k}$. One has the formula

$$\mathcal{G} \otimes_k \bar{k} = \prod_{\bar{v}:k_v \to \bar{k}} \mathcal{G}_{\bar{v}}.$$

3.2. Affine Springer Fibers. — We retain the notations from 1.3. In particular, we have a scheme $\mathfrak{c}$ over X_v obtained by outer twisting of the space of characteristic polynomials $\mathbb{c}$ of $\mathbb{g}$, as in Chevalley's theorem 1.1.1.

Define

$$\mathfrak{c}^{\heartsuit}(\mathcal{O}_v) = \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$$

to be the set of $\mathcal{O}_v$–points of $\mathfrak{c}$ whose generic fiber is regular semisimple. Recall that we have the Kostant section $\epsilon : \mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}}$ (see 1.3.4). For every $a \in \mathfrak{c}^{\heartsuit}(\mathcal{O}_v)$, one obtains a Kostant point

$$[\epsilon](a) \in [\mathfrak{g}^{\mathrm{reg}}/G],$$

which consists of the trivial G–torsor E_0 on X_v and a section

$$\gamma_0 \in \Gamma(X_v, \mathrm{ad}(E_0))$$

whose characteristic polynomial is a.

For each $a \in \mathfrak{c}^{\heartsuit}(\mathcal{O}_v)$, we define the affine Springer fiber $\mathcal{M}_v(a)$ as follows. The functor $\mathcal{M}_v(a)$ assigns to every k–scheme S the groupoid of pairs (E, ϕ), where E is a G–torsor on $X_v \hat{\times} S$ and ϕ is a section of $\mathrm{ad}(E)$, equipped with an isomorphism with (E_0, γ_0) on $X_v^{\bullet} \hat{\times} S$. In particular,

$$[\chi](E, \phi) = [\chi](E_0, \gamma_0) = a.$$

Note that by construction the G–torsor E_0 is trivial, so that E determines a point of the affine Grassmannian.

Proposition 3.2.1. — *The forgetful functor $(E, \phi) \mapsto E$ defines a morphism from the affine Springer fiber $\mathcal{M}_v(a)$ to the affine Grassmannian $\mathcal{G}_v$ that is a closed immersion. In particular, $\mathcal{M}_v(a)$ is strictly representable by an ind–scheme. Moreover, the reduced subscheme $\mathcal{M}_v^{\mathrm{red}}(a)$ underlying $\mathcal{M}_v(a)$ is representable by a scheme locally of finite type.*

Proof. — The first assertion is immediate. The second assertion was proved by Kazhdan and Lusztig [**36**]. □

Consider the set of k–points of $\mathcal{M}_v(a)$. Let (E, ϕ) be an object in $\mathcal{M}_v(a, k)$. Since the generic fibers of E and E_0 are identified, the data of E corresponds to a class $g \in G(F_v)/G(\mathcal{O}_v)$. Since ϕ is identified with γ_0 on the generic fiber, for ϕ to define a section on X_v of $\mathrm{ad}(E) \otimes D$, it is necessary and sufficient that $\mathrm{ad}(g)^{-1}\gamma_0 \in \mathfrak{g}(\mathcal{O}_v)$. Hence,

$$\mathcal{M}_v(a, k) = \{g \in G(F_v)/G(\mathcal{O}_v) \mid \mathrm{ad}(g)^{-1}\gamma_0 \in \mathfrak{g}(\mathcal{O}_v)\}.$$

We now consider a trivial variation of the above construction in the presence of a line bundle D' on X_v. This variation will be necessary to realize the passage between affine Springer fibers and Hitchin fibers.

Let $D = D'^{\otimes 2}$ and let

$$h_D : X_v \to \mathbf{B}\mathbb{G}_m$$

be the morphism to the classifying stack of $\mathbb{G}_m$ corresponding to the line bundle D. Fix a morphism

$$h_a : X_v \to [\mathfrak{c}/\mathbb{G}_m]$$

which fits into the commutative diagram

$$\begin{tikzcd} X_v \arrow[r, "h_a"] \arrow[dr, "h_D"'] & {[\mathfrak{c}/\mathbb{G}_m]} \arrow[d] \\ & \mathbf{B}\mathbb{G}_m \end{tikzcd}$$

Let $[\epsilon]^{D'}(a)$ be the Kostant point of a constructed as in Lemma 2.2.5. Denote by $a^\bullet$ and $h_a^\bullet$ the restrictions of a and h_a to $X_v^\bullet$, and denote by $[\epsilon]^{D'}(a^\bullet)$ the associated Kostant point.

Definition 3.2.2. — *The affine Springer fiber $\mathscr{M}_v(a)$ is defined as the functor that assigns to every k–scheme S the set $\mathscr{M}_v(a, S)$ of isomorphism classes of morphisms*

$$h_{E,\phi} : X_v \hat{\times} S \to [\mathfrak{g}/G \times \mathbb{G}_m]$$

that fit into the commutative diagram

$$\begin{tikzcd} X_v \hat{\times} S \arrow[r, "h_{E,\phi}"] \arrow[dr, "h_a"'] & {[\mathfrak{g}/G \times \mathbb{G}_m]} \arrow[d, "{[\chi]}"] \\ & {[\mathfrak{c}/\mathbb{G}_m]} \end{tikzcd}$$

and which come equipped with an isomorphism between the restriction of $h_{E,\phi}$ to $X_v^\bullet \hat{\times} S$ and the Kostant point $[\epsilon]^{D'}(a^\bullet)$.

The same definition holds when one replaces k by $\bar{k}$ and X_v by $\bar{X}_{\bar{v}}$ for every embedding $\bar{v} : k_v \to \bar{k}$.

3.3. Symmetries of an Affine Springer Fiber. — The regular centralizer allows one to define the symmetry group of an affine Springer fiber. Let $a \in \mathfrak{c}^\heartsuit(\mathscr{O}_v)$ and let $h_a : X_v \to \mathfrak{c}$ be the corresponding morphism. Set

$$J_a = h_a^* J,$$

the pullback of the regular centralizer.

Consider the Picard groupoid $\mathscr{P}_v(J_a)$ over $\mathrm{Spec}(k)$ which assigns to every k–scheme S the Picard groupoid $\mathscr{P}_v(J_a, S)$ of J_a–torsors on $X_v \hat{\times} S$ equipped with a trivialization on $X_v^\bullet \hat{\times} S$. One may check that for every k–scheme S, $\mathscr{P}_v(J_a, S)$ is a discrete Picard category. Moreover, the functor which assigns to S the group of isomorphism classes of $\mathscr{P}_v(J_a, S)$ is representable by an ind–group scheme over k, which we denote by $\mathscr{P}_v(J_a)$. The group of k–points $\mathscr{P}_v(J_a)(k)$ is canonically identified with the quotient

$$J_a(F_v)/J_a(\mathscr{O}_v).$$

If the fibers of J_a are connected, then the set of k–points of $\mathscr{P}_v(J_a)$ is identified with the quotient $J_a(F_v)/J_a(\mathscr{O}_v)$.

Lemma 2.1.1 allows one to define an action of $\mathscr{P}_v$ on $\mathscr{M}_v$. Indeed, for every $(E, \phi) \in \mathscr{M}_v(S)$, there is a homomorphism of group sheaves over $X_v \hat{\times} S$

$$J_a \longrightarrow \underline{\mathrm{Aut}}(E, \phi)$$

which is induced by Lemma 2.1.1. This allows one to twist (E, ϕ) by a J_a–torsor on $X_v \hat{\times} S$ that is trivialized on $X_v^\bullet \hat{\times} S$.

On the level of k–points, we can describe this action explicitly. To simplify the exposition, assume that J_a has connected fibers. The action of the group

$$\mathscr{P}_v(J_a)(k) = J_a(F_v)/J_a(\mathscr{O}_v)$$

on the set

$$\mathscr{M}_v(a)(k) = \{g \in G(F_v)/G(\mathscr{O}_v) \mid \mathrm{ad}(g)^{-1}\gamma_0 \in \mathfrak{g}(\mathscr{O}_v)\}$$

is described as follows. By 2.1.1, there is a canonical isomorphism from $J_a(F_v)$ to the centralizer $G_{\gamma_0}(F_v)$ of γ_0, which we denote by $j \mapsto \theta(j)$. Then $J_a(F_v)$ acts on the set of $g \in G(F_v)$ such that $\mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_v)$ by

$$j \cdot g = \theta(j)g.$$

In order for this to induce an action of $J_a(F_v)/J_a(\mathcal{O}_v)$ on $\mathcal{M}_v(a)(k)$, it is necessary and sufficient that for every $g \in \mathcal{M}_v(a)(k)$, we have

$$\theta(J_a(\mathcal{O}_v)) \subset \mathrm{ad}(g)G(\mathcal{O}_v).$$

Let $\gamma = \mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_v)$. By 2.1.1, the isomorphism

$$\mathrm{ad}(g)^{-1} \circ \theta : I_a \to G_{\gamma_0}$$

extends to a homomorphism of group schemes over X_v

$$\mathrm{ad}(g)^{-1} \circ \theta : J_a \longrightarrow I_\gamma,$$

which in particular implies that

$$\theta(J_a(\mathcal{O}_v)) \subset \mathrm{ad}(g)(I_{\gamma_0}(\mathcal{O}_v)) \subset \mathrm{ad}(g)G(\mathcal{O}_v).$$

We now consider the subfunctor $\mathcal{M}_v^{\mathrm{reg}}(a)$ of the affine Springer fiber $\mathcal{M}_v(a)$ consisting of those morphisms $h_{E,\phi} : X_v \to [\mathfrak{g}/G\times\mathbb{G}_m]$ that factor through the open substack $[\mathfrak{g}^{\mathrm{reg}}/G\times\mathbb{G}_m]$. Clearly, $\mathcal{M}_v^{\mathrm{reg}}(a)$ is an open subset of $\mathcal{M}_v(a)$.

Lemma 3.3.1. — *The open subset $\mathcal{M}_v^{\mathrm{reg}}(a)$ is a principal homogeneous space under the action of $\mathcal{P}_v(J_a)$.*

Proof. — This is a consequence of Lemma 2.2.1. □

Let $\bar v : k_v \to \bar k$ be an embedding. For every $a \in \mathfrak{c}(\bar{\mathcal{O}}_{\bar v}) \cap \mathfrak{c}^{\mathrm{rs}}(\bar F_{\bar v})$, one has the symmetry group $\mathcal{P}_{\bar v}(J_a)$ defined over $\bar k$ of the affine fiber $\mathcal{M}_{\bar v}(a)$. If $a \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$, then we have the formula

$$\mathcal{P}_v(J_a) \otimes_k \bar k = \prod_{\bar v : k_v \to \bar k} \mathcal{P}_{\bar v}(J_a).$$

In the rest of this chapter, we will work over $\bar k$ and discuss the geometric properties of $\mathcal{P}_{\bar v}(J_a)$. Recall that over $\bar X_{\bar v}$, G is split.

3.4. Projective Quotient of an Affine Springer Fiber. — Let $a \in \mathfrak{c}(\bar{\mathcal{O}}_{\bar v})$ whose generic fiber lies in $\mathfrak{c}^{\mathrm{rs}}(\bar F_{\bar v})$. According to Kazhdan and Lusztig, the functor $\mathcal{M}_{\bar v}(a)$ has an underlying reduced scheme $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$ which is locally of finite type (see Proposition 3.2.1 and [**36**]). They also showed that $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$, after taking a suitable quotient by a discrete group, is a projective scheme. We now state their result more precisely.

Let Λ be the maximal free quotient of $\pi_0(\mathcal{P}_{\bar v}(J_a))$. Choose an arbitrary lifting

$$\Lambda \to \mathcal{P}_{\bar v}(J_a)$$

which induces an action of Λ on $\mathcal{M}_{\bar v}(a)$ and on $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$.

Proposition 3.4.1. — *The discrete group Λ acts freely on $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$, and the quotient $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)/\Lambda$ is a projective $\bar k$–scheme.*

Proof. — Proposition 1 of [**36**, page 138] shows that Λ acts freely on $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$ and that by truncating the reduced Springer fiber $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)$ one obtains projective schemes which surject onto the quotient $\mathcal{M}_{\bar v}^{\mathrm{red}}(a)/\Lambda$. It follows that the quotient is also a projective scheme. □

3.5. Approximation. — According to a well-known theorem of Harish-Chandra, the semisimple regular orbital integrals are locally constant. In this section, we will show a geometric variant, stronger than this theorem.

Proposition 3.5.1. — *Let $a \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$, and let $\mathcal{M}_v(a)$ be the affine Springer fiber and $\mathcal{P}_v(J_a)$ the symmetry group of $\mathcal{M}_v(a)$. Then there exists an integer N such that for every finite extension k' of k, and for every*

$$a' \in \mathfrak{c}(\mathcal{O}_v \otimes_k k')$$

with

$$a \equiv a' \mod \epsilon_v^N,$$

the affine Springer fiber $\mathcal{M}_v(a')$, endowed with the action of $\mathcal{P}_v(J_{a'})$, is isomorphic to $\mathcal{M}_v(a) \otimes_k k'$ endowed with the action of $\mathcal{P}_v(J_a) \otimes_k k'$.

Proof. — Consider the cameral cover $\widetilde{X}_{a,v}$ of X_v associated to a, defined by the cartesian diagram

$$\begin{tikzcd} \widetilde{X}_{a,v} \arrow[r] \arrow[d, "\pi_a"'] & \mathfrak{t} \\ X_v \arrow[r, "a"'] & \mathfrak{c}. \end{tikzcd}$$

The morphism π_a is finite, flat, and generically étale. The cameral cover $\widetilde{X}_{a,v}$ is endowed with an action of W coming from its action on $\mathfrak{t}$. We also associate to $a' \in \mathfrak{c}(\mathcal{O}_v \otimes_k k')$ its cameral cover $\widetilde{X}_{a',v} \to X_v \otimes_k k'$.

Lemma 3.5.2. — *Let $a \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$. Then there exists an integer N such that for every finite extension k' of k, and for every*

$$a' \in \mathfrak{c}(\mathcal{O}_v \otimes_k k')$$

with

$$a \equiv a' \mod \epsilon_v^N,$$

the cameral cover $\widetilde{X}_{a',v}$ of $X_v \otimes_k k'$ endowed with the W–action is isomorphic to the cameral cover $\widetilde{X}_{a,v} \otimes_k k'$ endowed with the W–action. Moreover, one may require that the isomorphism lifts the evident isomorphism modulo ϵ_v^N.

Proof. — This is a special case of a lemma of Artin–Hironaka [**3**, Lemma 3.12]. The second assertion is implicit in Artin's proof. □

Assume that the cameral covers $\widetilde{X}_{a,v}$ and $\widetilde{X}_{a',v}$ of $X_v \otimes_k k'$ are isomorphic. We now show that $\mathcal{M}_v(a) \otimes_k k'$, with the action of $\mathcal{P}_v(J_a) \otimes_k k'$, and $\mathcal{M}_v(a')$, with the action of $\mathcal{P}_v(J_{a'})$, are isomorphic. Replacing k by k', we may assume $k = k'$.

To do this, we need a description of the affine Springer fiber in terms of the regular centralizer. Let $\gamma_0 = \epsilon(a) : X_v \to \mathfrak{g}$ be the Kostant section associated to a. Then there is a canonical isomorphism between J_a and the X_v–group scheme $I_{\gamma_0} = \gamma_0^* I$, where I is the group scheme of centralizers over $\mathfrak{g}$. The obvious homomorphism $I_{\gamma_0} \to G$ induces an injective homomorphism of vector bundles on X_v

$$\mathrm{Lie}(I_{\gamma_0}) \to \mathfrak{g}.$$

The following lemma shows that the affine Springer fiber $\mathcal{M}_v(a)$ depends only on the commutative Lie subalgebra $\mathrm{Lie}(I_{\gamma_0})$ of $\mathfrak{g}$.

Lemma 3.5.3. — *Let $g \in G(F_v)$. Then* $\mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_v)$ *if and only if* $\mathrm{ad}(g)^{-1}\mathrm{Lie}(I_{\gamma_0}) \subset \mathfrak{g}(\mathcal{O}_v)$.

Proof. — Since $\gamma_0 \in \mathrm{Lie}(I_{\gamma_0})$, if $\mathrm{ad}(g)^{-1}\mathrm{Lie}(I_{\gamma_0}) \subset \mathfrak{g}(\mathcal{O}_v)$ then immediately $\mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_v)$. Conversely, suppose $g \in G(F_v)$ such that $\gamma = \mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_v)$. Let $I_\gamma = \gamma^* I$. The fact (see 2.1.1) that the generic isomorphism

$$\mathrm{ad}(g)^{-1} \circ \theta : I_a \longrightarrow I_{\gamma_0}$$

extends to a homomorphism $J_a \longrightarrow I_\gamma$ implies, in particular, that

$$\mathrm{ad}(g)^{-1}\mathrm{Lie}(I_{\gamma_0}) \subset \mathfrak{g}(\mathcal{O}_v).$$

□

To complete the proof of Proposition 3.5.1, it suffices to prove the following lemma. □

Lemma 3.5.4. — *Let $a, a' \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$ be such that $a \equiv a' \mod \epsilon_v$. Suppose there exists an isomorphism between the cameral covers $\widetilde{X}_{a,v}$ and $\widetilde{X}_{a',v}$ (with their W–action) that lifts the evident isomorphism modulo ϵ_v^N. Let $\gamma_0 = \epsilon(a)$ be the Kostant section for a and $\gamma'_0 = \epsilon(a')$ the Kostant section for a'. Let I_{γ_0} and $I_{\gamma'_0}$ be the centralizer group schemes in G of γ_0 and γ'_0. Then there exists $g \in G(\mathcal{O}_v)$ such that*

$$\mathrm{ad}(g)^{-1} I_{\gamma_0} = I_{\gamma'_0}.$$

Proof. — This lemma is a consequence of a result of Donagi and Gaitsgory [**23**, Theorem 11.8]. For the reader's convenience, we extract the part useful for our purpose. Since the group schemes I_{γ_0} and $I_{\gamma'_0}$ are completely determined by the cameral covers $\widetilde{X}_{a,v}$ and $\widetilde{X}_{a',v}$ (see 2.4.7), the isomorphism between $\widetilde{X}_{a,v}$ and $\widetilde{X}_{a',v}$ induces an isomorphism

$$\iota : I_{\gamma_0} \to I_{\gamma'_0}.$$

This isomorphism transports $\gamma_0 \in \mathrm{Lie}(I_{\gamma_0})$ to an element $\iota(\gamma_0) \in \mathrm{Lie}(I_{\gamma'_0})$. Since the special fibers of $\iota(\gamma_0)$ and γ'_0 coincide, the section $\iota(\gamma_0) : X_v \longrightarrow \mathfrak{g}$ factors through the regular locus $\mathfrak{g}^{\mathrm{reg}}$. One deduces that

$$I_{\iota(\gamma_0)} = I_{\gamma'_0}$$

as subgroup schemes of G.

We have two sections

$$\gamma_0, \iota(\gamma_0) : X_v \longrightarrow \mathfrak{g}^{\mathrm{reg}}$$

which have the same characteristic polynomial a and are congruent modulo ϵ_v. Since the morphism

$$G \times_X \mathfrak{g}^{\mathrm{reg}} \longrightarrow \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{c}} \mathfrak{g}^{\mathrm{reg}}$$

is smooth, there exists $g \in G(\mathcal{O}_v)$ such that $g \equiv 1 \mod \epsilon_v$ and such that $\mathrm{ad}(g^{-1})(\gamma_0) = \iota(\gamma_0)$. Hence,

$$\mathrm{ad}(g)^{-1}(I_{\gamma_0}) = I_{\iota(\gamma_0)} = I_{\gamma'_0}.$$

This is what we wanted. □

3.6. The Linear Case. — We now examine the affine Springer fibers for the linear group, following Laumon [**51**]. (See also [**59**] for a similar discussion in the case of classical groups.) Let $G = \mathrm{GL}(r)$ with $r < p$.

We retain the notations fixed at the beginning of this chapter. A point

$$a \in \mathfrak{c}(\bar{\mathcal{O}}_{\bar{v}})$$

is represented by a monic polynomial of degree r in the variable t

$$P(a,t) = t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r \in \bar{\mathcal{O}}_{\bar{v}}[t].$$

Form the finite, flat $\bar{\mathcal{O}}_{\bar{v}}$–algebra of rank r

$$B = \bar{\mathcal{O}}_{\bar{v}}[t]/P(a,t)$$

and let

$$E = B \otimes_{\bar{\mathcal{O}}_{\bar{v}}} \bar{F}_{\bar{v}}.$$

The hypothesis $a \in \mathfrak{c}^{\heartsuit}(\bar{\mathcal{O}}_{\bar{v}})$ implies that E is a finite étale $\bar{F}_{\bar{v}}$–algebra of dimension r. One may write E as a product

$$E = E_1 \times \cdots \times E_s$$

of s separable extensions of $\bar{F}_{\bar{v}}$ with $s \leq r$.

3.6.1. — In this situation, the affine Springer fiber may be described in terms of lattices. The $\bar{k}$–points of the affine Springer fiber $\mathcal{M}_{\bar{v}}(a)$ are the set of B–lattices in E, i.e., the B–modules $\mathcal{L}$ contained in the $\bar{F}_{\bar{v}}$–vector space E which are $\bar{\mathcal{O}}_{\bar{v}}$–lattices. The $\bar{k}$–points of the regular part $\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$ consist of the B–lattices in E that are free B–modules. The $\bar{k}$–points of the symmetry group $\mathcal{P}_{\bar{v}}(J_a)$ are given by the group $E^{\times}/B^{\times}$.

3.6.2. — The normalization $B^{\flat}$ of B is the ring of integers of E. Then there is an exact sequence

$$1 \to (B^{\flat})^{\times}/B^{\times} \to E^{\times}/B^{\times} \to E^{\times}/(B^{\flat})^{\times} \to 1,$$

where $(B^{\flat})^{\times}/B^{\times}$ is the group of $\bar{k}$–points of the neutral component of the group $\mathcal{P}_{\bar{v}}(J_a)$. Consequently, the group of connected components $\pi_0(\mathcal{P}_{\bar{v}}(J_a))$ is canonically isomorphic to $E^{\times}/(B^{\flat})^{\times}$, which is a free abelian group equipped with a basis indexed by the set of connected components of $\mathrm{Spec}(B^{\flat})$.

3.6.3. — In this case, the dimension of $\mathcal{P}_{\bar{v}}(J_a)$ equals the Serre δ–invariant

$$\delta_{\bar{v}}(a) = \dim_k(B^{\flat}/B).$$

Moreover, this integer can be expressed in terms of the discriminant. Let

$$d_{\bar{v}}(a) = \mathrm{val}_{\bar{v}}(\mathfrak{D}(a))$$

be the $\bar{v}$–adic valuation of the discriminant of a. Using [**70**, III.3 Proposition 5 and III.6 Corollary 1] and the hypothesis $r < p$, one obtains the formula

$$\delta_{\bar{v}}(a) = \frac{d_{\bar{v}}(a) - c_{\bar{v}}(a)}{2},$$

where $c_{\bar{v}}(a) = r - s$.

3.7. Dimension. — In [**36**], Kazhdan and Lusztig showed that

$$\dim(\mathcal{M}_{\bar{v}}(a)) = \dim(\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)).$$

In fact, one can deduce a more precise statement from their results.

Proposition 3.7.1. — *The complement of the regular open subset $\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$ in the affine Springer fiber $\mathcal{M}_{\bar{v}}(a)$ has dimension strictly less than that of $\mathcal{M}_{\bar{v}}(a)$.*

Proof. — To discuss the dimension, one may neglect the nilpotents in the structure sheaves of $\mathcal{M}_{\bar{v}}(a)$. As recalled in Section 3.2, $\mathcal{M}_{\bar{v}}(a)$ is the subvariety of the affine Grassmannian consisting of those $g \in G(F_{\bar{v}})/G(\mathcal{O}_{\bar{v}})$ such that $\mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\mathcal{O}_{\bar{v}})$.

Following Kazhdan and Lusztig, consider also the subvariety $\mathcal{B}_{\bar{v}}(a)$ of the affine flag variety fixed by γ_0, i.e., those $g \in G(F_{\bar{v}})/\mathrm{Iw}_{\bar{v}}$ such that $\mathrm{ad}(g)^{-1}\gamma_0 \in \mathrm{Lie}(\mathrm{Iw}_{\bar{v}})$, where $\mathrm{Iw}_{\bar{v}}$ is an Iwahori subgroup of $G(\mathcal{O}_{\bar{v}})$. In [**36**], Kazhdan and Lusztig proved that the affine Springer fibers for Iwahori subgroups are equidimensional. The equidimensionality of classical Springer fibers was previously proved by Spaltenstein [**72**].

The morphism $\mathcal{B}_{\bar{v}}(a) \to \mathcal{M}_{\bar{v}}(a)$ is finite over $\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$. The fibers over points $x \in \mathcal{M}_{\bar{v}}(a) \setminus \mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$ are nonempty and have dimension at least one. The equidimensionality of $\mathcal{B}_{\bar{v}}(a)$ then implies that the irreducible components of $\mathcal{M}_{\bar{v}}(a) - \mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$ are of strictly smaller dimension than $\dim(\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a))$. □

Corollary 3.7.2. — *If $\dim(\mathcal{M}_v(a)) = 0$, then the dense open $\mathcal{M}_v^{\mathrm{reg}}(a)$ equals $\mathcal{M}_v(a)$.*

Kazhdan and Lusztig also conjectured a formula expressing the above dimension in terms of the discriminant and an effective defect term related to monodromy, which was later proved by Bezrukavnikov in [**8**]. Let us recall this formula.

Keep the notations fixed at the beginning of this chapter. Let

$$a : \bar{X}_{\bar{v}} \to \mathfrak{c}$$

be a morphism whose image is not contained in the discriminant divisor $\mathfrak{D}_G$. By pulling back $\mathfrak{D}_G$, one obtains an effective Cartier divisor on $\bar{X}_{\bar{v}}$ supported on its closed point. Its degree is a natural number that we denote by

$$d_{\bar{v}}(a) := \deg_{\bar{v}}(a^*\mathfrak{D}_G).$$

By pulling back the finite étale cover $\pi : \mathfrak{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ via the morphism $a : \bar{X}_{\bar{v}}^{\bullet} \to \mathfrak{c}^{\mathrm{rs}}$, one obtains a W–torsor π_a over $\bar{X}_{\bar{v}}^{\bullet}$. Recall that over $\bar{X}_{\bar{v}}$ the quasi-split form splits so that, after base change to $\bar{X}_{\bar{v}}$, we have $W = \mathbb{W}$. By choosing a geometric point of this torsor over the geometric point $\bar{\eta}_{\bar{v}}$ of $\bar{X}_{\bar{v}}$, one obtains a homomorphism

$$\pi_a^{\bullet} : I_v \to \mathbb{W}. \tag{3.7.3}$$

Since the characteristic does not divide the order of $\mathbb{W}$, $\pi_a^{\bullet}$ factors through the tame quotient I_v^{tame} of I_v, so that its image is a cyclic subgroup. Denote

$$c_{\bar{v}}(a) := \dim(\mathfrak{t}) - \dim(\mathfrak{t}^{\pi_a^{\bullet}(I_v)}). \tag{3.7.4}$$

The following result was proved by Bezrukavnikov in [**8**].

Proposition 3.7.5. — *We have the equalities*

$$\dim(\mathscr{M}_{\bar{v}}^{\mathrm{reg}}(a)) = \dim(\mathscr{P}_{\bar{v}}(J_a)) = \frac{d_{\bar{v}}(a) - c_{\bar{v}}(a)}{2}.$$

We now set

$$\delta_{\bar{v}}(a) = \dim(\mathscr{P}_{\bar{v}}(J_a))$$

and refer to it as the *local δ–invariant*.

3.8. Néron Model. — The dimension of $\mathscr{P}_{\bar{v}}(J_a)$ can be computed in another way using the Néron model of J_a. In fact, the Néron model allows us to fully analyze the structure of $\mathscr{P}_{\bar{v}}(J_a)$. According to Bosch, Lütkebohmert, and Raynaud [**10**], there exists a unique smooth group scheme of finite type $J_a^{\flat}$ over $\bar{X}_{\bar{v}}$ with the same generic fiber as J_a and which is maximal with this property; that is, for every other smooth group scheme J' of finite type over $\bar{X}_{\bar{v}}$ with the same generic fiber, there exists a canonical homomorphism $J' \to J_a^{\flat}$ inducing the identity on the generic fiber. In particular, there is a canonical homomorphism $J_a \to J_a^{\flat}$. On the level of points, this homomorphism defines the inclusions

$$J_a(\bar{\mathcal{O}}_{\bar{v}}) \subset J_a^{\flat}(\bar{\mathcal{O}}_{\bar{v}}) \subset J_a(\bar{F}_{\bar{v}}),$$

where $J_a^{\flat}(\bar{\mathcal{O}}_{\bar{v}})$ is the maximal bounded subgroup of $J_a(F_{\bar{v}})$. We call $J_a^{\flat}$ the *Néron model* of J_a. (In the terminology of [**10**], $J_a^{\flat}$ is called the finite type Néron model, to distinguish it from their Néron model which is only locally of finite type.)

Replacing in the definition of $\mathscr{P}_{\bar{v}}(J_a)$ the group scheme J_a by its Néron model $J_a^{\flat}$, we obtain an ind–group scheme $\mathscr{P}_{\bar{v}}(J_a^{\flat})$ over $\bar{k}$. The homomorphism of group schemes $J_a \to J_a^{\flat}$ induces a homomorphism of ind–groups

$$\mathscr{P}_{\bar{v}}(J_a) \to \mathscr{P}_{\bar{v}}(J_a^{\flat})$$

which gives a decomposition of $\mathscr{P}_{\bar{v}}(J_a)$.

Lemma 3.8.1. — *The group $\mathscr{P}_{\bar{v}}(J_a^\flat)$ is homeomorphic to a free abelian group of finite type. The homomorphism*

$$p_{\bar{v}} : \mathscr{P}_{\bar{v}}(J_a) \to \mathscr{P}_{\bar{v}}(J_a^\flat)$$

is surjective. Its kernel $\mathscr{R}_{\bar{v}}(a)$ is an affine group scheme of finite type over $\bar{k}$.

Proof. — Since $J_a^\flat(\bar{\mathcal{O}}_{\bar{v}})$ is the maximal bounded subgroup of the torus $J_a(\bar{F}_{\bar{v}})$, the quotient $J_a(\bar{F}_{\bar{v}})/J_a^\flat(\bar{\mathcal{O}}_{\bar{v}})$ is a free abelian group of finite type. The homomorphism

$$\mathscr{P}_{\bar{v}}(J_a)(\bar{k}) = J_a(\bar{F}_{\bar{v}})/J_a(\bar{\mathcal{O}}_{\bar{v}}) \to J_a(\bar{F}_{\bar{v}})/J_a^\flat(\bar{\mathcal{O}}_{\bar{v}}) = \mathscr{P}_{\bar{v}}(J_a^\flat)(\bar{k})$$

is clearly surjective.

For a sufficiently large integer N, $J_a(\mathcal{O}_{\bar{v}})$ contains the kernel of the homomorphism $J_a^\flat(\bar{\mathcal{O}}_{\bar{v}}) \to J_a^\flat(\bar{\mathcal{O}}_{\bar{v}}/\varepsilon_{\bar{v}}^N\bar{\mathcal{O}}_{\bar{v}})$. It follows that $\mathscr{R}_{\bar{v}}(a)$ is a quotient of the Weil restriction

$$\prod_{\mathrm{Spec}(\bar{\mathcal{O}}_{\bar{v}}/\varepsilon_{\bar{v}}^N\bar{\mathcal{O}}_{\bar{v}})/\mathrm{Spec}(\bar{k})} J_a^\flat \otimes_{\bar{\mathcal{O}}_{\bar{v}}} (\bar{\mathcal{O}}_{\bar{v}}/\varepsilon_{\bar{v}}^N\bar{\mathcal{O}}_{\bar{v}}),$$

which is an affine smooth $\bar{k}$–group scheme of finite type. Hence the lemma. □

The Néron model can be explicitly constructed using the normalization of the cameral cover. Let $\widetilde{X}_{a,\bar{v}}$ be the inverse image of $\mathfrak{t} \to \mathfrak{c}$ via the morphism $a : \bar{X}_{\bar{v}} \to \mathfrak{c}$. Consider the normalization $\widetilde{X}_{a,\bar{v}}^\flat$ of $\widetilde{X}_{a,\bar{v}}$, which is a finite flat scheme over $\bar{X}_{\bar{v}}$ endowed with a W–action. Since the residue field of $\bar{X}_{\bar{v}}$ is algebraically closed, $\widetilde{X}_{a,\bar{v}}^\flat$ is a complete regular one-dimensional semi-local scheme.

Proposition 3.8.2. — *Let $\widetilde{X}_{a,\bar{v}}^\flat = \mathrm{Spec}(\widetilde{\mathcal{O}}_{\bar{v}}^\flat)$ be the normalization of $\widetilde{X}_{a,\bar{v}}$. Then the Néron model $J_a^\flat$ of J_a is the subgroup of fixed points under the diagonal W–action in the Weil restriction of the torus $T \times_{\bar{X}_{\bar{v}}} \widetilde{X}_{a,\bar{v}}^\flat$ from $\widetilde{X}_{a,\bar{v}}^\flat$ to $\bar{X}_{\bar{v}}$, that is,*

$$J_a^\flat = \prod_{\widetilde{X}_{a,\bar{v}}^\flat/\bar{X}_{\bar{v}}} \big(T \times_{\bar{X}_{\bar{v}}} \widetilde{X}_{a,\bar{v}}^\flat\big)^W.$$

Proof. — Denote by $\pi_a^\flat$ the morphism $\widetilde{X}_{a,\bar{v}}^\flat \to \bar{X}_{\bar{v}}$. As in Lemma 2.4.1, the subfunctor of fixed points under the diagonal W–action on the Weil restriction

$$\prod_{\widetilde{X}_{a,\bar{v}}^\flat/\bar{X}_{\bar{v}}} \big(T \times_{\bar{X}_{\bar{v}}} \widetilde{X}_{a,\bar{v}}^\flat\big)$$

is a smooth group scheme of finite type over $\bar{X}_{\bar{v}}$. It is more convenient to work with the sheaf $(\pi_{a*}^\flat T)^W$, which represents the Weil restriction above.

According to the Galois description of the regular centralizer (see 2.4), the pullback $\pi_a^{\flat *}J_a$ over

$$\widetilde{X}_{a,\bar{v}}^{\flat\bullet} = \widetilde{X}_{a,\bar{v}}^\flat \times_{\bar{X}_{\bar{v}}} \bar{X}_{\bar{v}}^\bullet$$

is canonically isomorphic to the torus $T \times_{\bar{X}_{\bar{v}}^\bullet} \widetilde{X}_{a,\bar{v}}^{\flat\bullet}$. Since the Néron model of $T \times_{\bar{X}_{\bar{v}}^\bullet} \widetilde{X}_{a,\bar{v}}^{\flat\bullet}$ is the torus $T \times_{\bar{X}_{\bar{v}}} \widetilde{X}_{a,\bar{v}}^\flat$, we have a canonical homomorphism

$$\pi_a^{\flat *}J_a \longrightarrow T \times_{\bar{X}_{\bar{v}}} \widetilde{X}_{a,\bar{v}}^\flat.$$

By adjunction, one obtains a homomorphism $J_a \longrightarrow \pi_{a*}^\flat T$. The same reasoning applies to any smooth group scheme of finite type having the same generic fiber as J_a. Hence, the lemma follows. □

Corollary 3.8.3. — *We have the formula*

$$\dim\big(\mathscr{P}_{\bar{v}}(J_a)\big) = \dim_{\bar{k}}\Big(\big(\mathfrak{t} \otimes_{\bar{\mathcal{O}}_{\bar{v}}} \widetilde{\mathcal{O}}_{\bar{v}}^\flat/\widetilde{\mathcal{O}}_{\bar{v}}\big)^W\Big).$$

This thus gives another formula for the local δ–invariant $\delta_{\bar v}(a)$. Note that the integer $c_{\bar v}(a)$ defined in (3.7.4) is equal to the drop in the toric rank of the Néron model $J_a^\flat$, that is, the difference between r and the toric rank of the special fiber of $J_a^\flat$.

3.9. Connected Components. — In this section we describe the group of connected components $\pi_0\big(\mathscr{P}_{\bar v}(J_a)\big)$.

Let $a : \bar X_{\bar v} \to \mathfrak{c}$ be a morphism whose image is not contained in the discriminant divisor. Then there exists a smooth group scheme J_a over $\bar X_{\bar v}$ whose generic fiber is a torus. Let J_a^0 be the open subgroup scheme of J_a consisting of the neutral components of its fibers. Over $\bar F_{\bar v}$, J_a is connected, so that the homomorphism $J_a^0 \to J_a$ induces an isomorphism over $\bar F_{\bar v}$. As a homomorphism of sheaves of abelian groups, it is injective, and its cokernel is supported on the special fiber of $\bar X_{\bar v}$ with fiber

$$\pi_0(J_{a,\bar v}) = J_a(\bar{\mathscr{O}}_{\bar v})/J_a^0(\bar{\mathscr{O}}_{\bar v}).$$

The inclusions

$$J_a^0(\bar{\mathscr{O}}_{\bar v}) \subset J_a(\bar{\mathscr{O}}_{\bar v}) \subset J_a(\bar F_{\bar v})$$

induce an exact sequence

$$1 \to \pi_0(J_{a,\bar v}) \to J_a(\bar F_{\bar v})/J_a^0(\bar{\mathscr{O}}_{\bar v}) \to J_a(\bar F_{\bar v})/J_a(\bar{\mathscr{O}}_{\bar v}) \to 1.$$

Thus, we have a surjective homomorphism

$$\mathscr{P}_{\bar v}(J_a^0) \to \mathscr{P}_{\bar v}(J_a) \tag{3.9.1}$$

whose kernel is the finite group $\pi_0(J_{a,\bar v})$. Hence, we obtain an exact sequence

$$\pi_0(J_{a,\bar v}) \to \pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big) \to \pi_0\big(\mathscr{P}_{\bar v}(J_a)\big) \to 1.$$

To determine $\pi_0\big(\mathscr{P}_{\bar v}(J_a)\big)$, it suffices to describe the group $\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big)$ and the image of the map

$$\pi_0(J_{a,\bar v}) \to \pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big).$$

As in Tate–Nakayama duality, the group of connected components $\pi_0(\mathscr{P}_a)$ is more easily expressed via duality. For every finitely generated abelian group Λ, we denote

$$\Lambda^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\Lambda])$$

the diagonalizable $\overline{\mathbb{Q}}_\ell$–group whose group of characters is Λ, and conversely, for every diagonalizable $\overline{\mathbb{Q}}_\ell$–group A, we denote by A^* its character group, which is a finitely generated abelian group.

To write explicit formulas, fix a trivialization of ρ_{Out} over $\bar X_v$. This allows one to identify W and $\mathbb{W}$. Let $\widetilde X^\bullet_{a,\bar v}$ be the inverse image of the finite étale cover $\pi : \mathfrak{t}^{\mathrm{rs}} \to \mathfrak{c}^{\mathrm{rs}}$ via the morphism $a : \bar X_{\bar v} \to \mathfrak{c}$. By choosing a geometric point of $\widetilde X_{a,\bar v}$ over the geometric point $\bar\eta_v$ of $\bar X_{\bar v}$, one obtains a homomorphism

$$\pi_a^\bullet : I_v \to \mathbb{W}.$$

Proposition 3.9.2. — *With the choice of a geometric point of $\widetilde X_{a,\bar v}$ over the geometric point $\bar\eta_v$ of $\bar X_{\bar v}$, there is a canonical isomorphism of diagonalizable groups*

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big)^* = \hat{\mathbb{T}}^{\pi_a^\bullet(I_v)}.$$

Similarly, we have an isomorphism

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a)\big)^* = \hat{\mathbb{T}}\big(\pi_a^\bullet(I_v)\big),$$

where $\hat{\mathbb{T}}\big(\pi_a^\bullet(I_v)\big)$ is the subgroup of $\hat{\mathbb{T}}^{\pi_a^\bullet(I_v)}$ consisting of those elements $\kappa \in \hat{\mathbb{T}}$ such that $\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big)$ is contained in the Weyl group of the neutral component $\hat{\mathbb{H}}$ of the centralizer of κ in $\hat{\mathbb{G}}$.

Proof. — Consider the group scheme of neutral components $J_a^{\flat,0}$ of the Néron model $J_a^\flat$ of J_a. (In the terminology of [**10**], this is the connected Néron model.) Since J_a^0 has connected fibers, the homomorphism $J_a^0 \to J_a^\flat$ factors through $J_a^{\flat,0}$.

Lemma 3.9.3. — *The homomorphism $\mathscr{P}_{\bar v}(J_a^0) \to \mathscr{P}_{\bar v}(J_a^{\flat,0})$ induces an isomorphism*

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big) \cong J_a(\bar F_{\bar v})/J_a^{\flat,0}(\bar{\mathscr{O}}_{\bar v}).$$

Proof. — Since both J_a^0 and $J_a^{\flat,0}$ have connected fibers, the homomorphism $\mathscr{P}_{\bar v}(J_a^0) \to \mathscr{P}_{\bar v}(J_a^{\flat,0})$ induces an isomorphism on the groups of connected components. Topologically, $\mathscr{P}_{\bar v}(J_a^{\flat,0})$ is the discrete group $J_a(\bar F_{\bar v})/J_a^{\flat,0}(\bar{\mathscr{O}}_{\bar v})$. □

For any torus A over $\bar F_{\bar v}$, consider the connected Néron model $A^{\flat,0}$ of A over $\bar{\mathscr{O}}_{\bar v}$ and the finitely generated abelian group $A(\bar F_{\bar v})/A^{\flat,0}(\bar{\mathscr{O}}_{\bar v})$. One may verify (see [**63**]) that the functor $A \mapsto A(\bar F_{\bar v})/A^{\flat,0}(\bar{\mathscr{O}}_{\bar v})$ satisfies the axioms of [**42**, Lemma 2.2]. Following Kottwitz, one obtains a general formula for $A(\bar F_{\bar v})/A^{\flat,0}(\bar{\mathscr{O}}_{\bar v})$; in particular, we obtain the following lemma.

Lemma 3.9.4. — *With the above notation, there is an isomorphism*

$$J_a(\bar F_{\bar v})/J_a^{\flat,0}(\bar{\mathscr{O}}_{\bar v}) \cong (\mathbb{X}_*)_{\pi_a^\bullet(I_v)}.$$

The conjunction of the previous two lemmas yields the isomorphism

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big) = (\mathbb{X}_*)_{\pi_a^\bullet(I_v)},$$

which, by duality, induces the first assertion of the proposition:

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a^0)\big)^* = \hat{\mathbb{T}}^{\pi_a^\bullet(I_v)}.$$

The proof of the second assertion uses the trick of z–extensions. Following [**44**, 7.5], there exists an exact sequence

$$1 \to G \to G_1 \to C \to 1$$

of reductive group schemes over X obtained by outer twisting an exact sequence of split reductive groups

$$1 \to \mathbb{G} \to \mathbb{G}_1 \to \mathbb{C} \to 1,$$

where $\mathbb{C}$ is a torus and $\mathbb{G}_1$ is a reductive group with connected center. Its dual $\hat{\mathbb{G}}_1$ has a simply connected derived subgroup. There exists a maximal torus $\mathbb{T}_1$ in $\mathbb{G}_1$ such that one has an exact sequence

$$1 \to \mathbb{T} \to \mathbb{T}_1 \to \mathbb{C} \to 1.$$

Replacing G by G_1, denote by $\mathfrak{c}_1$ the corresponding space of characteristic polynomials. The homomorphism $G \to G_1$ induces a morphism

$$\alpha : \mathfrak{c} \to \mathfrak{c}_1.$$

Over $\mathfrak{c}_1$, we have the regular centralizer J_1, which is a smooth group scheme with connected fibers since the center of G_1 is connected. There is an exact sequence of smooth commutative group schemes

$$1 \to J \to \alpha^* J_1 \to C \to 1.$$

Let $a : \bar X_{\bar v} \to \mathfrak{c}$ be such that the restriction $a|_{\bar X_{\bar v}^\bullet}$ has image in $\mathfrak{c}^{\mathrm{rs}}$. Denote by $\alpha(a) : \bar X_{\bar v} \to \mathfrak{c}_1$ the composition of a with α. Over $\bar X_{\bar v}$, we have an exact sequence

$$1 \to J_a \to J_{1,\alpha(a)} \to C \to 1,$$

with $J_a = a^* J$ and $(J_1)_{\alpha(a)} = \alpha(a)^* J_1$.

Proposition 3.9.5. — *The homomorphism*

$$\pi_0\big(\mathscr{P}_{\bar v}(J_a)\big) \to \pi_0\big(\mathscr{P}_{\bar v}(J_{1,\alpha(a)})\big)$$

is injective.

Proof. — Since $J \to \alpha^* J_1$ is a closed immersion (hence proper), we have

$$J_a(\bar{\mathscr{O}}_{\bar{v}}) = J_a(\bar{F}_{\bar{v}}) \cap (J_1)_{\alpha(a)}(\bar{\mathscr{O}}_{\bar{v}}).$$

It follows that the homomorphism

$$j_{\alpha(a)} : J_a(\bar{F}_{\bar{v}})/J_a(\bar{\mathscr{O}}_{\bar{v}}) \to J_{1,\alpha(a)}(\bar{F}_{\bar{v}})/J_{1,\alpha(a)}(\bar{\mathscr{O}}_{\bar{v}})$$

is injective. Since C is a torus over $\bar{X}_{\bar{v}}$, the group

$$C(\bar{F}_{\bar{v}})/C(\bar{\mathscr{O}}_{\bar{v}})$$

is a free abelian group of finite type and discrete. The neutral component of

$$J_{1,\alpha(a)}(\bar{F}_{\bar{v}})/J_{1,\alpha(a)}(\bar{\mathscr{O}}_{\bar{v}})$$

therefore lies in the image of $j_{\alpha(a)}$. Consequently, $j_{\alpha(a)}$ induces an isomorphism between the neutral component of $J_a(\bar{F}_{\bar{v}})/J_a(\bar{\mathscr{O}}_{\bar{v}})$ and that of $J_{1,\alpha(a)}(\bar{F}_{\bar{v}})/J_{1,\alpha(a)}(\bar{\mathscr{O}}_{\bar{v}})$. It follows that the homomorphism $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big) \to \pi_0\big(\mathscr{P}_{\bar{v}}(J_{1,\alpha(a)})\big)$ is injective. □

Corollary 3.9.6. — *There is a canonical isomorphism between $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$ and the image of the homomorphism*

$$\pi_0\big(\mathscr{P}_{\bar{v}}(J_a^0)\big) \to \pi_0\big(\mathscr{P}_{\bar{v}}(J_{1,\alpha(a)})\big).$$

Proof. — In addition to the previous proposition, it suffices to invoke the surjectivity of $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a^0)\big) \to \pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$. □

By choosing a geometric point of $\widetilde{X}^{\bullet}_{a,\bar{v}}$, one may identify the group $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$ with the image of the homomorphism

$$(\mathbb{X}_*)_{I_v} \to (\mathbb{X}_{1,*})_{I_v},$$

where $\mathbb{X}_{1,*} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}_1)$. Note that the equality

$$\pi_0\big(\mathscr{P}_{\bar{v}}(J_{1,\alpha(a)})\big) = (\mathbb{X}_{1,*})_{I_v}$$

follows from the connectedness of the fibers of J_1 and Lemma 3.9.4. Dualizing, one obtains an isomorphism between the subgroup $\mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(\mathscr{P}_{\bar{v}}(J_a)])$ of $\hat{\mathbb{T}}^{I_v}$ and the image of the homomorphism

$$\hat{\mathbb{T}}_1^{I_v} = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[(\mathbb{X}_{1,*})_{I_v}]) \to \mathrm{Spec}\Big(\overline{\mathbb{Q}}_\ell[(\mathbb{X}_*)_{I_v}]\Big) = \hat{\mathbb{T}}^{I_v}.$$

It remains now to show that the image of $\hat{\mathbb{T}}_1^{I_v}$ in $\hat{\mathbb{T}}^{I_v}$ is exactly the subgroup of those elements $\kappa \in \hat{\mathbb{T}}$ such that $\pi_a^\bullet(I_v)$ is contained in the Weyl group of the neutral component $\hat{\mathbb{H}}$ of the centralizer of κ in $\hat{\mathbb{G}}$. For every $\kappa \in \hat{\mathbb{T}}$ and for every $\kappa_1 \in \hat{\mathbb{T}}_1$ mapping to κ, the centralizer $\hat{\mathbb{H}}_1$ of κ_1 in $\hat{\mathbb{G}}_1$ is connected and its image in $\hat{\mathbb{G}}$ is $\hat{\mathbb{H}}$. (Indeed, $\hat{\mathbb{G}}_1$ has a simply connected derived subgroup so that the centralizer of a semisimple element of $\hat{\mathbb{G}}_1$ is connected.) The desired result follows. □

We now present another, more explicit description of $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$, though it is ultimately less convenient.

Proposition 3.9.7. — *With the choice of a geometric point of $\widetilde{X}_{a,\bar{v}}$ over the geometric point $\bar{\eta}_v$ of $\bar{X}_{\bar{v}}$, there is a canonical isomorphism between $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$ and the cokernel of the composite of two maps*

$$\pi_0(Z_{\mathbb{G}}) \longrightarrow \pi_0\big(\mathbb{T}^{\pi_a^\bullet(I_v)}\big) \longrightarrow (\mathbb{X}_*)_{\pi_a^\bullet(I_v)},$$

where the first is induced by the evident homomorphism $Z_{\mathbb{G}} \to \mathbb{T}^{\pi_a^\bullet(I_v)}$ and the second is the canonical isomorphism of $\pi_0\big(\mathbb{T}^{\pi_a^\bullet(I_v)}\big)$ onto the torsion subgroup of $(\mathbb{X}_*)_{\pi_a^\bullet(I_v)}$.

Proof. — One knows that $\pi_0\big(\mathscr{P}_{\bar{v}}(J_a)\big)$ is the cokernel of the homomorphism $\pi_0(J_{a,\bar{v}}) \to \pi_0\big(\mathscr{P}_{\bar{v}}(J_a^0)\big)$. By 2.3.2, $\pi_0(J_{a,\bar{v}})$ and $\pi_0(Z_{\mathbb{G}})$ have the same image in $(\mathbb{X}_*)_{\pi_a^\bullet(I_v)}$. The proposition follows. □

3.10. Density of the Regular Orbit. — We refer to Section 4.16 for an outline of the proof of the following proposition. It will be deduced from its global analogue which will be proved in detail later. Only the global analogue will be used in the remainder of the paper.

Proposition 3.10.1. — *The open subset $\mathscr{M}_v^{\mathrm{reg}}(a)$ is dense in $\mathscr{M}_v(a)$.*

Note that we already know that the closed complement $\mathscr{M}_v(a) - \mathscr{M}_v^{\mathrm{reg}}(a)$ has dimension strictly less than $\dim(\mathscr{M}_v^{\mathrm{reg}}(a))$ (see Proposition 3.7.1).

Corollary 3.10.2. — *The set of irreducible components of the affine Springer fiber $\mathscr{M}_v(a)$ is canonically in bijection with the group of connected components of the group $\mathscr{P}_v(J_a)$.*

Proof. — Indeed, one has an isomorphism $\mathscr{P}_v(J_a) \to \mathscr{M}_v^{\mathrm{reg}}(a)$ by letting $\mathscr{P}_v(J_a)$ act on the Kostant point. □

3.11. The Case of an Endoscopic Group. — Let H be an endoscopic group of G in the sense of 1.8. Let $a_H \in \mathfrak{c}_H(\bar{\mathscr{O}}_{\bar v})$ with image $a \in \mathfrak{c}(\bar{\mathscr{O}}_{\bar v}) \cap \mathfrak{c}^{\mathrm{rs}}(\bar{F}_{\bar v})$. There is no direct relation between the affine Springer fibers $\mathscr{M}_{H,v}(a_H)$ and $\mathscr{M}_v(a)$. However, there is an obvious relation between their symmetry groups.

Thus, we have two $\bar{X}_{\bar v}$–group schemes $J_a = a^*J$ and $J_{H,a_H} = a_H^* J_H$, which are related by a homomorphism

$$\mu_{a_H} : J_a \to J_{H,a_H},$$

which is an isomorphism over the punctured disc $\bar{X}_{\bar v}^{\bullet}$. This homomorphism is constructed by pulling back the homomorphism μ constructed in 2.5.1 via a_H.

Let $\mathscr{R}_{H,\bar v}^G(a_H)$ be the affine algebraic group over $\bar k$ whose group of $\bar k$–points is

$$\mathscr{R}_{H,\bar v}^G(a_H)(\bar k) = J_{H,a_H}(\bar{\mathscr{O}}_{\bar v}) / J_a(\bar{\mathscr{O}}_{\bar v}).$$

Then we have the exact sequence

$$1 \to \mathscr{R}_{H,\bar v}^G(a_H) \to \mathscr{P}_{\bar v}(J_a) \to \mathscr{P}_{\bar v}(J_{H,a_H}) \to 1. \tag{3.11.1}$$

Lemma 3.11.2. — *We have*

$$\dim\Big(\mathscr{R}_{H,\bar v}^G(a_H)\Big) = \dim\Big(\mathscr{P}_{\bar v}(J_a)\Big) - \dim\Big(\mathscr{P}_{\bar v}(J_{H,a_H})\Big) = r_{H,\bar v}^G(a_H),$$

where

$$r_{H,\bar v}^G(a_H) = \deg_{\bar v}\big(a_H^* \mathfrak{R}_G^H\big),$$

with the resultant divisor $\mathfrak{R}_H^G$ on $\mathfrak{c}_H$ defined in 1.10.3.

Proof. — By the exact sequence (3.11.1), we have

$$\dim\Big(\mathscr{R}_{H,\bar v}^G(a_H)\Big) = \dim\Big(\mathscr{P}_{\bar v}(J_a)\Big) - \dim\Big(\mathscr{P}_{\bar v}(J_{H,a_H})\Big).$$

Since the homomorphism $J_{H,a_H} \to J_a$ is an isomorphism over the generic fiber, we have

$$c_{\bar v}(a_H) = c_{\bar v}(a),$$

where $c_{\bar v}(a_H)$ and $c_{\bar v}(a)$ are the Galois invariants appearing in Bezrukavnikov's dimension formula (see Proposition 3.7.5). Applying that formula to a and a_H, we find

$$\dim\Big(\mathscr{R}_{H,\bar v}^G(a_H)\Big) = \deg_{\bar v}(a^* \mathfrak{D}_G) - \deg_{\bar v}(a_H^* \mathfrak{D}_H).$$

It now suffices to invoke 1.10.3 to conclude. □

Now let $a_H \in \mathfrak{c}_H(\mathscr{O}_v)$ with image

$$a \in \mathfrak{c}(\mathscr{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v).$$

With the same definition as above, one obtains a group $\mathscr{R}_v(a_H)$ defined over k. Then we have the formula

$$\mathscr{R}_{H,v}^G(a_H) \otimes_k \bar k = \prod_{\bar v : k_v \to \bar k} \mathscr{R}_{\bar v}(a)$$

which implies the dimension formula

$$\dim\big(\mathcal{R}^G_{H,v}(a_H)\big) = \deg(k_v/k)\cdot\deg_v\big(a_H^*\mathfrak{R}^H_G\big).$$

Identifying $\mathcal{P}_v(J_a)$ with the open subset $\mathcal{M}_v^{\mathrm{reg}}(a)$ of $\mathcal{M}_v(a)$ and $\mathcal{P}_v(J_{H,a_H})$ with the open subset $\mathcal{M}_{H,v}(a_H)$, and by taking the closure of the graph of the homomorphism μ_{a_H} in $\mathcal{M}_v(a)\times\mathcal{M}_v(a_H)$, one obtains an interesting correspondence between these two affine Springer fibers. We will not use this correspondence in this work.

4. The Hitchin Fibration

In his seminal paper [**34**], Hitchin observed that the cotangent bundle of the moduli space of stable bundles on a curve forms a completely integrable Hamiltonian system. To achieve this, he explicitly constructed a family of commuting functions with respect to the Poisson bracket, in number equal to half the dimension of this cotangent bundle. These functions define a morphism from the cotangent bundle to an affine space whose generic fiber is essentially an abelian variety. This particularly elegant morphism is the Hitchin fibration.

We will adopt a different viewpoint by considering the fibers of the Hitchin fibration as the global analogue of the affine Springer fibers. In particular, we replace the canonical bundle of the curve with a line bundle of very high degree. In doing so, we lose the symplectic form while retaining a fibration that bears the same overall appearance as the original Hitchin fibration.

As observed in [**57**], the point counting over a finite field of this generalized Hitchin fibration formally yields the geometric side of the trace formula for the Lie algebra. This continues to serve as our guide in this chapter for studying the geometric properties of the Hitchin fibration. The point counting will be reviewed in Chapter 8.

Our preferred tool for exploring the geometry of the Hitchin fibration

$$f:\mathcal{M}\longrightarrow\mathcal{A}$$

is a Picard stack

$$\mathcal{P}\longrightarrow\mathcal{A}$$

acting on $\mathcal{M}$. This action is based on the construction of the regular centralizer (see Section 2). In particular, one shows that there exists an open subset $\mathcal{M}^{\mathrm{reg}}$ of $\mathcal{M}$ (see 4.3.3) on which $\mathcal{P}$ acts simply transitively and which is dense in each fiber $\mathcal{M}_a$ (see 4.16.1). One can analyze the structure of $\mathcal{P}_a$ in detail thanks to the cameral curve and the theory of Néron models. In particular, this allows one to define an open subset $\mathcal{A}^\diamondsuit$ over which $\mathcal{M}$ is essentially an abelian scheme.

We will also perform some useful computations for later use, such as the computation of the dimension (see Section 4.13), that of the group of connected components of $\mathcal{P}_a$ (see Section 4.10), and that of the group of automorphisms of Higgs bundles (see Section 4.11).

The key point of this chapter is the product formula (see 4.15.1) which establishes the relationship between the Hitchin fiber and the affine Springer fibers. This formula was proved in [**57**]. It plays a crucial role in the chapter on point counting (see 8) and is also implicitly used in the proof of the support theorem in Chapter 7.

Finally, we establish a link between the Hitchin fibration for G and that for an endoscopic group. There exists a canonical morphism

$$\nu:\mathcal{A}_H\longrightarrow\mathcal{A},$$

and if $a_H\mapsto a$, then there is a canonical homomorphism

$$\mu:\mathcal{P}_a\longrightarrow\mathcal{P}_{H,a_H}.$$

However, there is no direct relation between the Hitchin fibers $\mathcal{M}_a$ and $\mathcal{M}_{H,a_H}$ but rather a correspondence deduced from μ by schematic closure. We will not use this correspondence further in the article, though it is likely worth exploring further.

Here are the notations that will be used in this chapter. We fix a smooth, proper, and geometrically connected curve X of genus g over a finite field k. Let $\bar{k}$ be an algebraic closure of k. We denote

$$\bar{X} = X \otimes_k \bar{k}.$$

We denote by F the field of rational functions on X. Let $|X|$ be the set of closed points of X. Each element $v \in |X|$ defines a valuation

$$v : F^\times \longrightarrow \mathbb{Z}.$$

We denote by F_v the completion of F with respect to this valuation, by $\mathcal{O}_v$ its ring of integers, and by k_v its residue field. We write $X_v = \mathrm{Spec}(\mathcal{O}_v)$ for the formal disk at v, and $X_v^\bullet = \mathrm{Spec}(F_v)$ for the punctured formal disk.

Let $\mathbb{G}$ be a Chevalley group whose Coxeter number h satisfies the inequality $2h < p$, where p is the characteristic of k. Let G be an X-group scheme, a quasi-split reductive form of $\mathbb{G}$, defined by an $\mathrm{Out}(\mathbb{G})$-torsor ρ_G over X. The group G is then equipped with a pinning $(T, B, \mathbf{x}_+)$. We now follow the notations of Section 1.3. In particular, we have an X-scheme of characteristic polynomials $\mathfrak{c}$, the Chevalley morphism

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{c},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, and a finite flat morphism

$$\pi : \mathfrak{t} \longrightarrow \mathfrak{c},$$

which over the open subset $\mathfrak{c}^{\mathrm{rs}}$ is a torsor under the finite étale group scheme W, obtained by twisting $\mathbb{W}$ by ρ_G.

We fix a line bundle D on X which is the square

$$D = D'^{\otimes 2}$$

of another line bundle D'. In general, we assume that the degree of D is greater than $2g$, where g is the genus of X (in contrast to [**34**] where D is the canonical bundle). We will indicate, however, the places where this assumption is truly necessary.

Note that $\mathbb{G}_m$ acts on $\mathfrak{g}$ and $\mathfrak{t}$ by homothety, and acts compatibly on $\mathfrak{c}$. We denote by $\mathfrak{g}_D$, $\mathfrak{t}_D$, and $\mathfrak{c}_D$ the objects twisted by the $\mathbb{G}_m$-torsor associated with the line bundle D.

4.1. Review of Bun_G**. —** We consider the stack Bun_G which assigns to every k-scheme S the groupoid of G-torsors on $X \times S$. The stack Bun_G is an algebraic stack in the sense of Artin *cf.* [**53**] and [**33**, Prop. 1]. The groupoid of k-points of Bun_G can be expressed as a disjoint union of double cosets

$$\mathrm{Bun}_G(k) = \bigsqcup_{\xi \in \ker^1(F,G)} \left[G^\xi(F) \backslash \check{\prod}_{v \in |X|} \frac{G(F_v)}{G(\mathcal{O}_v)} \right]$$

over the set $\ker^1(F, G)$ of G-torsors over F that are locally trivial. Here, G^ξ denotes the strong inner form of G over F corresponding to the class $\xi \in \ker^1(F, G)$, and the symbol $\check{\prod}$ denotes a restricted product.

For every $v \in |X|$, as in Section 3.1, one has the affine Grassmannian $\mathcal{G}_v$ and a morphism

$$\zeta : \mathcal{G}_v \longrightarrow \mathrm{Bun}_G,$$

which consists of gluing the G-torsor on $X_v \hat{\times} S$, equipped with a trivialization on $X_v^\bullet \hat{\times} S$, with the trivial G-torsor on $(X - v) \times S$. The existence of this formal gluing was proved first by

Beauville and Laszlo in the case of vector bundles *cf.* [**4**], and in the general case by Heinloth [**33**, Lem. 5]. On the level of k-points, this morphism is the obvious functor

$$G(F_v)/G(\mathcal{O}_v) \longrightarrow \left[G(F)\backslash \check{\prod}_{v\in|X|} \frac{G(F_v)}{G(\mathcal{O}_v)}\right],$$

which sends an element $g_v \in G(F_v)/G(\mathcal{O}_v)$ to the tuple consisting of g_v and the neutral elements of $G(F_{v'})/G(\mathcal{O}_{v'})$ for all places $v' \neq v$.

4.2. Construction of the Fibration. — Recall the definition of the Hitchin moduli space.

Definition 4.2.1. — *The total Hitchin space is the fibered groupoid $\mathcal{M}$ which assigns to every k-scheme S the groupoid $\mathcal{M}(S)$ of pairs (E,ϕ) consisting of a G-torsor E on $X\times S$ and a section*

$$\phi \in \mathrm{H}^0\Big(X\times S, \mathrm{ad}(E)\otimes_{\mathcal{O}_X} D\Big),$$

where $\mathrm{ad}(E)$ *is the Lie algebra bundle obtained by twisting* $\mathfrak{g}$, *endowed with the adjoint action, by the G-torsor E.*

4.2.2. — Equivalently, one may say that $\mathcal{M}(S)$ is the groupoid of maps $h_{E,\phi}$ fitting into the commutative diagram

$$X\times S \longrightarrow [\mathfrak{g}_D/G].$$

This stack is a vector bundle over the algebraic stack Bun_G, which classifies the G-torsors on X, and hence it is itself an algebraic stack locally of finite type.

4.2.3. — The Chevalley characteristic morphism

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{c}$$

induces a morphism

$$[\chi] : [\mathfrak{g}_D/G] \longrightarrow \mathfrak{c}_D.$$

Thus, one obtains a morphism

$$f : \mathcal{M} \longrightarrow \mathcal{A},$$

where for any k-scheme S, $\mathcal{A}(S)$ is the groupoid of maps a fitting into the commutative diagram

$$a : X\times S \longrightarrow \mathfrak{c}_D.$$

Equivalently, $\mathcal{A}$ is the space of sections of the vector bundle $\mathfrak{c}_D$ over X. It follows that $\mathcal{A}$ is a finite-dimensional k-vector space.

4.2.4. — Given a square root D' of D, one obtains a section

$$\epsilon_{D'} : \mathcal{A} \longrightarrow \mathcal{M}$$

of the morphism $f : \mathcal{M} \longrightarrow \mathcal{A}$ using the construction of 2.2.5. This section is essentially the same as the ones that Hitchin constructed more explicitly in the case of classical groups. It is called the *Kostant–Hitchin section*.

4.2.5. — Writing D in the form $D = \mathcal{O}_X(\sum_{v\in|X|} d_v v)$, the set $\mathfrak{c}_D(k)$ is identified with a finite subset of $\mathfrak{c}(F)$. By [**57**, 1.3], for $a \in \mathfrak{c}_D(k)$ regular semisimple and anisotropic, the number of k-points of the fiber $\mathcal{M}_a$ is expressed in terms of sums of global orbital integrals

$$\sum_{\xi\in\ker^1(F,G)} \sum_{\gamma\in\mathfrak{g}^{\xi}(F)/\sim,\ \chi(\gamma)=a} \mathbf{O}_\gamma(1_D), \tag{4.2.6}$$

which is part of the formula *cf.* 1.13.2. In [**57**] we constructed a geometric interpretation of the stabilization process of this formula *cf.* 1.13. This process of counting and stabilization will be examined more systematically in Chapter 8.

In the remainder of this chapter we shall recall this geometry and study it in more detail.

4.3. Symmetries of a Hitchin Fiber. — As with affine Springer fibers, the construction of the natural symmetries of a Hitchin fiber is based on Lemma 2.1.1.

***4.3.1.* —** For every S-point a of $\mathscr{A}$, there is a morphism

$$h_a : X \times S \longrightarrow [\mathfrak{c}/\mathbb{G}_m].$$

We denote by $J_a = h_a^* J$ the pull-back of J from $[\mathfrak{c}/\mathbb{G}_m]$, and we consider the Picard groupoid $\mathscr{P}_a(S)$ of J_a-torsors over $X \times S$. As a varies, this construction defines a Picard groupoid $\mathscr{P}$ fibered over $\mathscr{A}$.

***4.3.2.* —** The homomorphism $\chi^* J \longrightarrow I$ from Lemma 2.2.1 induces, for every S-point (E, ϕ) over a, a homomorphism

$$J_a \longrightarrow \mathrm{Aut}_{X\times S}(E, \phi) = h_{E,\phi}^* I.$$

Consequently, one may twist (E, ϕ) by any J_a-torsor. This defines an action of the Picard groupoid $\mathscr{P}_a(S)$ on the groupoid $\mathscr{M}_a(S)$. Allowing a to vary, one obtains an action of $\mathscr{P}$ on $\mathscr{M}$ relative to the base $\mathscr{A}$.

As with the affine Springer fibers, we consider the open subset $\mathscr{M}^{\mathrm{reg}}$ of $\mathscr{M}$ whose points are the morphisms $h_{E,\phi} : X \times S \longrightarrow [\mathfrak{g}_D/G]$ that factor through the open subset $[\mathfrak{g}_D^{\mathrm{reg}}/G]$.

***Proposition 4.3.3.* —** *$\mathscr{M}^{\mathrm{reg}}$ is an open subset of $\mathscr{M}$ with nonempty fibers over $\mathscr{A}$. Moreover, it is a torsor under the action of $\mathscr{P}$.*

Proof. — For every $a \in \mathscr{A}(\bar{k})$, the point $[\epsilon]^{D'}(a)$ constructed in *cf.* 2.2.5 lies in the regular open subset. This shows that the morphism $\mathscr{M}^{\mathrm{reg}} \to \mathscr{A}$ has nonempty fibers. It follows from Lemma 2.2.1 that $\mathscr{M}_a^{\mathrm{reg}}$ is a torsor under $\mathscr{P}_a$. □

***4.3.4.* —** The Kostant–Hitchin section 4.2.4

$$\epsilon_{D'} : \mathscr{A} \longrightarrow \mathscr{M}$$

factors through the open subset $\mathscr{M}^{\mathrm{reg}}$ since the Kostant section

$$\epsilon : \mathfrak{c} \longrightarrow \mathfrak{g}$$

factors through $\mathfrak{g}^{\mathrm{reg}}$.

***Proposition 4.3.5.* —** *The Picard stack $\mathcal{P}$ is smooth over $\mathcal{A}$.*

Proof. — Since J_a is a smooth commutative group scheme, the obstruction to deforming a J_a-torsor lies in the group $\mathrm{H}^2(\bar{X}, \mathrm{Lie}(J_a))$. This group vanishes because $\bar{X}$ has dimension one. □

4.4. The Linear Case. — We now analyze the fibers of $\mathcal{M}$ and $\mathcal{P}$ over a point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$. Let us begin with the case of the linear group: let $G = \mathrm{GL}(r)$. In this case one can describe the Hitchin fibers using spectral curves following Hitchin [**34**] and Beauville–Narasimhan–Ramanan [**5**]. A similar treatment works for the classical groups (see [**34**] and [**59**]).

***4.4.1.* —** In the case $G = \mathrm{GL}(r)$, the affine space $\mathcal{A}$ is the vector space

$$\mathcal{A} = \bigoplus_{i=1}^{r} \mathrm{H}^0(X, D^{\otimes i}).$$

A point $a = (a_1, \dots, a_r) \in \mathcal{A}(\bar{k})$ determines a spectral curve Y_a drawn on the total space Σ_D of the line bundle D. This curve is given by the equation

$$t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r = 0.$$

We define an open subset $\mathcal{A}^{\heartsuit} \subset \mathcal{A}$ whose geometric points $a \in \mathcal{A}^{\heartsuit}(\bar{k})$ define a reduced spectral curve. If $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, the Hitchin fiber $\mathcal{M}_a$ is the groupoid $\overline{\mathrm{Pic}}(Y_a)$ of torsion-free O_{Y_a}-modules of rank one (see [**5**]). The fiber $\mathcal{P}_a$ is the groupoid $\mathrm{Pic}(Y_a)$ of invertible O_{Y_a}-modules. The group $\mathrm{Pic}(Y_a)$ acts on $\overline{\mathrm{Pic}}(Y_a)$ by tensor product, and $\mathrm{Pic}(Y_a)$ appears as an open subset in $\overline{\mathrm{Pic}}(Y_a)$.

4.4.2. — If the spectral curve Y_a is smooth then there is no difference between $\mathrm{Pic}(Y_a)$ and $\overline{\mathrm{Pic}}(Y_a)$, i.e., $\mathcal{P}_a$ acts simply transitively on $\mathcal{M}_a$. Moreover, in this case, the structure of $\mathcal{P}_a$ is as simple as possible. The group of connected components of $\mathcal{P}_a$ is isomorphic to $\mathbb{Z}$ via the degree map because Y_a is connected. The neutral component of $\mathcal{P}_a$ is the quotient of the Jacobian of Y_a (which is an abelian variety) by the trivial $\mathbb{G}_m$-action.

4.4.3. — Let $\xi : Y_a^\flat \to Y_a$ be the normalization of Y_a. The pull-back functor induces a homomorphism

$$\xi^* : \mathrm{Pic}(Y_a) \to \mathrm{Pic}(Y_a^\flat)$$

which in turn induces an isomorphism

$$\pi_0\big(\mathrm{Pic}(Y_a)\big) \cong \pi_0\big(\mathrm{Pic}(Y_a^\flat)\big) = \mathbb{Z}^{\pi_0(Y_a^\flat)}.$$

The kernel of ξ^* is an affine commutative group of dimension

$$\delta_a = \dim \mathrm{H}^0\Big(Y_a, \frac{\xi_* O_{Y_a^\flat}}{O_{Y_a}}\Big).$$

4.4.4. — By construction, Y_a is a curve drawn on a smooth surface so that in particular it has only planar singularities. According to Altman, Iarrobino, and Kleiman [**1**], $\mathrm{Pic}(Y_a)$ is then a dense open subset of $\overline{\mathrm{Pic}}(Y_a)$.

We now generalize the discussion above to a general reductive group.

4.5. Cameral Curve. — Consider the cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \mathfrak{t}_D \arrow[d, "\pi"] \\ X \times \mathcal{A} \arrow[r] & \mathfrak{c}_D \end{tikzcd}$$

where the bottom morphism sends a pair (x, a) with $x \in X$ and $a \in \mathcal{A}$ to the point $a(x) \in \mathfrak{c}_D$. The left morphism, obtained by base change from π, is a finite flat morphism with an action of W.

Taking the fiber at each point $a \in \mathcal{A}(\bar{k})$, we obtain the cameral cover $\pi_a : \widetilde{X}_a \to \bar{X}$ of Donagi. In what follows, we restrict to those parameters a for which π_a is generically an étale cover. These parameters form an open subset of $\mathcal{A}$ described as follows.

Consider the inverse image U of $\mathfrak{c}_D^{\mathrm{rs}} \subset \mathfrak{c}_D$ in $X \times \mathcal{A}$. Since the morphism $U \to \mathcal{A}$ is smooth, its image is an open subset of $\mathcal{A}$ which we denote by $\mathcal{A}^\heartsuit$. Its $\bar{k}$-points are described as

$$\mathcal{A}^\heartsuit(\bar{k}) = \{a \in \mathcal{A}(\bar{k}) \mid a(\bar{X}) \not\subset \mathrm{discrim}_{G,D}\}.$$

If $\deg(D) > 2g$, the open subset $\mathcal{A}^\heartsuit$ is nonempty. (See 4.7.1 for a stronger statement.)

Lemma 4.5.1. — *For every geometric point $a \in \mathcal{A}^\heartsuit(\bar{k})$, the cover*

$$\pi_a : \widetilde{X}_a \longrightarrow X \otimes_k \bar{k}$$

is generically a W-torsor. Moreover, $\widetilde{X}_a$ is a reduced curve.

Proof. — By definition of $\mathcal{A}^\heartsuit$, the intersection U_a of U with the fiber $X \times \{a\}$ is a nonempty open subset of $\bar{X}$. By construction, π_a is a W-torsor over this dense open. Since π_a is a finite flat morphism, $\pi_{a*}O_{\widetilde{X}_a}$ is a torsion-free $O_{\bar{X}}$-module. If it is generically reduced, then it is reduced everywhere. □

4.5.2. — For every $a \in \mathscr{A}^\heartsuit(\bar{k})$ we have an injective arrow of sheaves for the étale topology

$$J_a \to J_a^1 = \big(\pi_{a,*}(\widetilde{X}_a \times_X T)\big)^W,$$

deduced from 2.4.2, whose kernel is a finite sheaf with finite support that is entirely explicit. This arrow makes it possible to control J_a, and hence $\mathscr{P}_a$, by means of the cameral curve.

The cameral curve $\widetilde{X}_a$ is equipped with a natural family of finite étale covers. Consider a reduction of the torsor ρ_G as in *cf.* 1.3.5, from which we adopt the notation. In particular, we have a finite étale Galois cover $\rho : X_\rho \to X$ with Galois group Θ_ρ which induces ρ_{Out} by the change of structure groups $\Theta_\rho \to \mathrm{Out}(\mathbb{G})$. Then there is a cartesian diagram

(4.5.3)
$$\begin{tikzcd} X_\rho \times \mathbf{t} \arrow[r, "\pi"] \arrow[d] & X_\rho \times \mathbf{c} \arrow[d] \\ \mathfrak{t} \arrow[r, "\pi"'] & \mathfrak{c}. \end{tikzcd}$$

For every $a \in \mathcal{A}(\bar{k})$, construct the cover $\pi_{\rho,a} : \widetilde{X}_{\rho,a} \to \bar{X}$ by forming the cartesian product

(4.5.4)
$$\begin{tikzcd} \widetilde{X}_{\rho,a} \arrow[r] \arrow[d] & X_\rho \times \mathbf{t}_D \arrow[d, "\pi_\rho"] \\ \bar{X} \arrow[r, "a"'] & \mathfrak{c}_D. \end{tikzcd}$$

Then one obtains a finite étale morphism $\widetilde{X}_{\rho,a} \to \widetilde{X}_a$ which is Galois with Galois group Θ_ρ.

One then has the formula
$$J^1_a = \pi_{\rho,a,*}(T)^{W_\rho}$$
following 2.4.4.

The above discussion applies also if one has a reduction $\bar{\rho}$ of the $\mathrm{Out}(\mathbb{G})$-torsor $\bar{\rho}_G$ on $\bar{X}$.

Proposition 4.5.5. — *Suppose that* $\deg(D) > 2g$. *Let* Θ *denote the image of the homomorphism* $\rho^\bullet_G : \pi_1(\bar{X}, \infty) \to \mathrm{Out}(\mathbb{G})$. *Suppose that* Θ *is a finite group of order prime to the characteristic. Then for every* $a \in \mathcal{A}^\infty(\bar{k})$ *one has*
$$\mathrm{H}^0(\bar{X}, J_a) = (Z_{\mathbb{G}})^\Theta$$
and
$$\mathrm{H}^0(\bar{X}, \mathrm{Lie}(J_a)) = \mathrm{Lie}(Z_{\mathbb{G}})^\Theta.$$
In particular, if over $\bar{X}$ *the center of* G *contains no split torus, then* $\mathcal{P}_a$ *is a Deligne-Mumford Picard stack.*

Proof. — The key point, that the curve $\widetilde{X}_{\rho,a}$ is connected, is deferred to 4.6.1; let us grant it for the moment. We then have
$$\mathrm{H}^0(\bar{X}, J^1_a) = \mathbb{T}^{\mathbb{W}},$$
which is a finite unramified group under the hypothesis that $Z_{\mathbb{G}}$ contains no split tori. The group $\mathrm{H}^0(\bar{X}, J_a)$ is a subgroup of it. Using the explicit description of $J \to J^1$ *cf.* 2.4.7, and the fact that the section a meets every irreducible component of the discriminant locus, one can show that
$$\mathrm{H}^0(\bar{X}, J_a) = Z^\Theta_{\mathbb{G}}.$$
Since this explicit description will not be used in the sequel, we leave to the reader the task of reconstructing the details. □

4.5.6. — It is not difficult to introduce a rigidifier in order to benefit from representability. Let $\infty \in X$ be a fixed point and let $\mathcal{A}^\infty$ be the open subset of $\mathcal{A}$ consisting of the points a which are regular semisimple at ∞. For every $a \in \mathcal{A}^\infty$, write $\mathcal{P}^\infty_a$ for the Picard stack classifying J_a-torsors equipped with a trivialization at ∞.

Proposition 4.5.7. — *The functor* $\mathcal{P}^\infty$ *is representable by a smooth group scheme locally of finite type over* $\mathcal{A}^\infty$. *For every* $a \in \mathcal{A}^\infty$, *the torus* $J_{a,\infty}$ *acts on* $\mathcal{P}^\infty$ *by modifying the rigidification, and induces a canonical isomorphism between the quotient stack* $[\mathcal{P}^\infty_a / J_{a,\infty}]$ *and* $\mathcal{P}_a$.

Proof. — The first assertion follows from the representability of the Picard scheme of the étale cover $\widetilde{X}_{\rho,a}$ of the cameral curve. The second assertion is immediate. □

4.6. Connected Cameral Curve. — Under the hypothesis $\deg(D) > 2g$ one can show that the cameral curve $\widetilde{X}_a$ is connected. But what we shall need in the sequel is a slightly stronger connectedness statement. Consider a connected finite étale Galois cover $\rho : \bar{X}_\rho \to \bar{X}$ which trivializes the torsor ρ_G. We have then constructed a cover

$$\pi_{\rho,a} : \widetilde{X}_{\rho,a} \to \bar{X}$$

as in 4.5.4.

Proposition 4.6.1. — *Suppose that* $\deg(D) > 2g$. *Then for every* $a \in \mathcal{A}^\heartsuit(\bar{k})$ *the curves* $\widetilde{X}_a$ *and* $\widetilde{X}_{\rho,a}$ *are connected.*

Proof. — After possibly replacing $\bar{X}$ by $\bar{X}_\rho$, it suffices to prove that the cameral curve $\widetilde{X}_a$ is connected in the case where G is split. In that case we appeal to the following theorem, due to Debarre [**17**, th. 1.4], which generalizes a theorem of Bertini.

Theorem 4.6.2. — *Let M be an irreducible variety and let $m : M \to \mathbb{P}$ be a morphism from M into a product of projective spaces $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. In each $\mathbb{P}^{n_i}$ we are given a linear subspace $L_i \subset \mathbb{P}^{n_i}$ such that for every subset $I \subset \{1, \dots, r\}$ one has*

$$\dim(p_I(m(M))) > \sum_{i\in I} \operatorname{codim}(L_i),$$

where p_I is the projection of $\mathbb{P}$ onto $\prod_{i\in I} \mathbb{P}^{n_i}$. Suppose also that m is proper over an open subset V of $\mathbb{P}$ and that $L = L_1 \times \cdots \times L_r$ is contained in V. Then $m^{-1}(L)$ is connected.

Here is how to apply this result to our particular case. Since G is split, $\mathfrak{c}_D$, as a scheme, is a fibered product over $\bar{X}$ of line bundles $D^{\otimes e_i}$. Each of these line bundles $D^{\otimes e_i}$ can be compactified into a projective line bundle

$$\bar{D}^{\otimes e_i} = \operatorname{Proj}_{\bar{X}}\big(\operatorname{Sym}_{\mathcal{O}_{\bar{X}}}(D^{\otimes -e_i} \oplus \mathcal{O}_{X_\rho})\big),$$

which is a projective $\bar{k}$-surface. We write $Z_i = \bar{D}^{e_i} - D^{e_i}$ for the divisor at infinity. The invertible bundle $\mathcal{O}(1)$ attached to this Proj is very ample by the hypothesis $\deg(D) > 2g$, and induces a projective embedding

$$\bar{D}^{\otimes e_i} \to \mathbb{P}^{n_i}$$

of this surface.

The locally closed subscheme $\prod_{i=1}^r D^{\otimes e_i}$ of $\mathbb{P} = \prod_{i=1}^r \mathbb{P}^{n_i}$ is a closed subscheme of V, where

$$V = \prod_{i=1}^r \mathbb{P}^{n_i} - \bigcup_{i=1}^r \Big(Z_i \times \prod_{j\neq i} \bar{D}^{\otimes e_j}\Big).$$

The same holds for $\mathfrak{c}_D$, which is a closed subscheme of $\prod_{i=1}^r D^{\otimes e_i}$. What plays the role of the irreducible variety M in Debarre's theorem is $\mathfrak{t}_D$, which is finite over $\mathfrak{c}_D$, so that M has dimension $r + 1$ and the composite morphism $m : M \to V$ is a proper morphism.

The component $a_i \in \mathrm{H}^0(\bar{X}, D^{\otimes e_i})$ of a defines a hyperplane L_i of $\mathbb{P}^{n_i}$, since

$$(a_i, 1) \in \mathrm{H}^0(\mathbb{P}^{n_i}, \mathcal{O}(1)) = \mathrm{H}^0(\bar{X}, D^{\otimes e_i}) \oplus \mathrm{H}^0(\bar{X}, \mathcal{O}_{\bar{X}}),$$

whose intersection with the surface $\bar{D}^{\otimes e_i}$ does not meet the divisor at infinity Z_i; in other words, it is contained in the open surface $D^{\otimes e_i}$. Since $L_i \cap Z_i = \emptyset$, the product $L = \prod_{i=1}^r L_i$ is contained in the open subset V.

Since $\widetilde{X}_a = m^{-1}(L)$, it only remains to check that for every subset $I \subset \{1, \dots, r\}$ one has

$$\dim(p_I(m(M))) > \sum_{i\in I} \operatorname{codim}(L_i).$$

But one checks immediately that the dimension of $p_I(m(M))$ equals $\sharp I + 1$, which completes the proof. □

4.7. Connected and Smooth Cameral Curve. — Let us begin by studying the open subset $\mathcal{A}^\diamondsuit$ where the fibers of $\mathcal{M}_a$ are as simple as possible. This open set is defined as follows. A point $a \in \mathcal{A}(\bar{k})$ belongs to this open if the section

$$h_a : \bar{X} \to \mathfrak{c}_D$$

meets the divisor $\mathrm{discrim}_D$ transversely. Here $\mathrm{discrim}_D$ denotes the divisor in $\mathfrak{c}_D$ obtained by twisting the divisor $\mathrm{discrim} \subset \mathfrak{c}$ by the $\mathbb{G}_m$-torsor L_D associated to the line bundle D.

Proposition 4.7.1. — *If* $\deg(D) > 2g$, *then the open subset* $\mathcal{A}^\diamondsuit$ *is nonempty.*

The proof is completely similar to the Bertini theorem due to Zariski. We begin by proving a lemma.

Lemma 4.7.2. — *Assume* $\deg(D) > 2g$, *where g is the genus of X. For every $x \in \bar{X}(\bar{k})$ defined by a maximal ideal $\mathfrak{m}_x$, the map*

$$\mathrm{H}^0(\bar{X}, \mathfrak{c}) \longrightarrow \mathfrak{c} \otimes_{O_{\bar{X}}} O_{\bar{X}}/\mathfrak{m}_x^2$$

is surjective. Here the vector bundle $\mathfrak{c}$ is regarded as a locally free O_X-module of rank r.

Proof. — By passing to the finite étale cover $\rho : X_\rho \to X$, the bundle $\mathfrak{c}$ becomes isomorphic to a direct sum

$$\rho^*\mathfrak{c} = \bigoplus_{i=1}^{r} \rho^* D^{\otimes e_i},$$

where the e_i are defined in 1.1.1 and in particular are integers at least 1. Since $\mathfrak{c}$ is a direct summand of $\rho_*\rho^*\mathfrak{c}$, it suffices to prove the surjectivity of

$$\mathrm{H}^0(\bar{X}', \rho^*\mathfrak{c}) \longrightarrow \rho^*\mathfrak{c} \otimes_{O_{\bar{X}}} O_{\bar{X}}/\mathfrak{m}_x^2.$$

It is enough to prove this surjectivity for each direct factor $\rho^* D^{\otimes e_i}$.

One may assume that X_ρ is geometrically connected. Let g' be its genus. Then $2g' - 2 = n(2g-2)$ where n is the degree of ρ. The surjectivity follows from the inequality

$$\deg(D^{\otimes e_i}) > 2ng = (2g'-2) + 2n.$$

This proves the lemma. □

Proof of Proposition 4.7.1. — In $\mathrm{discrim}_{G,D}$ there is a smooth open subset

$$\mathrm{discrim}_D - \mathrm{discrim}_D^{\mathrm{sing}},$$

which is the complement of a closed subset $\mathrm{discrim}_D^{\mathrm{sing}}$ of codimension 2 in $\mathfrak{c}_D$.

Consider the subscheme Z_1 of

$$\left(\mathrm{discrim}_{G,D} - \mathrm{discrim}_{G,D}^{\mathrm{sing}}\right) \times \mathcal{A}$$

consisting of pairs (c, a) such that the section $a(\bar{X})$ passes through the point c and intersects the divisor $\mathrm{discrim}_G$ at that point with multiplicity at least 2. By the lemma above,

$$\dim(Z_1) \leq \dim(\mathcal{A}) - 1,$$

so that the projection $Z_1 \to \mathcal{A}$ is not surjective.

Consider the subscheme Z_2 of $\mathrm{discrim}_G^{\mathrm{sing}} \times \mathcal{A}$ consisting of pairs (c, a) such that the section $a(\bar{X})$ passes through c. Again by the lemma,

$$\dim(Z_2) \leq \dim(\mathcal{A}) - 1,$$

so that the union of the images of Z_1 and Z_2 is contained in a proper closed subscheme of $\mathcal{A}$. Hence, there exists a point $a \in \mathcal{A}^\heartsuit$ such that the section $a(\bar{X})$ does not meet the singular locus $\mathrm{discrim}_{G,D}^{\mathrm{sing}}$ of the discriminant and meets the smooth part of this divisor transversely. □

Lemma 4.7.3. — *A point $a \in \mathcal{A}(\bar{k})$ belongs to the open subset $\mathcal{A}^{\diamondsuit}(\bar{k})$ if and only if the cameral curve $\widetilde{X}_a$ is smooth.*

Proof. — Suppose that $a \in \mathcal{A}^{\diamondsuit}(\bar{k})$, *i.e.*, the section $a(\bar{X}) \subset \mathfrak{c}_D$ meets the divisor $\mathrm{discrim}_{G,D}$ transversely. We show that the inverse image of this section on the cover $X_\rho \times \mathfrak{t}_D$ is smooth. Outside the divisor $\mathrm{discrim}_{G,D}$, this cover is étale so that there is nothing to verify. The condition that $a(\bar{X}) \subset \mathfrak{c}_D$ meets $\mathrm{discrim}_{G,D}$ transversely implies in particular that it does not meet the singular locus $\mathrm{discrim}_{G,D}^{\mathrm{sing}}$ of $\mathrm{discrim}_{G,D}$. A pair $(v, x) \in \mathfrak{t}_D(\bar{k})$ consists of a point $v \in X(\bar{k})$ and a point x in the fiber of $\mathfrak{t}_D$ over v. Over v, the group G is split so that one may speak of the root hyperplanes in the fiber of $\mathfrak{t}_D$ over v. If $(v, x) \in \mathfrak{t}_D(\bar{k})$ lies above a point where $a(\bar{X})$ intersects

$$\mathrm{discrim}_{G,D} - \mathrm{discrim}_{G,D}^{\mathrm{sing}},$$

then x belongs to a unique root hyperplane. One may then reduce to the case of a semisimple group of rank one. In that case a direct calculation shows that the formal completion of $\widetilde{X}_a$ at (v, x) is of the form

$$\bar{k}[[\epsilon_v]][t]/(t^2 - \epsilon_v^m),$$

where ϵ_v is a uniformizer of $\bar{X}$ at the point v and m is the intersection multiplicity of $a(\bar{X})$ with $\mathrm{discrim}_{G,D}$ at that point. In the transverse case $m = 1$, this implies that $\widetilde{X}_a$ is smooth at $(\tilde{v}, x)$.

Now suppose that $a \notin \mathcal{A}^{\diamondsuit}(\bar{k})$. If $a(\bar{X})$ meets the smooth part of $\mathrm{discrim}_{G,D}$ with multiplicity greater than one, the calculation above shows that $\widetilde{X}_a$ is not smooth. Now suppose that $a(\bar{X})$ meets the locus $\mathrm{discrim}_{G,D}^{\mathrm{sing}}$ at a point $v \in \bar{X}$. Assume that $\widetilde{X}_a$ is smooth at the point $(v, x) \in \mathfrak{t}_D(\bar{k})$ above v. Then the point x belongs to at least two distinct root hyperplanes so that the local monodromy group $\pi_a^\bullet(I_v)$ (see 3.7.3) contains two distinct involutions. This is impossible because, under the assumption that the characteristic of k does not divide the order of $\mathbb{W}$, the local monodromy group $\pi_a^\bullet(I_v)$ is cyclic. □

Corollary 4.7.4. — *Let $\bar{X}_{\bar{\rho}} \to \bar{X}$ be a connected finite étale Galois cover which splits G. Suppose that* $\deg(D) > 2g$. *Then for every $a \in \mathcal{A}^{\diamondsuit}(\bar{k})$, the curve $\widetilde{X}_{\bar{\rho},a}$ is irreducible.*

Corollary 4.7.5. — *If $a \in \mathscr{A}^{\diamondsuit}(\bar{k})$, then the image of $\pi_a^\bullet$ is $\mathbb{W} \rtimes \Theta$.*

Proof. — By Proposition 4.6.1, $\widetilde{X}_{\rho,a}$ is connected. Since $\widetilde{X}_a$ is smooth, the étale cover $\widetilde{X}_{\rho,a}$ is also smooth. Therefore, it is irreducible. □

Definition 4.7.6. — *A* $\bar{k}$-abelian stack *is the quotient of a $\bar{k}$-abelian variety by the trivial action of a diagonalizable group.*

A typical example of an abelian stack is the stack classifying line bundles of degree zero on a smooth, projective, geometrically connected curve. If this curve is equipped with an action of a finite group whose order is prime to the characteristic, the associated Prym stacks are abelian stacks.

Proposition 4.7.7. — *For every $a \in \mathcal{A}^{\diamondsuit}(\bar{k})$, the regular locus $\mathcal{M}_a^{\mathrm{reg}}$ equals the whole fiber $\mathcal{M}_a$, so that $\mathcal{M}_a$ is a torsor under $\mathcal{P}_a$. The neutral component of $\mathcal{P}_a$ is an abelian stack.*

We refer to [**57**, Proposition 4.2] for the proof of the first assertion. The proof of the second assertion is postponed until the end of Section 4.8.

4.7.8. — For every $a \in \mathscr{A}^{\heartsuit}(\bar{k})$ we have seen that the inertia group of $\mathscr{P}_a$ is identified with $Z_{\mathbb{G}}$. We shall see in 4.10.4 that for $a \in \mathscr{A}^{\diamondsuit}(\bar{k})$ the group of connected components $\pi_0(\mathscr{P}_a)$ is identified with $Z_{\hat{\mathbb{G}}}$. One may observe that if G_1 and G_2 are dual to one another and if $a \in \mathscr{A}_{G_1}^{\diamondsuit}(\bar{k}) = \mathscr{A}_{G_2}^{\diamondsuit}(\bar{k})$, then the inertia group of $\mathscr{P}_{G_1,a}$ is isomorphic to the group of connected components of $\mathscr{P}_{G_2,a}$, and vice versa.

4.8. Global Néron Model. — In this section we fix a point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$ and denote by U the inverse image of the open set $\mathfrak{c}_D^{\mathrm{rs}}$ under the morphism $a : \bar{X} \to \mathfrak{c}_D^{\mathrm{rs}}$.

As in the local case (§3.8), the structure of the Picard stack $\mathcal{P}_a$ of J_a-torsors on $\bar{X}$ can be analyzed using the Néron model $J_a^\flat$ of J_a. This is a smooth group scheme of finite type over $\bar{X}$ equipped with a homomorphism $J_a \to J_a^\flat$ that is an isomorphism over U. It is further characterized by the following property: for every smooth group scheme J' of finite type on $\bar{X}$ with a homomorphism $J_a \to J'$ that is an isomorphism on U, there exists a unique homomorphism $J' \to J_a^\flat$ making the obvious triangle commute. The existence of this Néron model is a result of Bosch, Lütkebohmert and Raynaud [**10**] (they call it the finite type Néron model to distinguish it from the locally finite type local Néron model).

This global Néron model is obtained from the local Néron models (see §3.8) as follows. For every $v \in \bar{X} - U$, let $\bar{X}_v$ be the formal completion of $\bar{X}$ at v and let $\bar{X}_v^\bullet = \bar{X}_v - \{v\}$. Consider the Néron model of the torus $J_a|_{\bar{X}_v^\bullet}$. By gluing these Néron models at the various points $v \in \bar{X} - U$ with the torus $J_a|_U$, one obtains a smooth commutative group scheme $J_a^\flat$ over $\bar{X}$ along with a homomorphism $J_a \to J_a^\flat$.

As in 3.8.2, $J_a^\flat$ can be expressed in terms of the normalization $\widetilde{X}_a^\flat$ of the cameral curve $\widetilde{X}_a$. The W-action on $\widetilde{X}_a$ induces an action on $\widetilde{X}_a^\flat$. Let $\pi_a^\flat : \widetilde{X}_a^\flat \to \bar{X}$ be the projection. The following is the global consequence of the local statement 3.8.2.

Corollary 4.8.1. — *$J_a^\flat$ is identified with the subgroup of fixed points under the diagonal action of W in*

$$\prod_{\widetilde{X}_a^\flat/\bar{X}} \left(T \times_{\bar{X}} \widetilde{X}_a^\flat\right).$$

A useful variant is as follows. Consider a reduction $\bar{\rho} : \bar{X}_{\bar{\rho}} \to \bar{X}$ of $\bar{\rho}_G$ with Galois group $\Theta_{\bar{\rho}}$. For every $a \in \mathcal{A}^\heartsuit(\bar{k})$, there is a finite flat cover

$$\pi_{\bar{\rho},a} : \widetilde{X}_{\bar{\rho},a} \to \bar{X}$$

defined as in (4.5.4), which is generically étale Galois with group $\mathbb{W} \rtimes \Theta_{\bar{\rho}}$. Let $\widetilde{X}_{\bar{\rho},a}^\flat$ be the normalization of $\widetilde{X}_{\bar{\rho},a}$, which is also equipped with an action of $\mathbb{W} \rtimes \Theta_{\bar{\rho}}$. Denote by $\pi_{\bar{\rho},a}^\flat$ its projection to $\bar{X}$. Then one has

$$J_a^\flat = \prod_{\widetilde{X}_{\bar{\rho},a}^\flat/\bar{X}} \left(\mathbb{T} \times \widetilde{X}_{\bar{\rho},a}^\flat\right)^{\mathbb{W}\rtimes\Theta_{\bar{\rho}}}.$$

Consider the Picard groupoid $\mathcal{P}_a^\flat$ of $J_a^\flat$-torsors. The homomorphism of group schemes $J_a \to J_a^\flat$ induces a homomorphism of Picard groupoids $\mathcal{P}_a \to \mathcal{P}_a^\flat$. This homomorphism essentially reflects the general structure of an algebraic group over an algebraically closed field as an extension of an abelian variety by an affine algebraic group (see [**66**]). The proof below is inspired by an argument of Raynaud [**64**].

Proposition 4.8.2. — (1) *The homomorphism $\mathcal{P}_a(\overline{k}) \to \mathcal{P}_a^\flat(\overline{k})$ is essentially surjective.*

(2) *The neutral component $(\mathcal{P}_a^\flat)^0$ of $\mathcal{P}_a^\flat$ is an abelian stack.*

(3) *The kernel $\mathcal{R}_a$ of $\mathcal{P}_a \to \mathcal{P}_a^\flat$ is a product of affine algebraic groups of finite type $\mathcal{R}_v(a)$, as defined in Lemma 3.8.1. These are trivial except at finitely many points $v \in |\bar{X}|$.*

Proof. — (1) By the very construction of the Néron models, the homomorphism $J_a \to J_a^\flat$ is injective as a homomorphism between sheaves of abelian groups for the étale topology on $\bar{X}$. Consider the short exact sequence

$$1 \to J_a \to J_a^\flat \to J_a^\flat/J_a \to 1, \tag{4.8.3}$$

where the quotient $J_a^\flat/J_a$ is supported on the finite closed subset $\bar{X} - U$. The long exact cohomology sequence shows that, since $\mathrm{H}^1(\bar{X}, J_a^\flat/J_a) = 0$, the map

$$\mathrm{H}^1(\bar{X}, J_a) \longrightarrow \mathrm{H}^1(\bar{X}, J_a^\flat)$$

is surjective.

(2) Let $\widetilde{X}_a$ be the inverse image of the cover $\mathfrak{t}_D \longrightarrow \mathfrak{c}_D$ via $h_a : \bar{X} \longrightarrow \mathfrak{c}_D$. Let $\widetilde{X}_a^\flat$ be the normalization of $\widetilde{X}_a$. By Proposition 4.8.1, the Néron model $J_a^\flat$ depends only on the cover $\widetilde{X}_a^\flat$. More precisely, $J_a^\flat$ consists of the fixed points under the diagonal action of W in the Weil restriction

$$\prod_{\widetilde{X}_a^\flat/\bar{X}} \left(T \times_{\bar{X}} \widetilde{X}_a^\flat\right).$$

It follows that up to isogeny, $\mathcal{P}_a^\flat$ is a factor of the Picard groupoid of T-torsors on $\widetilde{X}_a^\flat$, which is isomorphic to the product of r copies of $\mathrm{Pic}(\widetilde{X}_a^\flat)$. Since $\widetilde{X}_a^\flat$ is a smooth projective curve (possibly disconnected), the neutral component of $\mathrm{Pic}(\widetilde{X}_a^\flat)$ is the quotient of a product of Jacobians by a product of $\mathbb{G}_m$ acting trivially.

(3) The last assertion also follows from the long exact sequence in cohomology arising from the short exact sequence (4.8.3). Since the kernel is defined as the category of J_a-torsors equipped with a trivialization of the induced $J_a^\flat$-torsor, one need not worry about the H^0 in the long exact sequence.

This completes the proof. □

Proof. — Returning now to the second assertion of Proposition 4.7.7: Let $a \in \mathcal{A}^\diamondsuit(\bar{k})$. Then the cameral curve $\widetilde{X}_a$ is smooth, so the cokernel of the homomorphism $J_a \longrightarrow J_a^\flat$ is a finite group sheaf supported on a finite subscheme of $\bar{X}$. It follows that $\mathcal{P}_a \longrightarrow \mathcal{P}_a^\flat$ is an isogeny of Picard groupoids (i.e., a homomorphism that is essentially surjective with finite kernel). Since $\mathcal{P}_a^\flat$ is an abelian stack, the same holds for $\mathcal{P}_a$. □

Lemma 4.8.4. — *Let $C_{\tilde{a}}$ be a connected component of $\widetilde{X}_{\rho,a}$ and let $W_{\tilde{a}}$ be the subgroup of elements of $\mathbb{W}$ stabilizing $C_{\tilde{a}}$. If $\mathbb{T}^{W_{\tilde{a}}}$ is finite and unramified, then $\mathscr{P}_a$ is a Deligne-Mumford abelian stack.*

4.9. Invariant δ_a. —

4.9.1. — An important numerical invariant that one may attach to $a \in \mathcal{A}^\heartsuit(\bar{k})$ is the delta invariant. For every $a \in \mathcal{A}^\heartsuit(\bar{k})$, consider the kernel

$$\mathcal{R}_a := \ker\left[\mathcal{P}_a \longrightarrow \mathcal{P}_a^\flat\right]. \tag{4.9.2}$$

It classifies the J_a-torsors on X whose induced $J_a^\flat$-torsor is trivialized. This is an affine algebraic group of finite type which decomposes as a product

$$\mathcal{R}_a = \prod_{v \in \bar{X} - U} \mathcal{R}_v(a),$$

where $\mathcal{R}_v(a)$ is the group defined in 3.8.1.

We define the *invariant* δ_a as the dimension of $\mathcal{R}_a$, that is,

$$\delta_a := \dim(\mathcal{R}_a) = \sum_{v \in \bar{X} - U} \delta_v(a).$$

Combining Proposition 4.8.2 and the dimension formula for the local symmetry group (see 3.8.3) yields the following formula for the global invariant δ.

4.9.3. — Recall that, on choosing a rigidifier *cf.* 4.5.6, one has defined a smooth commutative algebraic group $\tilde{\mathscr{P}}_a^0$ surjecting onto the neutral component $\mathscr{P}_a^0$. By Chevalley's structure theorem one has a canonical exact sequence

$$1 \longrightarrow \tilde{\mathscr{R}}_a \longrightarrow \tilde{\mathscr{P}}_a^0 \longrightarrow A_a \longrightarrow 1,$$

where A_a is an abelian variety. Under the hypothesis of 4.8.4 this is in fact an étale isogeny. There follows the formula

$$\dim(A_a) = d - \delta_a.$$

In this case, and in this case only, is it justified to call δ_a the dimension of the affine part of $\mathscr{P}_a$.

Corollary 4.9.4. — *For every $a \in \mathcal{A}^\heartsuit(\bar{k})$, the invariant δ_a defined above equals*

$$\delta_a = \dim \mathrm{H}^0\Big(\bar{X},\, \mathfrak{t} \otimes_{O_{\bar{X}}} \Big(\frac{\pi^\flat_{a*} O_{\widetilde{X}^\flat_a}}{\pi_{a*} O_{\widetilde{X}_a}}\Big)\Big)^W.$$

Again, there is another formula that expresses the global invariant δ in terms of the discriminant corrected by local monodromy invariants, as in Bezrukavnikov's formula.

Recall that the discriminant $\mathrm{discrim}_G$ is a homogeneous polynomial of degree $\sharp\,\Phi$ on $\mathfrak{c}$, where $\sharp\,\Phi$ is the number of roots in Φ. Hence, for every $a \in \mathcal{A}^\heartsuit(\bar{k})$, $a^*\,\mathrm{discrim}_{G,D}$ is a divisor linearly equivalent to $D^{\otimes(\sharp\Phi)}$, so that

$$\deg\big(a^*\,\mathrm{discrim}_{G,D}\big) = \sharp\,\Phi \deg(D).$$

Write

$$a^*\,\mathrm{discrim}_{G,D} = d_1 v_1 + \cdots + d_n v_n,$$

where $v_1, \ldots, v_n$ are pairwise distinct points of $\bar{X}$ and d_i is the multiplicity at v_i. For every $i = 1, \ldots, n$, denote by c_i the drop in the toric rank of $J^\flat_a$ at the point v_i. The following proposition is an immediate consequence of 3.7.5.

Proposition 4.9.5. — *We have the equality*

$$2\delta_a = \sum_{i=1}^{n} (d_i - c_i) = \sharp\,\Phi \deg(D) - \sum_{i=1}^{n} c_i.$$

4.10. The Group $\pi_0(\mathcal{P}_a)$. — In this paragraph we shall describe the group of connected components of $\mathscr{P}_a$ in the spirit of Tate-Nakayama duality. To do so it is necessary to make a certain number of choices and to fix some notation.

Fix a point $\infty \in X(\bar{k})$. Consider the open subset $\mathscr{A}^\infty$ of $\mathscr{A} \otimes_k \bar{k}$ consisting of the points $a \in \mathscr{A}(\bar{k})$ such that $a(\infty) \in \mathfrak{c}^{\mathrm{rs}}_D$. It is an open subscheme of $\mathscr{A}^\heartsuit \otimes_k \bar{k}$. If ∞ is defined over k, the open subset $\mathscr{A}^\infty$ is also defined over k.

Fix a point $a \in \mathscr{A}^\infty(\bar{k})$. Let U be the maximal open subset of $\bar{X}$ above which the cameral cover $\widetilde{X}_a \to \bar{X}$ is étale; by construction $\infty \in U$. Suppose that G is defined by a continuous homomorphism

$$\rho^\bullet_G : \pi_1(X, \infty) \to \mathrm{Out}(\mathbb{G}).$$

Having chosen a geometric point $\tilde{\infty}$ of the cameral curve $\widetilde{X}_a$ above ∞, we have, as in 1.4.5, a continuous homomorphism $\pi^\bullet_{\tilde{a}}$ fitting into a commutative diagram

(4.10.1)
$$\begin{tikzcd}
\pi_1(U,\infty) \arrow[r, "\pi^\bullet_{\tilde{a}}"] \arrow[d] & \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) \arrow[d] \\
\pi_1(\bar{X},\infty) \arrow[r, "\rho^\bullet_G"'] & \mathrm{Out}(\mathbb{G})
\end{tikzcd}$$

where $\tilde{a} = (a, \tilde{\infty})$. We write $W_{\tilde{a}}$ for the image of $\pi^\bullet_{\tilde{a}}$ in $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$, and $I_{\tilde{a}}$ for the image of the kernel of $\pi_1(U, \infty) \to \pi_1(\bar{X}, \infty)$. The commutativity of the diagram above implies $I_{\tilde{a}} \subset \mathbb{W}$.

Let now J^0_a be the open subgroup scheme of neutral components of J_a. Consider the Picard stack $\mathscr{P}'_a$ of J^0_a-torsors on $\bar{X}$. The homomorphism of sheaves of groups $J^0_a \to J_a$ induces a homomorphism of Picard stacks $\mathscr{P}'_a \to \mathscr{P}_a$.

Lemma 4.10.2. — *The homomorphism $\mathscr{P}'_a \to \mathscr{P}_a$ is surjective with finite kernel. The same holds for the homomorphism $\pi_0(\mathscr{P}'_a) \to \pi_0(\mathscr{P}_a)$ deduced from it.*

Proof. — We have a short exact sequence of sheaves

$$0 \to J_a^0 \to J_a \to \pi_0(J_a) \to 0,$$

where $\pi_0(J_a)$ is a sheaf with finite support whose fiber at a point $v \in \bar{X}$ is the group $\pi_0(J_{a,v})$ of connected components of the fiber $J_{a,v}$ of J_a. From it we deduce a long exact sequence

$$\mathrm{H}^0(\bar{X}, \pi_0(J_a)) \to \mathrm{H}^1(\bar{X}, J_a^0) \to \mathrm{H}^1(\bar{X}, J_a) \to \mathrm{H}^1(\bar{X}, \pi_0(J_a)) = 0.$$

The vanishing of the last term results from the fact that $\pi_0(J_a)$ is supported on a scheme of dimension zero. We deduce the surjectivity of $\mathscr{P}'_a \to \mathscr{P}_a$, with kernel $\mathrm{H}^0(\bar{X}, \pi_0(J_a))$. The finiteness of this kernel comes from the finiteness of the fibers of $\pi_0(J_a)$. The assertion on $\pi_0(\mathscr{P}'_a) \to \pi_0(\mathscr{P}_a)$ follows immediately. □

Instead of the abelian groups $\pi_0(\mathscr{P}'_a)$ and $\pi_0(\mathscr{P}_a)$, it is more convenient to describe the dual diagonalizable groups. Dually, one has an inclusion of diagonalizable groups

$$\pi_0(\mathscr{P}_a)^* \subset \pi_0(\mathscr{P}'_a)^*,$$

where we have written

$$\pi_0(\mathscr{P}_a)^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(\mathscr{P}_a)]), \qquad \pi_0(\mathscr{P}'_a)^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(\mathscr{P}'_a)]).$$

We have used the exponent $(\_)^*$ to denote the duality between finitely generated abelian groups and diagonalizable groups of finite type over $\overline{\mathbb{Q}}_\ell$.

Proposition 4.10.3. — *For every $\tilde{a} = (a, \tilde{\infty})$ as above, one has isomorphisms of diagonalizable groups*

$$\pi_0(\mathscr{P}'_a)^* = \hat{\mathbb{T}}^{W_{\tilde{a}}}$$

and

$$\pi_0(\mathscr{P}_a)^* = \hat{\mathbb{T}}(I_{\tilde{a}}, W_{\tilde{a}}),$$

where $\hat{\mathbb{T}}(I_{\tilde{a}}, W_{\tilde{a}})$ is the subgroup of $\hat{\mathbb{T}}^{W_{\tilde{a}}}$ consisting of the elements κ such that $W_{\tilde{a}} \subset (\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ and $I_{\tilde{a}} \subset \mathbb{W}_{\mathbb{H}}$, where $\mathbb{W}_{\mathbb{H}}$ is the Weyl group of the neutral component of the centralizer of κ in $\hat{\mathbb{G}}$.

Proof. — By [**57**, Cor. 6.7] one has a canonical isomorphism

$$(\mathbb{X}_*)_{W_{\tilde{a}}} \longrightarrow \pi_0(\mathscr{P}'_a)$$

from the group of $W_{\tilde{a}}$-coinvariants of $\mathbb{X}_* = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ onto the group of connected components of $\mathscr{P}'_a$. This is essentially a particular case of a lemma of Kottwitz [**42**, Lemma 2.2]. Dually, one has an isomorphism of diagonalizable groups

$$\pi_0(\mathscr{P}'_a)^* = \hat{\mathbb{T}}^{W_{\tilde{a}}},$$

where $\hat{\mathbb{T}}$ is the $\overline{\mathbb{Q}}_\ell$-dual torus of $\mathbb{T}$.

Write $U = a^{-1}(\mathfrak{c}_D^{\mathrm{rs}})$. As in the proof of 4.10.2, we have an exact sequence

$$\mathrm{H}^0(\bar{X}, \pi_0(J_a)) \to \pi_0(\mathscr{P}'_a) \to \pi_0(\mathscr{P}_a) \to 0,$$

where $\mathrm{H}^0(\bar{X}, J_a/J_a^0) = \bigoplus_{v \in \bar{X} - U} \pi_0(J_{a,v})$, with $\pi_0(J_{a,v})$ the group of connected components of the fiber of J_a at v. For every $v \in \bar{X} - U$ one has an analogous local exact sequence

$$\pi_0(J_{a,v}) \to \pi_0(\mathscr{P}_v(J_a^0)) \to \pi_0(\mathscr{P}_v(J_a)) \to 0$$

compatible with the global one. Consider the dual sequences of diagonalizable groups

$$0 \to \pi_0(\mathscr{P}_v(J_a))^* \to \pi_0(\mathscr{P}_v(J_a^0))^* \to \pi_0(J_{a,v})^*.$$

The subgroup

$$\pi_0(\mathscr{P}_a)^* \subset \pi_0(\mathscr{P}'_a)^*$$

is then the intersection of the inverse images of the subgroups

$$\pi_0(\mathscr{P}_v(J_a))^* \subset \pi_0(\mathscr{P}_v(J_a^0))^*$$

for all $v \in \bar{X} - U$. The proposition now follows from 3.9.2. □

Corollary 4.10.4. — *For $a \in \mathscr{A}^{\diamond}(\bar{k})$ one has $\pi_0(\mathscr{P}_a) = Z_{\hat{\mathbb{G}}}$.*
Proof. — For $\tilde{a} = (a, \tilde{\infty})$ with $a \in \mathscr{A}^{\diamond}(\bar{k})$, we have seen that $W_{\tilde{a}} = \mathbb{W} \rtimes \Theta$ and $I_{\tilde{a}} = \mathbb{W}$ *cf.* 4.7.5. The corollary therefore follows from 4.10.3. □

4.10.5. — Let $\mathscr{A}^{\mathrm{ani}}(\bar{k})$ be the subset of those $a \in \mathscr{A}^{\heartsuit}(\bar{k})$ for which $\pi_0(\mathscr{P}_a)$ is finite. By 4.10.3, this is equivalent to the finiteness of $\hat{\mathbb{T}}^{W_{\tilde{a}}}$. We shall show in 5.4.5 that $\mathscr{A}^{\mathrm{ani}}(\bar{k})$ is the set of $\bar{k}$-points of an open subset $\mathscr{A}^{\mathrm{ani}}$ of $\mathscr{A}^{\heartsuit}$.

4.11. Automorphisms. — Let $(E, \phi) \in \mathcal{M}(\bar{k})$ be a point lying over $a \in \mathcal{A}^{\heartsuit}(\bar{k})$. We now establish bounds for the group of automorphisms $\mathrm{Aut}(E, \phi)$ in terms of a. Let U be the open subset of $\bar{X}$ where a is regular semisimple.

Consider the sheaf of automorphisms $\underline{\mathrm{Aut}}(E, \phi)$ that associates to every $\bar{X}$-scheme S the group $\mathrm{Aut}((E, \phi)|_S)$. This sheaf is represented by the group scheme

$$I_{(E,\phi)} = h^*_{(E,\phi)} I,$$

which is the pull-back of the centralizer I on $\mathfrak{g}$ via the map $h_{(E,\phi)} : \bar{X} \to [\mathfrak{g}_D/G]$. The restriction of $I_{(E,\phi)}$ to the open U is a torus, but over $\bar{X}$ the group scheme $I_{(E,\phi)}$ is neither smooth nor flat. However, one may consider its smoothening in the sense of [**10**]. According to loc. cit., there exists a unique smooth group scheme $I^{\mathrm{lis}}_{(E,\phi)}$ over $\bar{X}$ such that for every smooth $\bar{X}$-scheme S, we have

$$\mathrm{Aut}((E, \phi)|_S) = \mathrm{Hom}_X(S, I^{\mathrm{lis}}_{(E,\phi)}).$$

The tautological morphism $I^{\mathrm{lis}}_{(E,\phi)} \to I_{(E,\phi)}$ is an isomorphism over U. Note that by this characterization,

$$\mathrm{Aut}(E, \phi) = \mathrm{H}^0\Big(\bar{X}, I^{\mathrm{lis}}_{(E,\phi)}\Big). \tag{4.11.1}$$

Since J_a is smooth, the canonical homomorphism $J_a \to I_{(E,\phi)}$ induces a homomorphism

$$J_a \longrightarrow I^{\mathrm{lis}}_{(E,\phi)},$$

which is an isomorphism over U. By the universal property of the Néron model, there is a homomorphism

$$I^{\mathrm{lis}}_{(E,\phi)} \longrightarrow J^{\flat}_a,$$

which is an isomorphism on U.
Proposition 4.11.2. — *For every $(E, \phi) \in \mathcal{M}(\bar{k})$ lying over a point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, we have canonical inclusions*

$$\mathrm{H}^0(\bar{X}, J_a) \subset \mathrm{Aut}(E, \phi) \subset \mathrm{H}^0(\bar{X}, J^{\flat}_a).$$

Proof. — It suffices to check that the maps $J_a \to I^{\mathrm{lis}}_{(E,\phi)}$ and $I^{\mathrm{lis}}_{(E,\phi)} \to J^{\flat}_a$ are injective as morphisms of sheaves for the étale topology. For this, it is enough to verify injectivity on the formal neighborhoods at each point $v \in \bar{X} - U$. Denote by $\bar{\mathcal{O}}_v$ the formal completion of $\mathcal{O}_{\bar{X}}$ at v and by $\bar{F}_v$ its fraction field. Then we have the inclusions

$$J_a(\bar{\mathcal{O}}_v) \subset I^{\mathrm{lis}}_{(E,\phi)}(\bar{\mathcal{O}}_v) \subset J^{\flat}_a(\bar{\mathcal{O}}_v)$$

as subgroups of $J_a(\bar{F}_v)$. □

To describe explicitly the upper bound $\mathrm{H}^0(\bar{X}, J^{\flat}_a)$, choose splittings as in the previous section. In particular, we have a point $\infty \in X(\bar{k})$ with $a(\infty) \in \mathfrak{c}^{\mathrm{rs}}_D$, and choose a splitting $\bar{\rho} : \bar{X}_{\bar{\rho}} \to \bar{X}$ (see 1.3.6). Then there is a finite flat cover

$$\pi_{\bar{\rho},a} : \widetilde{X}_{\bar{\rho},a} \to \bar{X},$$

which is generically étale Galois with Galois group $\mathbb{W} \rtimes \Theta_{\bar\rho}$. Choosing a point $\tilde\infty_{\bar\rho} \in \widetilde{X}_{\bar\rho,a}$ above ∞ defines a subgroup $W_{\tilde a} \subset \mathbb{W} \rtimes \Theta_{\bar\rho}$ and a normal subgroup $I_{\tilde a} \subset W_{\tilde a}$ as in §4.10. Then, by 4.8.1 and 2.4.4, we have

$$\mathrm{H}^0(\bar X, J_a^\flat) = \mathbb{T}^{W_{\tilde a}}.$$

We thus deduce the following corollary.

Corollary 4.11.3. — *For every $(E,\phi) \in \mathcal{M}(\bar k)$ lying over a point $a \in \mathcal{A}^\heartsuit(\bar k)$,* $\mathrm{Aut}(E,\phi)$ *is identified with a subgroup of $\mathbb{T}^{W_{\tilde a}}$.*

Recent work of Frenkel and Witten suggests that this bound is not optimal. In fact, one should have

$$\mathrm{Aut}(E,\phi) \subset \mathbb{T}(I_{\tilde a}, W_{\tilde a}),$$

where $\mathbb{T}(I_{\tilde a}, W_{\tilde a})$ is the subgroup of $\mathbb{T}^{W_{\tilde a}}$ defined as in Proposition 4.10.3 (with $\hat{\mathbb{T}}$ replaced by $\mathbb{T}$). Moreover, equality is expected to be achieved at the most singular points of the fiber $\mathcal{M}_a$.

4.11.4. — Over the open subset $\mathscr{A}^{\mathrm{ani}}$ *cf.* 4.10.5, the group $\hat{\mathbb{T}}^{W_{\tilde a}}$ is finite. The same holds for $\mathbb{T}^{W_{\tilde a}}$. Assuming the order of $\mathbb{W} \rtimes \Theta$ prime to the characteristic, $\mathbb{T}^{W_{\tilde a}}$ will be finite unramified. The restriction of $\mathscr{M}$ to $\mathscr{A}^{\mathrm{ani}}$ is therefore Deligne-Mumford, by the corollary above.

4.12. Polarizable Tate Module. — Let us place ourselves under the hypothesis of 4.5.5, so that $\mathscr{P}$ is a smooth Deligne-Mumford Picard stack over $\mathscr{A}^\heartsuit$. Write $\mathscr{P}^0$ for the Picard substack of neutral components of $\mathscr{P}$, and let $g : \mathscr{P}^0 \to \mathscr{A}^\heartsuit$ be the structural morphism, which is smooth of relative dimension d. Consider the Tate module

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}^0) = \mathrm{H}^{2d-1}(g_!\overline{\mathbb{Q}}_\ell),$$

which is a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathscr{A}^\heartsuit$.

The more elementary way of presenting this sheaf is to use rigidifiers. As in 4.5.6, we have defined over the open subset $\mathscr{A}^\infty$ smooth group schemes with connected fibers P^{-1} and P^0, with P^{-1} affine, such that there is an exact sequence

$$1 \to P^{-1} \to P^0 \to \mathscr{P}^0|_{\mathscr{A}^\infty} \to 1.$$

There follows an exact sequence of $\overline{\mathbb{Q}}_\ell$-Tate modules

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^{-1}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}^0|_{\mathscr{A}^\infty}) \to 0.$$

For every geometric point $a \in \mathscr{A}^\infty(\bar k)$, write A_a for the maximal abelian quotient of P_a^0. Since P_a^{-1} is affine, the arrow $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_a^0) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$ factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}_a^0)$. From this we deduce an exact sequence

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_a) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}_a^0) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a) \to 0,$$

which in fact does not depend on the choice of rigidifier. Thus, although no canonical Chevalley decomposition is available for the Deligne-Mumford Picard stack $\mathscr{P}_a^0$, one does have a canonical decomposition of its $\overline{\mathbb{Q}}_\ell$-Tate module.

Proposition 4.12.1. — *There exists an alternating form*

$$\psi : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}^\heartsuit) \times \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathscr{P}^\heartsuit) \to \overline{\mathbb{Q}}_\ell(-1)$$

such that at every geometric point $a \in \mathscr{A}^\heartsuit(\bar k)$ the form ψ_a vanishes on the affine part $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_a)$ and induces a perfect pairing on the abelian part $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$.

The proof of this proposition is based on Weil pairing theory which we will recall for the reader's convenience. Let S be a strictly Henselian local scheme. Let $c : C \to S$ be a proper flat morphism with geometrically reduced fibers of dimension one.

Suppose that C is connected. Consider the Stein factorization $C \to S' \to S$ where $C \to S'$ is a proper morphism with connected non-empty fibers and $S' \to S$ is a finite morphism. Since the special fiber of c is reduced, the morphism $S' \to S$ is finite and étale. Since C is connected,

S' is also connected. Since we assumed that S is strictly Henselian, we then have $S' = S$. In other words, the fibers of $c : C \to S$ are connected.

Consider the S-Artin stack $\mathrm{Pic}_{C/S}$ which associates to any S-scheme Y the groupoid of invertible bundles on $C \times_S Y$. It is smooth over S. Consider its neutral component $\mathrm{Pic}^0_{C/S}$. For any point $L \in \mathrm{Pic}_{X/S}(Y)$, for any $y \in Y$, we can define the Euler-Poincaré characteristic $\chi_y(L)$ of the restriction of L to C_y. If Y is connected, this integer is independent of y and we denote it $\chi(L)$. If $L \in \mathrm{Pic}^0_{C/S}$, we have $\chi(L) = \chi(\mathcal{O}_C)$.

For any pair of $L, L' \in \mathrm{Pic}^0_{C/S}$, we define their Weil pairing by the formula

$$\begin{aligned}\langle L, L'\rangle_{C/S} = {} & \det(\mathrm{R}c_*(L \otimes L')) \otimes \det(\mathrm{R}c_* L)^{\otimes -1}\\ & \otimes \det(\mathrm{R}c_* L')^{\otimes -1} \otimes \det(\mathrm{R}c_* \mathcal{O}_C)\end{aligned}$$

using the determinant of cohomology. If t is an automorphism of L which is then a scalar, t acts on $\det(Rc_* L)$ by the scalar $t^{\chi(L)}$. Using the equalities

$$\chi(L \otimes L') = \chi(L) = \chi(L') = \chi(\mathcal{O}_C)$$

for $L, L' \in \mathrm{Pic}^0_{X/S}$, we verify that for any pair of scalars (t, t') the action of t on L and the action of t' on L' induce the identity on $\langle L, L'\rangle_{C/S}$.

If N is an invertible integer on S and L is an invertible bundle equipped with an isomorphism $\iota_L : L^{\otimes N} \to \mathcal{O}_C$, we have an isomorphism

$$\langle L, L'\rangle^{\otimes N}_{C/S} = \mathcal{O}_S.$$

If furthermore L' is also equipped with an isomorphism $\iota_{L'} : L'^{\otimes N} \to \mathcal{O}_C$, we have another isomorphism $\langle L, L'\rangle^{\otimes N}_{C/S} = \mathcal{O}_S$. The difference between these two isomorphisms defines an N-th root of unity. This root depends only on the isomorphism classes of L and L' by virtue of the discussion on the effect of scalars.

Suppose now that C is a connected projective curve over an algebraically closed field k. The stack Pic^0_C is then the quotient of a connected commutative algebraic k-group Jac_C by $\mathbb{G}_m$ acting trivially. The above construction defines an alternating form

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Jac}_C) \times \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Jac}_C) \longrightarrow \overline{\mathbb{Q}}_\ell(-1)$$

which is non-degenerate when C is smooth. To conclude this digression, we still need to consider the behavior of the Weil pairing with respect to normalization.

Lemma 4.12.2. — *Let C be a reduced projective curve over an algebraically closed field k. Let $C^\flat$ be its normalization and $\xi : C^\flat \to C$ be the normalization morphism. Let L, L' be two invertible bundles on C. Then there exists a canonical isomorphism of one-dimensional k-vector spaces*

$$\langle L, L'\rangle_C = \langle \xi^* L, \xi^* L'\rangle_{C^\flat}.$$

Proof. — We have an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \xi_* \mathcal{O}_{C^\flat} \longrightarrow \mathcal{D} \longrightarrow 0$$

where $\mathcal{D}$ is an $\mathcal{O}_C$-module of finite length supported on a finite collection of points $\{c_1, \ldots, c_n\}$ of C. Let $\mathcal{D}_i$ denote the direct factor of $\mathcal{D}$ supported on c_i and d_i its length. The multiplicativity of the determinant then provides us with an isomorphism

$$\det(c^\flat_* \mathcal{O}_{C^\flat}) = \det(c_* \xi_* \mathcal{O}_{C^\flat}) = \det(c_* \mathcal{O}_C) \otimes \bigotimes_{i=1}^{r} \wedge^{d_i} \mathcal{D}_i$$

where $c : C \to \mathrm{Spec}(k)$ and $c^\flat : C^\flat \to \mathrm{Spec}(k)$ denote the structural morphisms.

Let L be an invertible bundle on C. By using the projection formula $\xi_*\xi^*L = (\xi_*\mathcal{O}) \otimes L$, we obtain the formula

$$\det(c^\flat_*\xi^*L) = \det(c_*L) \otimes \bigotimes_{i=1}^{r} (L_{c_i}^{\otimes d_i} \otimes \wedge^{d_i}\mathcal{D}_i)$$

where L_{c_i} is the fiber of L at c_i. By applying this formula to $L \otimes L'$, L and to L', we obtain the lemma. $\square$

We are now able to prove Proposition 4.12.1.

Proof. — Recall that for any point $a \in \mathcal{A}^\heartsuit$, $\mathcal{P}_a$ is the Picard stack of J_a-torsors on X. Let $\pi_a : \tilde{X}_a \to X$ be the cameral cover associated with a. According to the Galois description of the regular centralizer, we have a homomorphism of group sheaves *cf.* 2.4.2

$$J_a \to \pi_{a,*}(T \times_X \tilde{X}_a)$$

where T is the maximal torus of G in the fixed pinning. By considering a splitting $X_\rho \to X$ of G, we obtain a finite étale cover $\tilde{X}_{\rho,a} \to \tilde{X}_a$ by base change. Let $\pi_{\rho,a} : \tilde{X}_{\rho,a} \to X$ be the composed morphism. We thus have a homomorphism

$$J_a \to \pi_{\rho,a,*}(\mathbb{T} \times \tilde{X}_{\rho,a}).$$

where $\mathbb{T}$ is a split torus. The datum of an ℓ^n-torsion point of $\mathcal{P}_a$ thus induces an ℓ^n-torsion point of $\mathrm{Pic}_{\tilde{X}_{\rho,a}} \otimes \mathbb{X}_*(\mathbb{T})$. By choosing an invariant symmetric form on $\mathbb{X}_*(\mathbb{T}) \otimes \overline{\mathbb{Q}}_\ell$ and using the Weil pairing on $\mathrm{Pic}^0_{\tilde{X}_{\rho,a}}$, we obtain an alternating form

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_a^0) \otimes \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_a^0) \longrightarrow \overline{\mathbb{Q}}_\ell(-1).$$

It remains to show that at each geometric point a, this symplectic form is zero on the affine part of $\mathrm{T}_\ell(\mathcal{P}_a)$ and induces a non-degenerate pairing on its abelian part. This follows from Lemma 4.12.2. $\square$

4.13. Dimensions. — We will use the same notation $\mathfrak{c}_D$ to denote both the vector bundle over X and the corresponding locally free O_X-module. If G is split, the O_X-module $\mathfrak{c}_D$ has the explicit expression

$$\mathfrak{c}_D = \bigoplus_{i=1}^{r} D^{\otimes e_i},$$

where $e_1, \dots, e_r$ are the natural numbers appearing in 1.1.1. In general, $\mathfrak{c}_D$ has this form after a finite étale Galois base change $\rho : X_\rho \to X$ that splits G. By definition, the vector bundle $\mathfrak{c}_D$ is

$$\mathfrak{c}_D = \mathrm{Spec}\Big(\mathrm{Sym}_{O_X}(\mathfrak{c}_D^*)\Big),$$

where $\mathrm{Sym}_{O_X}(\mathfrak{c}_D^*)$ is the symmetric algebra sheaf of the dual O_X-module $\mathfrak{c}_D^*$.

Lemma 4.13.1. — *If* $\deg(D) > 2g-2$, *then* $\mathcal{A}$ *is an affine* k*-space of dimension*

$$\dim(\mathcal{A}) = \frac{\sharp\Phi \deg(D)}{2} + r(1 - g + \deg(D)),$$

where r *is the rank of* $\mathbb{G}$ *and* $\sharp\,\Phi$ *is the number of its roots.*

Proof. — Let $\rho : X_\rho \to X$ be the finite étale Galois cover that splits G. Then $\rho^*\mathfrak{c}_D$ is isomorphic to the direct sum $\bigoplus_{i=1}^r \rho^*D^{\otimes e_i}$. It follows that

$$\deg(\mathfrak{c}_D) = (e_1 + \cdots + e_r)\deg(D).$$

By the Riemann–Roch theorem, we have

$$\dim \mathrm{H}^0(X, \mathfrak{c}_D) - \dim \mathrm{H}^1(X, \mathfrak{c}_D) = (e_1 + \cdots + e_r)\deg(D) + r(1-g).$$

According to Kostant, the integers $e_i - 1$ are the exponents of the root system Φ, so that

$$e_1 + \cdots + e_r = r + \frac{\sharp\Phi}{2}.$$

Thus, it remains only to prove that $\mathrm{H}^1(X, \mathfrak{c}_D) = 0$. Since $\deg(D) > 2g - 2$ and ρ is finite étale, we have

$$\deg(\rho^* D) > \deg(\rho^*\Omega_{X/k}) = \deg(\Omega_{X_\rho/k}),$$

from which the vanishing of $\mathrm{H}^1(X_\rho, \rho^* D^{\otimes e_i})$ follows. To prove the vanishing of $\mathrm{H}^1(X, \mathfrak{c}_D)$, note that $\mathrm{H}^1(X, \mathfrak{c}_D)$ is a direct summand of $\mathrm{H}^1(X_\rho, \rho^*\mathfrak{c}_D)$. □

Thus, if $\deg(D)$ is a fixed integer greater than $2g - 2$, the dimension of the Hitchin base $\mathcal{A}$ does not depend on D nor on the quasi-split form.

Proposition 4.13.2. **—** *For every $a \in \mathcal{A}(\bar{k})$, there is a canonical isomorphism*

$$\mathrm{Lie}(J_a) = \mathfrak{c}_D^* \otimes D.$$

In the proof below, when $f : X \to Y$ and $\mathcal{L}$ is an $\mathcal{O}_Y$-module, we simply write $\mathcal{L}$ to also denote its pull-back $f^*\mathcal{L}$. This abuse of notation should not cause confusion since the scheme on which the module lives is always clear from context.

Proof. — According to 2.4.7, the homomorphism $J_a \to J_a^1$ induces an isomorphism $\mathrm{Lie}(J_a) \to \mathrm{Lie}(J_a^1)$ on Lie algebras. By the construction of J^1, $\mathrm{Lie}(J_a^1)$ can be computed using the cameral cover $\pi_a : \widetilde{X}_a \to \bar{X}$ by the formula

$$\mathrm{Lie}(J_a^1) = \Big((\pi_a)_*\mathfrak{t}\Big)^W.$$

Recall that the cameral cover is defined by forming the cartesian diagram

$$\begin{tikzcd} \widetilde{X}_a \arrow[r] \arrow[d, "\pi_a"'] & \mathfrak{t}_D \arrow[d, "\pi"] \\ \bar{X} \arrow[r, "a"'] & \mathfrak{c}_D. \end{tikzcd}$$

Since π is finite and flat, the formation of $\pi_*\mathfrak{t}$ commutes with any base change, in particular $(\pi_a)_*\mathfrak{t} = a^*\pi_*\mathfrak{t}$. Hence, it suffices to compute $(\pi_*\mathfrak{t})^W$.

Since $\mathfrak{c}_D$ is the invariant quotient of $\mathfrak{t}_D$ by the action of W, the same holds for the total spaces of the tangent bundles $T_{\mathfrak{t}_D/X}$ and $T_{\mathfrak{c}_D/X}$. Thus,

$$(\pi_*\Omega_{\mathfrak{t}_D/X})^W = \Omega_{\mathfrak{c}_D/X}.$$

Moreover, as $\mathfrak{t}_D$ is a vector bundle over X, we have $\Omega_{\mathfrak{t}_D/X} = \mathfrak{t}_{D^{-1}}$. Similarly, $\Omega_{\mathfrak{c}_D/X} = \mathfrak{c}_D^*$. We ultimately obtain an equality of $\mathcal{O}_{\mathfrak{c}_D}$-modules

$$(\pi_*\mathfrak{t}_{D^{-1}})^W = \mathfrak{c}_D^*,$$

so that

$$(\pi_*\mathfrak{t})^W = \mathfrak{c}_D^* \otimes D.$$

Pulling back this equality via the section $a : \bar{X} \to \mathfrak{c}_D$, we obtain $\mathrm{Lie}(J_a) = \mathfrak{c}_D^* \otimes_{\mathcal{O}_X} D$. □

In the case where G is split, the lemma allows one to express $\mathrm{Lie}(J_a)$ in terms of D and the exponents of the root system Φ. If $e_1, \ldots, e_r$ are the degrees of the homogeneous invariant polynomials (see 1.1.1), then

$$\mathrm{Lie}(J_a) = D^{-e_1+1} \oplus \cdots \oplus D^{-e_r+1}.$$

If G is not split, $\mathrm{Lie}(J_a)$ becomes isomorphic to the direct sum on a finite étale Galois cover of X that splits G. In particular,

$$\deg(\mathrm{Lie}(J_a)) = \sum_{i=1}^{r}(-e_i+1)\deg(D) = -\frac{\sharp\,\Phi\,\deg(D)}{2}.$$

Corollary 4.13.3. — *For every $a \in \mathcal{A}^{\heartsuit}(\bar{k})$,*

$$\dim(\mathcal{P}_a) = \frac{\sharp\,\Phi\,\deg(D)}{2} + r(g-1).$$

Proof. — We have

$$\dim(\mathcal{P}_a) = \dim \mathrm{H}^1(\bar{X}, \mathrm{Lie}(J_a)) - \dim \mathrm{H}^0(\bar{X}, \mathrm{Lie}(J_a)),$$

so that the stated equality follows from the Riemann–Roch formula. □

4.13.4. — We shall write $d = \sharp\,\Phi\deg(D)/2 + r(g-1)$ for the relative dimension of $\mathscr{P}$ over $\mathscr{A}$. Comparing with the formula 4.13.1

$$\dim(\mathcal{A}) = \frac{\sharp\,\Phi\,\deg(D)}{2} + r(1-g+\deg(D)),$$

we obtain

$$\dim(\mathcal{P}) = (r+\sharp\,\Phi)\deg(D).$$

In 4.16.1 we will show that $\mathcal{M}_a^{\mathrm{reg}}$ is dense in $\mathcal{M}_a$ so that one obtains the same dimension formulas for $\mathcal{M}$

$$\dim(\mathcal{M}_a) = \frac{\sharp\,\Phi\,\deg(D)}{2} + r(g-1)$$

and

$$\dim(\mathcal{M}) = (r+\sharp\,\Phi)\deg(D).$$

4.14. Deformation Calculation. — The deformations of Higgs bundles have been studied by Biswas and Ramanan in [**9**]. For the reader's convenience we shall take up their computation again.

Recall first the standard computation of the deformations of a torsor under a smooth group. Let S be a k-scheme and G a smooth group scheme over S. Let $\mathbf{B}G$ be the classifying stack of G. The universal G-torsor $\mathbf{E}G$ is then S over $[S/G]$. Denote

$$\pi_{\mathbf{E}G} : \mathbf{E}G \longrightarrow \mathbf{B}G$$

the tautological G-torsor. Consider the distinguished triangle of cotangent complexes

$$\pi_{\mathbf{E}G}^* L_{\mathbf{B}G/S} \longrightarrow L_{\mathbf{E}G/S} \longrightarrow L_{\mathbf{E}G/\mathbf{B}G} \longrightarrow \pi_{\mathbf{E}G}^* L_{\mathbf{B}G/S}[1].$$

Since $S = \mathbf{E}G$, the cotangent complex $L_{\mathbf{E}G/S}$ is zero, while $L_{\mathbf{E}G/\mathbf{B}G}$ is the vector bundle $\mathfrak{g}^*$ placed in degree 0. It follows that

$$L_{\mathbf{E}G/\mathbf{B}G} \cong \pi_{\mathbf{E}G}^* L_{\mathbf{B}G/S}[1].$$

Hence, we deduce an isomorphism

$$\mathfrak{g}^*[-1] \cong \pi_{\mathbf{E}G}^* L_{\mathbf{B}G/S},$$

which, by descent along $\pi_{\mathbf{E}G}$, induces an isomorphism

$$\left(\mathbf{E}G \wedge^G \mathfrak{g}^*\right)[-1] \cong L_{\mathbf{B}G/S}.$$

Thus, the cotangent complex $L_{\mathbf{B}G/k}$ of the classifying stack of G is the vector bundle obtained by twisting $\mathfrak{g}^*$ by the torsor $\mathbf{E}G$, placed in degree 1.

Therefore, for any S-scheme X and any G-torsor E on X corresponding to a map $h_E : X \to \mathbf{B}G$, the obstruction to deforming E lies in

$$\mathrm{H}^1\Big(X, \underline{\mathrm{RHom}}(h_E^* L_{\mathbf{B}G/S}, \mathcal{O}_X)\Big) = \mathrm{H}^2(X, E \wedge^G \mathfrak{g}).$$

If this obstruction vanishes, the deformations form a torsor under

$$\mathrm{H}^0\Big(X, \underline{\mathrm{RHom}}(h_E^* L_{\mathbf{B}G/S}, \mathcal{O}_X)\Big) = \mathrm{H}^1(X, E \wedge^G \mathfrak{g}),$$

and the group of infinitesimal automorphisms is $\mathrm{H}^0(X, E \wedge^G \mathfrak{g})$.

Now let V be a vector bundle on S with a G-action and consider the quotient stack $[V/G]$. The G-torsor $\pi_V : V \to [V/G]$ defines a morphism $[\nu] : [V/G] \to \mathbf{B}G$, which fits into a cartesian diagram

$$\begin{tikzcd} V \arrow[r, "\nu"] \arrow[d, "\pi_V"'] & \mathbf{E}G \arrow[d, "\pi_{\mathbf{E}G}"] \\ {[V/G]} \arrow[r, "{[\nu]}"'] & \mathbf{B}G. \end{tikzcd}$$

Consider the distinguished triangle of cotangent complexes

$$\pi_V^* L_{[V/G]/S} \longrightarrow L_{V/S} \longrightarrow L_{V/[V/G]} \longrightarrow \pi_V^* L_{[V/G]/k}[1].$$

Here, $L_{V/S}$ is a constant vector bundle of value $\nu^* V^*$ placed in degree 0 (with V^* the dual of V) and $L_{V/[V/G]} = \nu^* L_{\mathbf{E}G/\mathbf{B}G}$ is the vector bundle $\nu^* \mathfrak{g}^*$ placed in degree 0. At each point $v \in V$, the G-action yields a linear map

$$\alpha_v : \mathfrak{g} \longrightarrow T_v V = V,$$

whose dual is the map

$$\alpha_v^* : (L_{V/S})_v = V^* \longrightarrow (L_{V/[V/G]})_v = \mathfrak{g}^*.$$

This map descends to $[V/G]$ as

$$\alpha_v^* \wedge^G \pi_V : \pi_V \wedge^G V^* \longrightarrow \pi_V \wedge^G \mathfrak{g}^*,$$

whose cone is isomorphic to $L_{[V/G]/S}$.

Apply the above to calculate the deformations of Hitchin pairs. Recall the notations fixed at the beginning of Section 3. In particular, G is the reductive group scheme over the curve X and $\mathfrak{g}$ is its Lie algebra, equipped with the adjoint action of G and with the $\mathbb{G}_m$-action by homothety. The line bundle D defines a $\mathbb{G}_m$-torsor L_D. Consider the stack $[\mathfrak{g}_D/G]$ obtained by twisting $\mathfrak{g}$ by L_D and then taking the quotient by G. Then the cotangent complex $L_{[\mathfrak{g}_D/G]/X}$ is identified with

$$L_{[\mathfrak{g}/G]/X} \wedge^{\mathbb{G}_m} L_D,$$

which is the cone of

$$\Big(\pi_{D,\mathfrak{g}} \wedge^G \mathfrak{g}^*\Big) \otimes D^{-1} \longrightarrow \pi_{D,\mathfrak{g}} \wedge^G \mathfrak{g}^*,$$

where $\pi_{D,\mathfrak{g}}$ is the obvious G-torsor over the quotient $[\mathfrak{g}_D/G]$.

Let (E, ϕ) be a Higgs bundle on X with values in $\bar{k}$. It corresponds to a morphism

$$h_{E,\phi} : \bar{X} \to [\mathfrak{g}_D/G].$$

The deformation of (E, ϕ) is controlled by the complex

$$\underline{\mathrm{RHom}}\Big(h_{E,\phi}^*\Big(L_D \wedge^{\mathbb{G}_m} L_{[\mathfrak{g}/G]/X}\Big), \mathcal{O}_X\Big),$$

which in turn identifies with

$$\mathrm{ad}(E, \phi) := [\mathrm{ad}(E) \longrightarrow \mathrm{ad}(E) \otimes D],$$

where

– $\mathrm{ad}(E)$ is the vector bundle $E \wedge^G \mathfrak{g}$;
– $\mathrm{ad}(E)$ is placed in degree -1, and $\mathrm{ad}(E) \otimes D$ is placed in degree 0;
– the differential is given by $x \mapsto [x, \phi]$.

Recall Proposition 5.3 of [**57**]. (The reader will note a difference in the shift of $\mathrm{ad}(E, \phi)$ relative to that reference.) We now give an alternative proof.

Theorem 4.14.1. — *Let $(E, \phi) \in \mathcal{M}(\bar{k})$ be a point lying over a point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$. Then the group $\mathrm{H}^1(X, \mathrm{ad}(E, \phi))$, in which the obstruction to deforming the pair (E, ϕ) lies, vanishes in either of the following cases:*

– $\deg(D) > 2g - 2$,
– $\deg(D) = 2g - 2$ *and* $a \in \mathcal{A}^{\mathrm{ani}}(\bar{k})$.

If either hypothesis is satisfied, then $\mathcal{M}$ is smooth at (E, ϕ).

Proof. — Fix a nondegenerate, symmetric, invariant form on $\mathfrak{g}$. Then the dual of the complex $\mathrm{ad}(E, \phi)$ is identified with

$$\mathrm{ad}(E, \phi)^* = [\mathrm{ad}(E) \otimes D^{-1} \longrightarrow \mathrm{ad}(E)],$$

with the two nonzero terms placed in degrees -1 and 0, and with differential $x \mapsto [x, \phi]$. The (-1)-st cohomology of this complex is isomorphic to

$$\mathrm{Lie}(I^{\mathrm{lis}}_{(E,\phi)}) \otimes D^{-1},$$

where $I^{\mathrm{lis}}_{(E,\phi)}$ is the smooth group scheme over $\bar{X}$ defined in §4.11. By Serre duality, $\mathrm{H}^1(X, \mathrm{ad}(E, \phi))$ is dual to

$$\mathrm{H}^0\Big(\bar{X}, \mathrm{Lie}(I^{\mathrm{lis}}_{(E,\phi)}) \otimes D^{-1} \otimes \Omega_{X/k}\Big).$$

Since we have an injection $\mathrm{Lie}(I^{\mathrm{lis}}_{(E,\phi)}) \to \mathrm{Lie}(J^{\flat}_a)$, it suffices to prove that

$$\mathrm{H}^0\Big(\bar{X}, \mathrm{Lie}(J^{\flat}_a) \otimes D^{-1} \otimes \Omega_{X/k}\Big) = 0.$$

This reduces to proving the following lemma. □

Lemma 4.14.2. — *For every $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, one has*

$$\mathrm{H}^0\Big(\bar{X}, \mathrm{Lie}(J^{\flat}_a) \otimes L\Big) = 0$$

for every line bundle L of strictly negative degree. The same conclusion holds under the hypothesis $a \in \mathcal{A}^{\mathrm{ani}}(\bar{k})$ when $\deg(L) \leq 0$.

Proof. — Choose a splitting $\bar{\rho} : \bar{X}_{\bar{\rho}} \to \bar{X}$ of G (see 1.3.6). Then there is a finite flat cover $\widetilde{X}_{\bar{\rho},a} \to \bar{X}$ which is generically étale Galois with group $\mathbb{W} \rtimes \Theta_{\bar{\rho}}$. Let $\widetilde{X}^{\flat}_{\bar{\rho},a}$ be the normalization of $\widetilde{X}_{\bar{\rho},a}$, and denote by $\pi^{\flat}_{\bar{\rho},a}$ its projection to $\bar{X}$. By 4.8.1,

$$\mathrm{Lie}(J^{\flat}_a) = (\pi^{\flat}_{\bar{\rho},a})_*\Big(\mathcal{O}_{\widetilde{X}^{\flat}_{\bar{\rho},a}} \otimes \mathfrak{t}\Big)^{\mathbb{W} \rtimes \Theta_{\bar{\rho}}},$$

so that

$$\mathrm{Lie}(J^{\flat}_a) \otimes L = (\pi^{\flat}_a)_*\Big((\pi^{\flat}_a)^* L \otimes \mathfrak{t}\Big)^{\mathbb{W} \rtimes \Theta_{\bar{\rho}}}.$$

If $\deg(L) < 0$, then $(\pi^{\flat}_a)^* L$ is a line bundle of strictly negative degree on each connected component of $\widetilde{X}^{\flat}_{\bar{\rho},a}$ and cannot have any nonzero global sections. If $\deg(L) = 0$, then $(\pi^{\flat}_a)^* L$ has nonzero global sections if and only if it is isomorphic to $\mathcal{O}_{\widetilde{X}^{\flat}_{\bar{\rho},a}}$. In that case,

$$\mathrm{H}^0\Big(\widetilde{X}^{\flat}_{\bar{\rho},a},\ ((\pi^{\flat}_{\bar{\rho},a})^* L \otimes \mathfrak{t})^{\mathbb{W} \rtimes \Theta_{\bar{\rho}}}\Big) = \mathfrak{t}^{W_{\tilde{a}}},$$

where $W_{\tilde{a}}$ is the subgroup of $\mathbb{W} \rtimes \Theta_{\bar{\rho}}$ defined as in §4.10. Under the hypothesis $a \in \mathcal{A}^{\mathrm{ani}}(\bar{k})$, the group $\mathfrak{t}^{W_{\tilde{a}}}$ vanishes. □

4.15. Product Formula. — We now recall the link between the Hitchin fibers and the affine Springer fibers, which are their local analogues.

Let $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, and let U be the inverse image of the regular semisimple locus $\mathfrak{c}^{\mathrm{rs}}$ under the map $a : \bar{X} \to \mathfrak{c}_D$. The gluing with the Kostant section defines a morphism

$$\prod_{v \in \bar{X}-U} \mathcal{M}_v(a) \longrightarrow \mathcal{M}_a.$$

Similarly, there is a group homomorphism $\prod_{v\in\bar{X}-U} \mathcal{P}_v(J_a) \longrightarrow \mathcal{P}_a$. These induce a morphism

$$\zeta : \prod_{v\in\bar{X}-U} \mathcal{M}_v(a) \wedge^{\prod_{v\in\bar{X}-U}\mathcal{P}_v(J_a)} \mathcal{P}_a \longrightarrow \mathcal{M}_a.$$

According to Theorem 4.6 of [**57**], this morphism induces an equivalence on the category of $\bar{k}$-points. (Notice that, compared to [**57**], the notations have been changed: what was denoted by $\mathcal{M}^{\bullet}_{v,a}$ there is now $\mathcal{M}_v(a)$, and the notation $\mathcal{M}_{v,a}$ does not appear here.)

***Proposition 4.15.1.** — For every $a \in \mathcal{A}^{\mathrm{ani}}(\bar{k})$, the quotient of*

$$\prod_{v\in\bar{X}-U} \mathcal{M}_v^{\mathrm{red}}(a) \times \mathcal{P}_a$$

by the diagonal action of $\prod_{v\in\bar{X}-U} \mathcal{P}_v^{\mathrm{red}}(J_a)$ is a proper Deligne–Mumford stack. Moreover, the morphism

$$\prod_{v\in\bar{X}-U} \mathcal{M}_v^{\mathrm{red}}(a) \wedge^{\prod_{v\in\bar{X}-U}\mathcal{P}_v^{\mathrm{red}}(J_a)} \mathcal{P}_a \longrightarrow \mathcal{M}_a$$

is a homeomorphism.

Proof. — The homomorphism $\mathcal{P}_v(J_a) \to \mathcal{P}_a$ induces a map $\pi_0(\mathcal{P}_v(J_a)) \to \pi_0(\mathcal{P}_a)$ on connected components. Since $a \in \mathcal{A}^{\mathrm{ani}}(\bar{k})$, $\pi_0(\mathcal{P}_a)$ is finite, and the kernel of this map is a subgroup of finite index in $\pi_0(\mathcal{P}_v(J_a))$. Thus, there exists a free abelian subgroup Λ_v of finite index in $\pi_0(\mathcal{P}_v(J_a))$ contained in the kernel.

Choose a lift $\Lambda_v \to \mathcal{P}_v(J_a)$. Since a is defined over a finite field, after possibly replacing Λ_v by a subgroup of finite index, we may assume that Λ_v is contained in the kernel of $\mathcal{P}_v(J_a) \to \mathcal{P}_a$.

The group $\prod_{v\in\bar{X}-U} \Lambda_v$ acts on $\prod_{v\in\bar{X}-U} \mathcal{M}_v^{\mathrm{red}}(a) \times \mathcal{P}_a$ by acting freely on the first factor and trivially on the second. The quotient is

$$\prod_{v\in\bar{X}-U} \Big(\mathcal{M}_v^{\mathrm{red}}(a)/\Lambda_v\Big) \times \mathcal{P}_a,$$

where each $\mathcal{M}_v^{\mathrm{red}}(a)/\Lambda_v$ is a projective $\bar{k}$-scheme (by Kazhdan–Lusztig; see 3.4.1).

It remains to quotient by $\prod_{v\in\bar{X}-U}\Big(\mathcal{P}_v(J_a)/\Lambda_v\Big)$. For each v, the homomorphism $\mathcal{R}_v(a) \to \mathcal{P}_v(J_a)/\Lambda_v$ is injective with finite kernel. Recall that we have a short exact sequence

$$1 \to \mathcal{R}_a \to \mathcal{P}_a \to \mathcal{P}_a^{\flat} \to 1,$$

where $\mathcal{P}_a^{\flat}$ is a proper Deligne–Mumford stack. Thus, the quotient of

$$\prod_{v\in\bar{X}-U} \Big(\mathcal{M}_v^{\mathrm{red}}(a)/\Lambda_v\Big) \times \mathcal{P}_a$$

by the diagonal action of $\mathcal{R}_a = \prod_v \mathcal{R}_v(a)$ is a locally trivial fibration over $\mathcal{P}_a^{\flat}$ with fibers isomorphic to

$$\prod_{v\in\bar{X}-U} \Big(\mathcal{M}_v^{\mathrm{red}}(a)/\Lambda_v\Big).$$

This quotient is then a proper Deligne–Mumford stack.

Finally, we must quotient by the finite group $\prod_v \mathcal{P}_v(J_a)/(\mathcal{R}_v(a)\times\Lambda_v)$. The final quotient is also a proper Deligne–Mumford stack.

Since $\mathcal{M}_a$ is a Deligne–Mumford stack (in particular, separated), the morphism

$$\zeta : \prod_{v\in\bar{X}-U} \mathcal{M}_v^{\mathrm{red}}(a) \wedge^{\prod_{v\in\bar{X}-U}\mathcal{P}_v^{\mathrm{red}}(J_a)} \mathcal{P}_a \longrightarrow \mathcal{M}_a$$

is proper. Since it induces an equivalence on $\bar{k}$-points, it is a homeomorphism. In particular, $\mathcal{M}_a$ is a proper Deligne–Mumford stack. □

One expects that the statement above extends to all $a\in\mathcal{A}^\heartsuit(\bar{k})$.

Corollary 4.15.2. — *For every $a \in \mathscr{A}^{\mathrm{ani}}(\bar{k})$, $\mathscr{M}_a$ is homeomorphic to a projective scheme. Moreover, for every $m\in\mathscr{M}_a(\bar{k})$, the stabilizer of m in $\mathscr{P}_a$ is an affine group.*

4.16. Density. — We now state and prove the global analogue of 3.10.1.

Proposition 4.16.1. — *For every geometric point $a\in\mathcal{A}^\heartsuit(\bar{k})$, the fiber $\mathcal{M}_a^{\mathrm{reg}}$ is dense in the fiber $\mathcal{M}_a$.*

Proof. — The product formula (Proposition 4.15.1) implies that the closed complement of $\mathcal{M}_a^{\mathrm{reg}}$ in $\mathcal{M}_a$ has strictly smaller dimension than $\mathcal{M}_a^{\mathrm{reg}}$. Since the total Hitchin stack $\mathcal{M}$ is smooth over k (by 4.14.1), the Hitchin fibers $\mathcal{M}_a$ are locally complete intersections. In particular, they cannot have irreducible components of strictly smaller dimension than themselves. This shows that $\mathcal{M}_a^{\mathrm{reg}}$ is dense in $\mathcal{M}_a$. □

The proof above is essentially the same as that of Altman, Iarrobino, and Kleiman in [**1**] for the density of the Jacobian in the compactified Jacobian of a reduced, irreducible projective curve with planar singularities.

Corollary 4.16.2. — *The regular part $\mathcal{M}_v^{\mathrm{reg}}(a)$ of the affine Springer fiber $\mathcal{M}_v(a)$ is dense.*

One constructs a global situation from the local one as in *cf.* 8.6. Then the local assertion follows from the global one via the product formula. (We omit further details as this corollary will not be used later in the article.)

Corollary 4.16.3. — *For every $a\in\mathcal{A}^\heartsuit(\bar{k})$, $\mathcal{M}_a$ is equidimensional of dimension*

$$\frac{\sharp\Phi\ \deg(D)}{2}+r(g-1).$$

Moreover, the set of irreducible components of $\mathcal{M}_a$ is identified with the group $\pi_0(\mathcal{P}_a)$.

Proof. — The dimension formula follows from Corollary 4.13.3. The identification of the set of irreducible components of $\mathcal{M}_a$ with the group $\pi_0(\mathcal{P}_a)$ is enabled by the Kostant section. □

Corollary 4.16.4. — *If* $\deg(D) > 2g-2$, *the morphism $f^\heartsuit : \mathcal{M}^\heartsuit \to \mathcal{A}^\heartsuit$ is flat of relative dimension d. Its fibers are geometrically reduced.*

Proof. — By 4.14.1, both $\mathcal{M}^\heartsuit$ and $\mathcal{A}^\heartsuit$ are smooth over k. To prove that f is flat, it suffices to show that the dimensions of the fibers satisfy

$$\dim(\mathcal{M}_a)=\dim(\mathcal{M})-\dim(\mathcal{A}).$$

This follows from the equalities $\dim(\mathcal{M}_a)=\dim(\mathcal{P}_a)$ and $\dim(\mathcal{M})=\dim(\mathcal{P})$, and since for $\mathcal{P}$ smooth over $\mathcal{A}$ (see 4.3.5) we have

$$\dim(\mathcal{P}^\heartsuit)=\dim(\mathcal{A}^\heartsuit)+\dim(\mathcal{P}_a).$$

As in the proof of Proposition 4.16.1, the fiber $\mathcal{M}_a$, being locally a complete intersection and admitting a dense smooth open subset, must be reduced. □

4.17. The Case of Endoscopic Groups. — Let (κ, ρ_κ) be an endoscopic datum for G over X (see 1.8.1). Let H be the associated endoscopic group. As in §1.9, there is a morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$. Twisting by D, one obtains a morphism

$$\nu : \mathfrak{c}_{H,D} \to \mathfrak{c}_D.$$

Taking X-valued points, we obtain a morphism (still denoted by)

$$\nu : \mathcal{A}_H \to \mathcal{A}.$$

It is known (see [**57**, 7.2]) that the restriction of this morphism to $\mathcal{A}^\heartsuit$ is finite and unramified.

***4.17.1.* —** Set

$$r_H^G(D) = \frac{\big(|\Phi| - |\Phi_H|\big)\deg(D)}{2}.$$

By Lemma 4.13.1, we have

$$\dim(\mathcal{A}) - \dim(\mathcal{A}_H) = r_H^G(D),$$

so that the image of $\mathcal{A}_H^{G-\heartsuit} = \nu^{-1}(\mathcal{A}^\heartsuit)$ in $\mathcal{A}^\heartsuit$ is a closed subscheme of codimension $r_H^G(D)$.

***4.17.2.* —** Over $\mathcal{A}_H$, there is the Hitchin fibration for the group H

$$f_H : \mathcal{M}_H \to \mathcal{A}_H.$$

There is also the Picard stack $\mathcal{P}_H \to \mathcal{A}_H$ acting on $\mathcal{M}_H$. There is no direct relation between $\mathcal{M}$ and $\mathcal{M}_H$, but $\mathcal{P}$ and $\mathcal{P}_H$ are simply related. Let $a_H \in \mathcal{A}_H(\bar{k})$ with image $a \in \mathcal{A}^\heartsuit(\bar{k})$. The homomorphism $\mu : \nu^* J \to J_H$ from 2.5.1 induces a homomorphism $J_a \to J_{H,a_H}$ which is generically an isomorphism. Thus, one obtains a surjective homomorphism

$$\mathcal{P}_a \to \mathcal{P}_{H,a_H}$$

with kernel

$$\mathcal{R}_{H,a_H}^G = \mathrm{H}^0\Big(\bar{X}, \frac{J_{H,a_H}}{J_a}\Big),$$

which is an affine group of dimension

$$\dim(\mathcal{R}_{H,a_H}^G) = \dim(\mathcal{P}_a) - \dim(\mathcal{P}_{H,a_H}).$$

According to Corollary 4.13.3, we also have

$$\dim(\mathcal{R}_{H,a_H}^G) = \frac{\big(|\Phi| - |\Phi_H|\big)\deg(D)}{2} = r_H^G(D).$$

***4.17.3.* —** Let $J_{H,a_H}^\flat$ be the Néron model of J_{H,a_H}. We have homomorphisms

$$J_a \to J_{H,a_H} \to J_{H,a_H}^\flat,$$

which are isomorphisms over a nonempty open of $\bar{X}$. Hence, $J_{H,a_H}^\flat$ is also the Néron model of J_a. Combining with (4.9.2), we obtain the exact sequence

$$1 \to \mathcal{R}_{H,a_H}^G \to \mathcal{R}_a \to \mathcal{R}_{H,a_H} \to 1.$$

We deduce the dimension formula

$$\delta_a - \delta_{H,a_H} = r_H^G(D),$$

where $\delta_a = \dim(\mathcal{R}_a)$ and $\delta_{H,a_H} = \dim(\mathcal{R}_{H,a_H})$ are the δ-invariants for G and H, respectively.

4.18. The Case of Paired Groups. — Let G_1 and G_2 be two X-group schemes that are paired in the sense of 1.12.4. Then there is an isomorphism $\mathfrak{c}_{G_1} \cong \mathfrak{c}_{G_2}$ (see 1.12.6). This yields an isomorphism $\mathfrak{c}_{G_1,D} \cong \mathfrak{c}_{G_2,D}$ and thus an isomorphism between the bases of the Hitchin fibrations,

$$\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_2.$$

There is no direct relation between the fibrations

$$f_1 : \mathcal{M}_1 \to \mathcal{A}_1 \quad \text{and} \quad f_2 : \mathcal{M}_2 \to \mathcal{A}_2$$

of G_1 and G_2, but there is a relation between the associated Picard stacks $\mathcal{P}_1$ and $\mathcal{P}_2$.

***Proposition 4.18.1.* —** *There exists a homomorphism between the $\mathcal{A}$-Picard stacks*

$$\mathcal{P}_1 \to \mathcal{P}_2,$$

which induces an isogeny between their neutral components.

Proof. — As in the proof of 1.12.6, there is an isomorphism $\mathfrak{t}_1 \cong \mathfrak{t}_2$ over the isomorphism $\mathfrak{c}_{G_1} \cong \mathfrak{c}_{G_2}$. By 2.4.7, we deduce an isomorphism between the neutral components $J^0_{G_1} \cong J^0_{G_2}$ of the regular centralizers of G_1 and G_2. Since J_{G_1} and J_{G_2} are group schemes of finite type, by composing the isomorphism $J^0_{G_1} \cong J^0_{G_2}$ with multiplication by a sufficiently divisible integer N, one obtains a homomorphism $J^0_{G_1} \to J^0_{G_2}$ which extends to a homomorphism $J_{G_1} \to J^0_{G_2}$ with finite kernel and cokernel. This yields a homomorphism between the Picard stacks

$$\mathcal{P}_1 \to \mathcal{P}_2,$$

which induces an isogeny between their neutral components. □

5. Stratification

In the remainder of this paper we will need two stratifications of the Hitchin base $\mathcal{A}$, one relative to the group of connected components of the fiber $\mathcal{P}_a$ and the other relative to the invariant δ_a, which is, in some sense, the dimension of the affine part of $\mathcal{P}_a$. These stratifications exist by general semicontinuity arguments. The purpose of this chapter is to understand how these two invariants relate to the cameral curve and to render the construction of these stratifications more explicit.

The basic observation is as follows: since these two invariants are completely determined by the cameral cover $\widetilde{X}_a \to \bar{X}$, in the case where the normalizations of the cameral curve vary in families the invariants δ_a and $\pi_0(\mathcal{P}_a)$ are locally constant. We will therefore define the stratification by family normalization having, over each stratum, a cameral curve which normalizes in family after a radicial base change of the stratum. This stratification is then finer than the stratifications relative to δ_a and to $\pi_0(\mathcal{P}_a)$. We will begin by discussing the case of $\mathrm{GL}(r)$, where one is led to the well-known problem of family normalization of curves on a surface.

A key result of this chapter is Proposition 5.7.2 which gives a lower bound for the codimension of the stratum $\mathcal{A}_\delta$ under a hypothesis on $\deg(D)$. In characteristic 0, one has a proof of this inequality based on the symplectic nature of the Hitchin fibration without resorting to the hypothesis on $\deg(D)$. We will present that argument in another paper where more emphasis is placed on characteristic 0. In positive characteristic that argument does not work. Here is how we will circumvent the obstacle. Goresky, Kottwitz, and MacPherson have computed similar codimensions in the local setting (see [**28**]). The global calculation reduces to the local calculation provided that $\deg(D)$ is large.

We will introduce an étale open subset $\tilde{\mathcal{A}}$ of $\mathcal{A}$ over which $\pi_0(\mathcal{P})$ becomes a quotient of a constant sheaf. With this extra rigidity, the endoscopic decomposition that will be studied in the next chapter will have a pleasant form over $\tilde{\mathcal{A}}$.

5.1. Family Normalizations of Spectral Curves. — In this chapter we introduce a stratification of the Hitchin base $\mathcal{A}$ adapted to the Picard stack $\mathcal{P}$. The general case will be modeled on the case of the linear group, which we now discuss.

In the case $G = \mathrm{GL}(r)$, one may associate to every point

$$a \in \mathcal{A}^{\heartsuit}(\bar{k})$$

a reduced curve Y_a drawn on the total space of the line bundle D (see 4.4). The symmetry group $\mathcal{P}_a$ is then the Picard stack $\mathrm{Pic}(Y_a)$ of invertible $\mathcal{O}_{Y_a}$-modules. The structure of $\mathrm{Pic}(Y_a)$ can be analyzed using the normalization of Y_a. Let

$$\xi : Y_a^{\flat} \to Y_a$$

be the normalization of Y_a. The functor

$$\mathcal{L} \mapsto \xi^* \mathcal{L},$$

which assigns to every invertible $\mathcal{O}_{Y_a}$-module its pull-back via ξ, defines a homomorphism

$$\mathrm{Pic}(Y_a) \to \mathrm{Pic}(Y_a^{\flat}).$$

The long exact sequence in cohomology associated with the short exact sequence of sheaves on Y_a

$$1 \to \mathcal{O}_{Y_a}^{\times} \to \xi_* \mathcal{O}_{Y_a^{\flat}}^{\times} \to \xi_* \mathcal{O}_{Y_a^{\flat}}^{\times} / \mathcal{O}_{Y_a}^{\times} \to 1$$

provides the following information. The kernel of ξ^*, which consists of the category of invertible $\mathcal{O}_{Y_a}$-modules $\mathcal{L}$ equipped with a trivialization of $\xi^* \mathcal{L}$, is an algebraic group whose set of points is

$$\mathrm{H}^0(Y_a, \xi_* \mathcal{O}_{Y_a^{\flat}}^{\times} / \mathcal{O}_{Y_a}^{\times}).$$

Moreover, the functor ξ^* is essentially surjective.

One can attach two invariants to this situation:

– the set $\pi_0(Y_a^{\flat})$ of connected components of $Y_a^{\flat}$;

– the integer

$$\delta_a = \dim \mathrm{H}^0(Y_a, \xi_* \mathcal{O}_{Y_a^{\flat}} / \mathcal{O}_{Y_a}),$$

called the Serre δ-invariant.

By taking the degree on the components of $Y_a^{\flat}$, one obtains a homomorphism

$$\pi_0(\mathrm{Pic}(Y_a)) \to \mathbb{Z}^{\pi_0(Y_a^{\flat})}$$

which is an isomorphism. The invariant δ measures the dimension of the affine group $\ker(\xi^*)$.

Teissier introduced in [**77**] the moduli space of family normalizations of a family of plane curves. First, recall the definition of a family normalization.

***Definition 5.1.1.* —** *Let $y : Y \to S$ be a proper, flat morphism with reduced one-dimensional fibers. A family normalization of Y is a proper birational morphism $\xi : Y^{\flat} \to Y$ that is an isomorphism over an open subset U of Y, which is dense in every fiber of Y over S, and such that the composite $y \circ \xi$ is a proper and smooth morphism.*

In a family normalization the invariants δ and $\pi_0(\mathcal{P})$ remain locally constant.

***Proposition 5.1.2.* —** (1) *The direct image $y_*(\xi_* \mathcal{O}_{Y^{\flat}} / \mathcal{O}_Y)$ is a finitely generated locally free $\mathcal{O}_S$-module.*

(2) *There exists a sheaf $\pi_0(Y^{\flat})$, locally constant for the étale topology on S, whose fiber at each geometric point $s \in S$ is the set of connected components of $Y_s^{\flat}$.*

Proof. — Let s be a geometric point of S. The restriction of $\xi_* \mathcal{O}_{Y^{\flat}} / \mathcal{O}_Y$ is supported on $Y_s - U_s$, which is a zero-dimensional scheme. It follows that

$$\mathrm{H}^1(Y_s, \xi_* \mathcal{O}_{Y^{\flat}} / \mathcal{O}_Y) = 0$$

and that

$$\dim \mathrm{H}^0(Y_s, \xi_* O_{Y^\flat}/O_Y)$$

is the difference between the arithmetic genus of Y_s and that of $Y_s^\flat$. This is thus a function locally constant in s. It remains to apply the base change theorem for coherent sheaves [**56**, 5, Corollary 2].

The second assertion is a consequence of the fact that $\xi_* \mathbb{Q}_\ell$ is a locally constant sheaf. □

We now restrict ourselves to the case of spectral curves. Let $\mathcal{B}$ be the moduli space of family normalizations of the spectral curves Y_a. It associates to every k-scheme S the groupoid $\mathcal{B}(S)$ of triples $(a, Y_a^\flat, \xi)$ where $a \in \mathcal{A}^\heartsuit(S)$ is an S-point of $\mathcal{A}^\heartsuit$, where $Y_a^\flat$ is a smooth projective curve over S, and where $\xi : Y_a^\flat \to Y_a$ is a family normalization of the spectral curve Y_a associated to a. The functor $\mathcal{B}$ is representable by a k-scheme of finite type.

The forgetful morphism $\mathcal{B} \to \mathcal{A}^\heartsuit$ induces a bijection on $\bar{k}$-points. Indeed, for every $a \in \mathcal{A}^\heartsuit(\bar{k})$, the normalization $Y_a^\flat$ of Y_a is uniquely determined. However, $\mathcal{B}$ has more connected components than $\mathcal{A}$. In fact, the two invariants $\pi_0(\widetilde{Y}_a)$ and δ_a are locally constant by Proposition 5.1.2. Let Ψ be the set of connected components of $\mathcal{B} \otimes_k \bar{k}$. For every $\psi \in \Psi$, there exists an integer $\delta(\psi)$ equal to δ_a for every $a \in \mathcal{B}_\psi(\bar{k})$.

Teissier [**77**], Diaz and Harris [**22**], Fantechi, Göttsche, and Van Straten [**25**], and Laumon [**51**] have proved the local analogue of this result. The passage from local to global currently requires the hypothesis on $\deg(D)$. We will use this result only as a guide. From the following section on, the letters $\mathcal{B}$, Ψ, and ψ will take on slightly different meanings.

5.2. Family Normalization of Cameral Curves. — In the case of classical groups one still has spectral curves [**34**], [**59**]. However, with these spectral curves one is forced to work case by case. We will in fact study the general case by examining the moduli space of normalizations of the cameral curves endowed with the W-action. This will even lead in the case of $\mathrm{GL}(r)$ to a different stratification, finer than that defined by the moduli space of normalizations of the spectral curves.

Let S be a k-scheme. An S-point a of $\mathcal{A}^\heartsuit$ defines a morphism

$$a : X \times S \to \mathfrak{c}_D.$$

By taking the inverse image of the covering $\pi : \mathfrak{t}_D \to \mathfrak{c}_D$, one obtains a finite flat cover $\widetilde{X}_a$ of $X \times S$ which in each fiber is generically a W-torsor.

Consider the functor $\mathcal{B}$ which assigns to every k-scheme S the groupoid of triples $(a, \widetilde{X}_a^\flat, \xi)$ where:

– $a \in \mathcal{A}^\heartsuit(S)$ is an S-point of $\mathcal{A}^\heartsuit$.

– $\widetilde{X}_a^\flat$ is a proper smooth curve over S, endowed with a W-action and a morphism $\pi_a^\flat : \widetilde{X}_a^\flat \to X \times S$ that is finite, flat, and, fiberwise, generically a W-torsor.

– $\xi : \widetilde{X}_a^\flat \to \widetilde{X}_a$ is a W-equivariant birational morphism which is a family normalization (see Definition 5.1.1) over an open subset U of Y that is dense in every fiber.

Let $b = (a, \widetilde{X}_a^\flat, \xi) \in \mathcal{B}(S)$ be a point of $\mathcal{B}$ over a connected S. Let $\mathrm{pr}_S : X \times S \to S$ be the projection. By Definition 5.1.1, the direct image $(\mathrm{pr}_S)_*(\xi_* O_{\widetilde{X}_a^\flat}/O_{X_a^\flat})$ is a finitely generated locally free O_S-module. The same holds for

$$((\mathrm{pr}_S)_*(\xi_* O_{\widetilde{X}_a^\flat}/O_{X_a^\flat}) \otimes_{O_X} \mathfrak{t})^W$$

under the hypothesis that the order of $\mathbb{W}$ is prime to the characteristic of k. Since S is connected, this locally free O_S-module has a constant rank which we denote by $\delta(b)$.

For every $a \in \mathcal{A}^\heartsuit(\bar{k})$, the cameral curve $\widetilde{X}_a$ has a unique normalization $\widetilde{X}_a^\flat$, which is then a smooth curve over $\bar{k}$. It follows that the forgetful morphism $\mathcal{B} \to \mathcal{A}^\heartsuit$ induces a bijection on $\bar{k}$-points. Moreover, it induces an injection on points over any field by the uniqueness of normalization. In characteristic p and over a non-perfect field, the normalization may fail to be smooth so that the above map may not be surjective when evaluated on a non-perfect field.

Proposition 5.2.1. — *The functor $\mathcal{B}$ defined above is representable by a k-scheme of finite type.*

Proof. — Consider the functor $\mathcal{B}'$ which assigns to every scheme S the set of pairs $(\widetilde{X}_a^\flat, \gamma)$ where:

– $\widetilde{X}_a^\flat$ is a proper smooth curve over S, endowed with a W-action and a morphism $\pi_a^\flat : \widetilde{X}_a^\flat \to X \times S$ as above.

– $\gamma : \widetilde{X}_a^\flat \to \mathfrak{t}_D \times S$ is a W-equivariant morphism such that the inverse image of the regular semisimple locus of $\mathfrak{t}_D$ is dense in each fiber.

Let $\mathcal{H}$ be the functor that assigns to every scheme S the set of isomorphism classes of smooth projective curves $\widetilde{X}_a^\flat$ over S, endowed with a W-action and a morphism $\pi_a^\flat : \widetilde{X}_a^\flat \to X \times S$ as above. This functor is representable by a quasi-projective k-scheme. The morphism

$$h : \mathcal{B}' \to \mathcal{H}$$

is also representable so that $\mathcal{B}'$ is representable by a quasi-projective k-scheme.

Moreover, there is a morphism $\mathcal{B} \to \mathcal{B}'$. It sends a point $b = (a, \widetilde{X}_a^\flat, \xi) \in \mathcal{B}(S)$ to the point $b' = (\widetilde{X}_a^\flat, \gamma)$, where γ is the composition of $\xi : \widetilde{X}_a^\flat \to \widetilde{X}_a$ with the closed immersion $\widetilde{X}_a \to \mathfrak{t}_D \times S$. To prove the representability of $\mathcal{B}$, it suffices to verify the following assertion. □

Lemma 5.2.2. — *The morphism $\mathcal{B} \to \mathcal{B}'$ is an isomorphism.*

Proof. — To show that the morphism $\mathcal{B} \to \mathcal{B}'$ is an isomorphism, we will construct an inverse. Let $b' = (\widetilde{X}_a^\flat, \gamma) \in \mathcal{B}'(S)$. Consider the W-invariant part of the push-forward $(\pi_a^\flat)_* \mathcal{O}_{\widetilde{X}_a^\flat}$. This is a finite $\mathcal{O}_{X\times S}$-algebra whose fiber in each S-fiber is generically isomorphic to $\mathcal{O}_X$. Since X is normal, it follows that

$$((\pi_a^\flat)_* \mathcal{O}_{\widetilde{X}_a^\flat})^W = \mathcal{O}_{X\times S}.$$

Using the equality $k[\mathbf{t}]^{\mathbb{W}} = \mathbf{c}$ (see 1.1.1), the W-equivariant morphism $\gamma : \widetilde{X}_a^\flat \to \mathfrak{t}_D$ induces a morphism

$$a : X \times S \to \mathfrak{c}_D.$$

Let $\widetilde{X}_a$ be the cameral curve associated to a. The morphism γ then factors through a morphism

$$\xi : \widetilde{X}_a^\flat \to \widetilde{X}_a,$$

which fiberwise over S is a normalization of X_a. Thus we have constructed a point

$$b = (a, \widetilde{X}_a^\flat, \xi) \in \mathcal{B}(S).$$

The morphism $\mathcal{B}' \to \mathcal{B}$ constructed in this way is inverse to the morphism $\mathcal{B} \to \mathcal{B}'$. □

Let Ψ be the set of connected components of $\mathcal{B} \otimes_k \bar{k}$. For every $\psi \in \Psi$, let $\mathcal{B}_\psi$ be the reduced connected component of $\mathcal{B} \otimes_k \bar{k}$ indexed by ψ. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the finite set Ψ. For every $\psi \in \Psi$, let $\mathcal{A}_\psi$ be the image of $\mathcal{B}_\psi$ in $\mathcal{A}^\heartsuit \otimes_k \bar{k}$. A priori, $\mathcal{A}_\psi$ is a constructible subset by Chevalley's theorem; in fact, it is a locally closed subset.

5.3. Stratification of the Étale Open Subset $\tilde{\mathcal{A}}$. — In the remainder of the article we shall not work directly on the Hitchin base $\mathcal{A}$, but on an étale open subset of it. This choice detracts from our claim to geometrize the stabilization of the trace formula 1.13, but it rids us of a number of incidental difficulties.

5.3.1. — Let $\infty \in X(\bar{k})$. Consider the étale open subset $\tilde{\mathcal{A}}$ of $\mathcal{A} \otimes_k \bar{k}$ whose points are the pairs $(a, \tilde{\infty})$ with $a \in \mathcal{A}$ such that the cameral cover $\widetilde{X}_a \to X$ is étale above ∞, and where $\tilde{\infty}$ is a point of $\widetilde{X}_a$ above ∞. Note that if $\infty \in X(k)$, then $\tilde{\mathcal{A}}$ has an evident k-structure.

The construction above can be reformulated diagrammatically as follows. The choice of the point $\infty \in X(\bar{k})$ defines a morphism $\mathcal{A} \otimes_k \bar{k} \to \mathfrak{c}_{D,\infty}$, where $\mathfrak{c}_{D,\infty}$ is the fiber of $\mathfrak{c}_D$ above

∞, which assigns to $a \in \mathcal{A}$ the point $a(\infty)$. Let $\mathcal{A}^\infty$ denote the inverse image of the regular semisimple open subset $\mathfrak{c}^{\mathrm{rs}}_{D,\infty}$ of $\mathfrak{c}_{D,\infty}$. We then in fact have a cartesian diagram

$$\begin{tikzcd} \tilde{\mathcal{A}} \arrow[r] \arrow[d] & \mathfrak{t}_{D,\infty} \arrow[d] \\ \mathcal{A}^\infty \arrow[r] & \mathfrak{c}_{D,\infty} \end{tikzcd}$$

which makes $\tilde{\mathcal{A}}$ a $\mathbb{W}_\infty$-torsor over $\mathcal{A}^\infty$, where $\mathbb{W}_\infty$ is the fiber above ∞ of the finite étale X-group scheme $\mathbb{W}$.

Lemma 5.3.2. — *Suppose that* $\deg(D) > 2g$. *Then* $\tilde{\mathcal{A}}$ *is smooth and geometrically irreducible.*

Proof. — Under the hypothesis $\deg(D) > 2g$, the linear map $\mathcal{A} \to \mathfrak{c}_{D,\infty}$ in the diagram above is surjective *cf.* 4.7.2. We deduce that the upper arrow of the diagram is a smooth morphism with connected fibers. Since $\mathfrak{t}_{D,\infty}$ is a vector space, $\tilde{\mathcal{A}}$ is smooth and geometrically irreducible. □

We now form the cartesian diagram

$$\tilde{\mathcal{B}} = \mathcal{B} \times_{\mathcal{A}} \tilde{\mathcal{A}},$$

where $\mathcal{B}$ is the moduli space of family normalizations of cameral curves *cf.* 5.2.1. This morphism defines a bijection at the level of geometric points. The image of a connected component of $\tilde{\mathcal{B}} \otimes_k \bar{k}$ in $\tilde{\mathcal{A}} \otimes_k \bar{k}$ is a constructible subset. A constructible subset is by definition a finite union of irreducible locally closed subsets. Let $\tilde{\mathcal{A}}'$ be one of these irreducible locally closed subsets and $\tilde{\mathcal{B}}'$ its inverse image. The morphism $\tilde{\mathcal{B}}' \to \tilde{\mathcal{A}}'$ induces a bijection at the level of geometric points; in particular it is quasi-finite. Applying Zariski's main theorem, one sees that there exists a dense open subset $\tilde{\mathcal{A}}''$ of $\tilde{\mathcal{A}}'$ above which the morphism $\tilde{\mathcal{B}}' \to \tilde{\mathcal{A}}'$ is finite and radicial. Subdividing further and proceeding by noetherian induction, one obtains a stratification into irreducible locally closed subsets

$$\tilde{\mathcal{A}} \otimes_k \bar{k} = \bigsqcup_{\psi \in \Psi} \tilde{\mathcal{A}}_\psi \tag{5.3.3}$$

such that, if $\tilde{\mathcal{B}}_\psi$ denotes the inverse image of $\tilde{\mathcal{A}}_\psi$ in $\tilde{\mathcal{B}}$, the morphism $\tilde{\mathcal{B}}_\psi \to \tilde{\mathcal{A}}_\psi$ is a finite radicial morphism.

Subdividing yet further if necessary, we may assume that the closure of a stratum $\tilde{\mathcal{A}}_\psi$ is a union of other strata. This makes it possible to define a partial order relation on the set Ψ of strata. Since $\tilde{\mathcal{A}}$ is geometrically irreducible, the set Ψ admits a maximal element, which we shall denote by ψ_G.

5.4. Monodromic Invariants.

5.4. Monodromic Invariants. — We shall now construct a monodromic invariant attached to each stratum of the stratification 5.3.3.

Recall that the form G of $\mathbb{G}$ is given by a homomorphism $\rho^\bullet_G : \pi_1(X,\infty) \to \mathrm{Out}(\mathbb{G})$. Let Θ denote the image of the geometric fundamental group $\pi_1(\bar{X},\infty)$ in $\mathrm{Out}(\mathbb{G})$, which is a subgroup of finite order.

5.4.1. — Let $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$. Let U be the maximal open subset of $\bar{X}$ above which the cameral cover $\widetilde{X}_a \to \bar{X}$ is étale; it contains ∞ in particular. By 1.3.6, we have a commutative diagram

$$\begin{tikzcd} \pi_1(U,\infty) \arrow[r, "\pi^\bullet_a"] \arrow[d] & \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) \arrow[d] \\ \pi_1(\bar{X},x) \arrow[r, "\rho^\bullet_G"'] & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

Let $W_{\tilde{a}}$ denote the image of $\pi_a^\bullet$ in $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$, and let $I_{\tilde{a}}$ be the image of the kernel of the homomorphism $\pi_1(U, \infty) \to \pi_1(\bar{X}, \infty)$. By construction, $W_{\tilde{a}}$ is contained in the finite subgroup $\mathbb{W} \rtimes \Theta$, and $I_{\tilde{a}}$ is a normal subgroup of $W_{\tilde{a}}$ contained in $W_{\tilde{a}} \cap \mathbb{W}$.

5.4.2. — Let us restate the definition above in the language of covers. Let $X_\rho \to \bar{X}$ be the connected finite étale Galois cover with Galois group Θ corresponding to the surjective homomorphism

$$\rho_G^\bullet : \pi_1(\bar{X}, \infty) \to \Theta.$$

By construction it carries a point ∞_ρ above ∞. Form the cartesian product

$$\widetilde{X}_{\rho,a} = \widetilde{X}_a \times_X X_\rho$$

of the cameral curve $\widetilde{X}_a$ with the finite étale cover $X_\rho \to X$. The curve $\widetilde{X}_{\rho,a}$ is then equipped with an action of $\mathbb{W} \rtimes \Theta$, and so is its normalization $\widetilde{X}^\flat_{\rho,a}$. Let $C_{\tilde{a}}$ be the connected component of $\widetilde{X}^\flat_{\rho,a}$ containing $\tilde{\infty}_\rho = (\tilde{\infty}, \infty_\rho)$. The group $W_{\tilde{a}}$ is then the subgroup of $\mathbb{W} \rtimes \Theta$ consisting of the elements that leave this component stable. The group $I_{\tilde{a}}$ is then the subgroup generated by the elements of $W_{\tilde{a}}$ admitting at least one fixed point in $C_{\tilde{a}}$. Since the projection of $C_{\tilde{a}}$ onto X_ρ, on which the group Θ_ρ acts freely, $I_{\tilde{a}}$ is contained in the kernel of the projection $W_{\tilde{a}} \to \Theta$. It follows also that $I_{\tilde{a}} \subset W_{\tilde{a}} \cap \mathbb{W}$.

Proposition 5.4.3. — *The map $\tilde{a} \mapsto (I_{\tilde{a}}, W_{\tilde{a}})$ is constant on each stratum $\tilde{\mathcal{A}}_\psi$ of the stratification 5.3.3.*

Proof. — By construction of the stratification 5.3.3, the morphism $\tilde{\mathcal{B}}_\psi \to \tilde{\mathcal{A}}_\psi$ is a finite radicial morphism. Above the connected scheme $\tilde{\mathcal{B}}_\psi$, the normalization $\widetilde{X}^\flat_a \to \widetilde{X}_a$ of the cameral curve goes over into families, as does the whole construction preceding the statement of this proposition. We therefore have a proper smooth relative curve

$$\widetilde{X}_{\rho,\psi} \to \tilde{\mathcal{B}}_\psi$$

equipped with an action of $\mathbb{W} \rtimes \Theta$ and with a section $\tilde{\infty}_\rho$. As in Lemma 5.1.2, there exists a locally constant sheaf $\pi_0(\widetilde{X}_{\rho,\psi}/\tilde{\mathcal{B}}_\psi)$ for the étale topology of $\tilde{\mathcal{B}}_\psi$ which interpolates the sets of irreducible components of the fibers of $\widetilde{X}_{\rho,\psi}/\tilde{\mathcal{B}}_\psi$. The group $\mathbb{W} \rtimes \Theta$ acts on this sheaf and induces a transitive action on its fibers. The datum of the section deduced from $\tilde{\infty}_\rho$ implies that this sheaf is constant. The proposition follows. □

5.4.4. — It follows from the preceding proposition that there is a map

$$\psi \mapsto (I_\psi, W_\psi)$$

defined on the set Ψ of strata of the stratification 5.3.3, compatible with the map $\tilde{a} \mapsto (I_{\tilde{a}}, W_{\tilde{a}})$ defined at the level of geometric points.

Lemma 5.4.5. — *Consider the set of pairs (I_1, W_1) consisting of a subgroup W_1 of $\mathbb{W} \rtimes \Theta$ and a normal subgroup I_1 of W_1, and the partial order on it defined by $(I_1, W_1) \le (I_2, W_2)$ if and only if $W_1 \subset W_2$ and $I_1 \subset I_2$. Then the map $\psi \mapsto (I_\psi, W_\psi)$ is an increasing map.*

Proof. — Let $S = \mathrm{Spec}(R)$ be a formal trait with generic point $\eta = \mathrm{Spec}(k(\eta))$ and geometric closed point $s = \mathrm{Spec}(k(s))$. Let $\tilde{a} : S \to \tilde{\mathcal{A}}$ be a morphism with $\tilde{a}(\eta) \in \tilde{\mathcal{A}}_\psi$ and $\tilde{a}(s) \in \tilde{\mathcal{A}}_{\psi'}$. We must show that

$$(I_{\psi'}, W_{\psi'}) \le (I_\psi, W_\psi).$$

Consider the cover $\widetilde{X}_{\rho,a}$ of $X \times S$ defined as the inverse image under a of the cover $X_\rho \times \mathfrak{t}_D \to \mathfrak{c}_D$. Consider the normalization $\widetilde{X}^\flat_{\rho,a}$ of $\widetilde{X}_{\rho,a}$. After possibly making a radicial base change of the trait, we may assume that the generic fiber $(\widetilde{X}^\flat_{\rho,a})_\eta$ is a smooth curve over $k(\eta)$, so that (I_ψ, W_ψ) can be computed from this generic fiber. By contrast, the special fiber $(\widetilde{X}^\flat_{\rho,a})_s$ is not normal in general, and one must take its normalization $(\widetilde{X}^\flat_{\rho,a})^\flat_s$ in order to compute $(I_{\psi'}, W_{\psi'})$.

The point $\tilde{a}$ defines a section of $(\widetilde{X}^{\flat}_{\rho,a})_{\eta}$. Let $C_{\tilde{a}}$ be the connected component of $(\widetilde{X}^{\flat}_{\rho,a})_{\eta}$ containing this section. The group W_{ψ} is then the subgroup of $\mathbb{W} \rtimes \Theta$ formed by the elements that leave this component stable. Let $C_{\tilde{a}(s)}$ be the connected component of $(\widetilde{X}^{\flat}_{\rho,a})^{\flat}_{s}$ containing the point defined by $\tilde{a}(s)$. The group $W_{\psi'}$ is the subgroup of $\mathbb{W} \rtimes \Theta$ formed by the elements that leave $C_{\tilde{a}(s)}$ stable. Since $C_{\tilde{a}(s)}$ is a connected component of the normalization of the special fiber of $C_{\tilde{a}}$, we have the inclusion $W_{\psi'} \subset W_{\psi}$. An element of $W_{\psi'}$ having a fixed point in $C_{\tilde{a}(s)}$ necessarily has a fixed point in $C_{\tilde{a}}$, whence the second inclusion $I_{\psi'} \subset I_{\psi}$. □

5.4.6. **—** For every pair (I_-, W_-) consisting of a subgroup W_- of $\mathbb{W} \rtimes \Theta$ and a normal subgroup I_- of W_- contained in $\mathbb{W}$, it follows from this lemma that the union of the strata $\tilde{\mathcal{A}}_{\psi}$ with $W_{\psi} \subset W_-$ and $I_{\psi} \subset I_-$ is a closed subscheme of $\tilde{\mathcal{A}}$. It follows also that the union of the strata $\tilde{\mathcal{A}}_{\psi}$ with $W_{\psi} = W_-$ and $I_{\psi} = I_-$ is an open subset of the closed subscheme just mentioned. We shall write $\tilde{\mathcal{A}}_{(I_-,W_-)}$ for this locally closed subscheme of $\tilde{\mathcal{A}}$. We then have the stratification

$$\tilde{\mathcal{A}} = \bigsqcup_{(I_-,W_-)} \tilde{\mathcal{A}}_{(I_-,W_-)}.$$

Recall that if $\infty \in X(k)$, then $\tilde{\mathcal{A}}$ is defined over k, but the stratification above is not necessarily defined over k.

5.4.7. **—** It follows also from Lemma 5.4.5 that the union of the strata $\tilde{\mathcal{A}}_{\psi}$ with $\psi \in \Psi$ such that $\mathfrak{t}^{W_{\psi}} = 0$ is an open subset of $\tilde{\mathcal{A}}$. By 4.10.3, a point $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$ lies in this open subset if and only if a belongs to the set $\mathcal{A}^{\mathrm{ani}}(\bar{k})$ defined in 4.10.5. The stratification of $\tilde{\mathcal{A}}$ induces on $\tilde{\mathcal{A}}^{\mathrm{ani}}$ a stratification

$$\tilde{\mathcal{A}}^{\mathrm{ani}} = \bigsqcup_{\psi \in \Psi^{\mathrm{ani}}} \tilde{\mathcal{A}}_{\psi},$$

where Ψ^{ani} is a subset of Ψ. If $\infty \in X(k)$, then $\tilde{\mathcal{A}}^{\mathrm{ani}}$ is defined over k, but the stratification above is not necessarily defined over k.

Lemma 5.4.8. **—** *Suppose that* $\deg(D) > 2g$. *Let* ψ_G *be the maximal element of* Ψ. *Then*

$$(I_{\psi_G}, W_{\psi_G}) = (\mathbb{W}, \mathbb{W} \rtimes \Theta).$$

Proof. — By construction, $W_{\psi} \subset \mathbb{W} \rtimes \Theta$ and $I_{\psi} \subset \mathbb{W}$. It therefore suffices to show that this bound is attained. We shall show that it is attained at a point $\tilde{a} = (a, \tilde{\infty})$ with a in the open subset $\mathcal{A}^{\diamondsuit}$, which we have seen is non-empty under the hypothesis $\deg(D) > 2g$ *cf.* 4.7.1.

By 4.7.5 we already know that $W_{\tilde{a}} = \mathbb{W} \rtimes \Theta$ if $a \in \mathcal{A}^{\diamondsuit}$. The subgroup $I_{\tilde{a}}$ is then a normal subgroup of $\mathbb{W}$. Since the curve $\widetilde{X}_{\rho,a}$ meets transversally all the walls of h_a in $X_{\rho} \times \mathfrak{t}$ associated with the roots α of $\mathfrak{g}$, the group $I_{\tilde{a}}$ is a normal subgroup of $\mathbb{W}$ containing all the reflections s_{α} associated with the walls h_{α}. It follows that $I_{\tilde{a}} = \mathbb{W}$. □

5.5. Description of $\pi_0(\mathcal{P})$. — In this paragraph we shall describe the sheaf $\pi_0(\mathcal{P})$ along the strata $\tilde{\mathcal{A}}_{(I_-,W_-)}$ of the stratification 5.4.6. Let us recall the definition of this sheaf. Since the Picard stack $\mathcal{P} \to \mathcal{A}^{\heartsuit}$ is smooth, there exists a unique sheaf $\pi_0(\mathcal{P})$ for the étale topology of $\mathcal{A}^{\heartsuit}$ whose fiber at a point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$ is the group $\pi_0(\mathcal{P}_a)$ of connected components of $\mathcal{P}_a$. This is a consequence of a theorem of Grothendieck *cf.* [**30**, 15.6.4]; see also [**57**, 6.2].

We shall also consider the intermediate problem of determining the sheaf $\pi_0(\mathcal{P}')$ of connected components of the Picard stack $\mathcal{P}'$ whose fiber at each point $a \in \mathcal{A}^{\heartsuit}(\bar{k})$ classifies the J^0_a-torsors on $\bar{X}$. The surjective homomorphism $\mathcal{P}' \to \mathcal{P}$ induces a surjective homomorphism $\pi_0(\mathcal{P}') \to \pi_0(\mathcal{P})$. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ denote the restrictions of $\mathcal{P}$ and $\mathcal{P}'$ to $\tilde{\mathcal{A}}$.

By 4.10.3, for every point $\tilde{a} \in \tilde{\mathcal{A}}(\bar{k})$ the fiber of $\pi_0(\mathcal{P}'_{\tilde{a}})$ admits the description

$$\pi_0(\mathcal{P}'_a) = (\hat{\mathbb{T}}^{W_{\tilde{a}}})^*,$$

where the exponent $(\_)^*$ denotes the duality between finitely generated abelian groups and diagonalizable groups of finite type over $\overline{\mathbb{Q}}_\ell$. Likewise, $\pi_0(\mathcal{P}_a)$ is identified with the quotient of $\pi_0(\mathcal{P}'_a)$ dual to the subgroup $\hat{\mathbb{T}}(I_{\tilde{a}}, W_{\tilde{a}})$ defined in 4.10.3. The following statement makes it possible to determine the sheaves $\pi_0(\tilde{\mathcal{P}}')$ and $\pi_0(\tilde{\mathcal{P}})$ completely.

Proposition 5.5.1. — *The surjective arrows of Proposition 4.10.3*

$$\mathbb{X}_* \to \pi_0(\mathcal{P}'_a) = (\mathbb{X}_*)_{W_{\tilde{a}}},$$

defined for every $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$, interpolate into a canonical surjective homomorphism from the constant sheaf $\mathbb{X}_*$ onto $\pi_0(\tilde{\mathcal{P}}')$.

Proof. — Let $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$. The geometric point $\tilde{\infty}_\rho = (\tilde{\infty}, \infty_\rho)$ of $\widetilde{X}_{\rho,a}$ makes it possible to identify the fiber of J_a at ∞ with the fixed torus $\mathbb{T}$ *cf.* 2.4.7. By [**57**, 6.8], this identification defines a homomorphism from the constant sheaf $\mathbb{X}_*$ into $\tilde{\mathcal{P}}'$, and hence a homomorphism

$$\mathbb{X}_* \times \tilde{\mathcal{A}} \to \pi_0(\tilde{\mathcal{P}}').$$

Fiber by fiber this is the surjective homomorphism of 4.10.3. □

5.5.2. — By means of this lemma one obtains an explicit description of the restrictions of $\pi_0(\mathcal{P}')$ and of $\pi_0(\mathcal{P})$ to $\tilde{\mathcal{A}}$. For every étale open subset U of $\tilde{\mathcal{A}}$, the stratification 5.4.6 induces on U a stratification

$$U = \bigsqcup_{(I_-, W_-)} U_{(I_-, W_-)}.$$

For every pair (I_1, W_1) consisting of a subgroup W_1 of $\mathbb{W} \rtimes \Theta$ and a normal subgroup I_1 of W_1, we say that U is a *small open subset of type* (I_1, W_1) if $U_{(I_1, W_1)}$ is the unique non-empty closed stratum in the stratification above. It is clear that the small open subsets of the various types form a base for the étale topology of $\tilde{\mathcal{A}}$, in the sense that every open subset can be covered by a family of small open subsets. In order to define a sheaf for the étale topology of $\tilde{\mathcal{A}}$, it therefore suffices to specify its sections on the small open subsets together with the transition arrows.

Consider the sheaves Π' and Π, which are quotients of the constant sheaf $\mathbb{X}_*$, defined as follows. For a small open subset U_1 of type (I_1, W_1) we set

$$\Gamma(U, \Pi') = (\hat{\mathbb{T}}^{W_1})^* = (\mathbb{X}_*)_{W_1},$$

$$\Gamma(U, \Pi) = \hat{\mathbb{T}}(I_1, W_1)^*.$$

Let U_2 be a small étale open subset of U_1 of type (I_2, W_2). Since $U_{(I_1, W_1)}$ is the unique non-empty closed stratum of U_1, we have the inequality

$$(I_1, W_1) \le (I_2, W_2).$$

We then have an evident inclusion of the subgroups of invariants of $\hat{\mathbb{T}}$ under W_1 and W_2

$$\hat{\mathbb{T}}^{W_2} \subset \hat{\mathbb{T}}^{W_1},$$

whence the transition arrow

$$\Gamma(U_1, \Pi') \to \Gamma(U_2, \Pi').$$

The definition of the transition arrows of the sheaf Π follows from the lemma below.

Lemma 5.5.3. — *If $(I_1, W_1) \le (I_2, W_2)$, then one has the inclusion*

$$\hat{\mathbb{T}}(I_2, W_2) \subset \hat{\mathbb{T}}(I_1, W_1).$$

Proof. — Let κ be an element of $\hat{\mathbb{T}}$, let $\hat{\mathbb{G}}_\kappa$ be its centralizer in $\hat{\mathbb{G}}$ and let $\hat{\mathbb{H}}$ be the neutral component of the latter. Let $(\mathbb{W} \rtimes \Theta)_\kappa$ be the centralizer of κ in $\mathbb{W} \rtimes \Theta$ and let $\mathbb{W}_{\mathbb{H}}$ be the Weyl group of $\hat{\mathbb{H}}$. If $\kappa \in \hat{\mathbb{T}}(I_2, W_2)$, then $I_2 \subset \mathbb{W}_{\mathbb{H}}$ and $W_2 \subset (\mathbb{W} \rtimes \Theta)_\kappa$. We deduce that $I_1 \subset \mathbb{W}_{\mathbb{H}}$ and $W_1 \subset (\mathbb{W} \rtimes \Theta)_\kappa$. □

The following statement is an immediate consequence of 5.5.1 and 4.10.3.

Corollary 5.5.4. — *The surjective homomorphism 5.5.1 induces, on passing to the quotient, isomorphisms $\pi_0(\mathcal{P}')|_{\tilde{\mathcal{A}}} = \Pi'$ and $\pi_0(\mathcal{P})|_{\tilde{\mathcal{A}}} = \Pi$.*

Above the open subset $\tilde{\mathcal{A}}^{\mathrm{ani}}$ of $\tilde{\mathcal{A}}$ cf. 5.4.7, these are sheaves of finite abelian groups.

5.6. Stratification with Constant δ**.** — For every $a \in \mathcal{A}^{\heartsuit}(\bar{k})$ we defined in 4.9.4 an integer $\delta(a)$. This induces a function $\tilde{a} \mapsto \delta(a)$ for $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$.

Lemma 5.6.1. — *The function $\tilde{a} \mapsto \delta(a)$ is constant on each stratum $\tilde{\mathscr{A}}_{\psi}$, for every $\psi \in \Psi$.*

Proof. — After the finite radicial base change $\tilde{\mathscr{B}}_{\psi} \to \tilde{\mathscr{A}}_{\psi}$, the restriction of the cameral curve to $\tilde{\mathscr{B}}_{\psi}$ admits a family normalization. Write $\pi_{\psi} : \widetilde{X}_{\psi} \to \tilde{\mathscr{B}}_{\psi}$ for the restriction of the cameral curve to $\tilde{\mathscr{B}}_{\psi}$, and $\xi : \widetilde{X}^{\flat}_{\psi} \to \widetilde{X}_{\psi}$ for the family normalization. By 5.1.2, the sheaf

$$\pi_{\psi,*}(\xi_* \mathcal{O}_{\widetilde{X}^{\flat}_{\psi}} / \mathcal{O}_{\widetilde{X}_{\psi}})$$

is a locally constant $\mathcal{O}_{\tilde{\mathscr{B}}_{\psi}}$-module of finite type. This module is equipped with an action of $\mathbb{W}$. Since the order of $\mathbb{W}$ is prime to the characteristic,

$$(\mathcal{O}_{\tilde{\mathscr{B}}_{\psi}} \otimes \mathfrak{t})^{\mathbb{W}}$$

is likewise locally constant. We conclude by the formula 4.9.4. □

Lemma 5.6.2. — *Let $\psi, \psi' \in \Psi$ with $\psi \geq \psi'$. Then $\delta(\psi) \leq \delta(\psi')$.*

Proof. — Let $S = \mathrm{Spec}(R)$ be a formal trait with generic point $\eta = \mathrm{Spec}(k(\eta))$ and special point $s = \mathrm{Spec}(k(s))$ (with $k(s)$ algebraically closed). Let $a : S \to \mathcal{A}^{\heartsuit}$ be an S-point of $\mathcal{A}^{\heartsuit}$ whose generic point lies in $\tilde{\mathcal{A}}_{\psi}$ and whose special point lies in $\tilde{\mathcal{A}}_{\psi'}$. Let

$$\widetilde{X}_a \to S \times X$$

be the associated cameral cover.

After possibly replacing η by an inseparable extension and S by its normalization in that extension, we may assume that the morphism $a : \eta \to \tilde{\mathcal{A}}_{\psi}$ factors through $\mathcal{B}_{\psi}$. Then the normalization $\widetilde{X}^{\flat}_{a,\eta}$ of the generic fiber $\widetilde{X}_{a,\eta}$ is a smooth curve.

Consider the normalization $\widetilde{X}^{\flat}_{a}$ of $\widetilde{X}_a$ in $\widetilde{X}^{\flat}_{a,\eta}$ and denote by ξ the morphism

$$\xi : \widetilde{X}^{\flat}_{a} \to \widetilde{X}_a.$$

The R-module

$$\mathrm{H}^0(\widetilde{X}_a, \xi_* O_{\widetilde{X}^{\flat}_a} / O_{\widetilde{X}_a})$$

is finite and flat over R (since R is a discrete valuation ring, flatness is equivalent to being torsion-free). Denote it by M, which is a finite flat R-module endowed with an action of W. By decomposing M into irreducible representations of $\mathbb{W}$ (since the order of $\mathbb{W}$ is prime to the characteristic), we have

$$\dim_{k(\eta)}\Big((\mathfrak{t} \otimes_{O_X} M_{\eta})^W\Big) = \dim_{k(s)}\Big((\mathfrak{t} \otimes_k M_s)^W\Big).$$

The special fiber of $\widetilde{X}^{\flat}_{a,s}$ of $\widetilde{X}^{\flat}_{a}$ is a partial normalization of $\widetilde{X}_a$. To compute the invariant $\delta(\psi')$, one must take the full normalization $(\widetilde{X}^{\flat}_{a,s})^{\flat}$ of $\widetilde{X}^{\flat}_{a,s}$. The $k(s)[\mathbb{W}]$-module M_s is then a direct summand of

$$\mathrm{H}^0\Big(O_{(\widetilde{X}^{\flat}_{a,s})^{\flat}} / O_{\widetilde{X}_{a,s}}\Big).$$

We deduce that

$$\delta(\psi') \geq \dim_{k(s)}\Big((\mathfrak{t} \otimes_k M_s)^W\Big),$$

which implies the lemma. □

Lemma 5.6.3. — *Let $P \to S$ be a smooth commutative group scheme of finite type. The function $s \mapsto \tau_s$, which assigns to a geometric point s the dimension of the abelian part of P_s, is lower semi-continuous. Conversely, the function $s \mapsto \delta_s$, which assigns to a geometric point s the dimension of the affine part of P_s, is upper semi-continuous.*

Proof. — We may assume that P has only connected fibers, and we may also assume that S is strictly henselian. Choose a prime number ℓ prime to the residue characteristic of S. The kernel $P[\ell]$ of multiplication by ℓ is representable by an étale quasi-finite subgroup scheme. For every geometric point s of S, the length of $P_s[\ell]$ is given by the formula

$$\lg(P_s[\ell]) = \mu_s \ell + \tau_s \ell^2,$$

where μ_s is the dimension of the multiplicative part of P_s and τ_s is the dimension of its abelian part. If s_0 denotes the geometric closed point of S and s_1 the geometric generic point, one has the inequality

$$\lg(P_{s_0}[\ell]) \leq \lg(P_{s_1}[\ell]),$$

since, $P[\ell]$ being étale over the henselian base S, every point in the special fiber extends to an S-section. The inequality

$$\mu_{s_0}\ell + \tau_{s_0}\ell^2 \leq \mu_{s_1}\ell + \tau_{s_1}\ell^2$$

holding for every prime number ℓ prime to the residue characteristic of S, we obtain

$$\tau_{s_0} \leq \tau_{s_1}.$$

The other inequality follows, since $\delta_s + \tau_s = \dim(P_s)$ does not depend on the point s. □

Lemma 5.6.4. — *Suppose that* $\deg(D) > 2g$. *Then* $\delta(\psi_G) = 0$, *where* ψ_G *is the maximal element of* Ψ.

Proof. — This is immediate from 4.9.4. □

For every integer $\delta \in \mathbb{N}$, the union

$$\overline{\tilde{\mathcal{A}}}_\delta := \bigsqcup_{\delta(\psi)\geq\delta} \tilde{\mathcal{A}}_\psi$$

is a closed subset of $\tilde{\mathcal{A}}$. We then define

$$\tilde{\mathcal{A}}_\delta := \overline{\tilde{\mathcal{A}}}_\delta - \overline{\tilde{\mathcal{A}}}_{\delta+1},$$

which is then a locally closed subscheme of $\tilde{\mathcal{A}}$. Thus, we obtain a stratification

$$\tilde{\mathcal{A}} = \bigsqcup_{\delta\in\mathbb{N}} \tilde{\mathcal{A}}_\delta \tag{5.6.5}$$

called *the stratification with constant* δ.

Consider the big stratum $\tilde{\mathcal{A}}_{\psi_G}$ for the maximal element ψ_G of Ψ. This stratum consists of points $a \in \mathcal{A}(\bar{k})$ such that the cameral curve $\widetilde{X}_a$ is smooth. In this case the invariant δ equals 0. In particular, the stratum $\tilde{\mathcal{A}}_0$ in the δ-constant stratification above is a nonempty open subset of $\mathcal{A}$. In general, it strictly contains the open set $\mathcal{A}_{\psi_G}$, as shown by the following example.

In the case $G = \mathrm{GL}_r$, the description of $\mathcal{P}_a$ in terms of the spectral curve implies that $\delta_a = 0$ if and only if the spectral curve Y_a is smooth. It may happen that the spectral curve is smooth with a branch point $y \in Y_a$ of order greater than 2. In that case the cameral curve $\widetilde{X}_a$ is not smooth. Thus, in the GL_r case the stratification by family normalization of cameral curves strictly refines the stratification by family normalization of spectral curves.

5.6.6. — One deduces from this a stratification of the open subset $\tilde{\mathscr{A}}^{\mathrm{ani}}$

$$\tilde{\mathscr{A}}^{\mathrm{ani}} = \bigsqcup_{\delta\in\mathbb{N}} \tilde{\mathscr{A}}^{\mathrm{ani}}_\delta.$$

Note that for every $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathscr{A}}_\delta^{\mathrm{ani}}(\bar{k})$, by 4.9.5, the dimension of the affine part of $\mathscr{P}_a^0$ equals δ.

5.7. Stratification by Radicial Valuations. — Let $\bar{v}$ be a geometric point of $\bar{X}$ and denote by $\bar{O}_{\bar{v}}$ the completion of $\bar{X}$ at $\bar{v}$ and by $\bar{F}_{\bar{v}}$ its field of fractions. By choosing a uniformizer $\varepsilon_{\bar{v}}$, one may identify $\bar{O}_{\bar{v}}$ with the ring $\bar{k}[[\varepsilon_{\bar{v}}]]$. We choose a trivialization of the restriction of ρ_G to $\bar{O}_{\bar{v}}$ that splits G and, in particular, provides an isomorphism $W = \mathbb{W}$.

We now briefly review the analysis of the stratification of $\mathfrak{c}^\heartsuit(O_{\bar{v}})$ by radicial valuations as in [**28**], due to Goresky, Kottwitz, and MacPherson. Their strata of radicial valuations are finer than our strata defined by family normalization and, a fortiori, finer than the strata with constant δ.

Let $a \in \mathfrak{c}^\heartsuit(\bar{O}_{\bar{v}})$ and let $J_a = a^*J$ be the smooth group scheme over $\bar{O}_{\bar{v}}$ associated to a. The generic fiber of J_a is a torus whose monodromy can be described using the cameral cover (see 2.4.7). Let $\bar{F}_{\bar{v}}^{\mathrm{sep}}$ be the separable closure of $\bar{F}_{\bar{v}}$. Let $x \in \mathfrak{t}(\bar{F}_{\bar{v}}^{\mathrm{sep}})$ be a $\bar{F}_{\bar{v}}^{\mathrm{sep}}$-point of $\mathfrak{t}$ mapping to $a \in \mathfrak{c}(\bar{F}_{\bar{v}})$. The choice of this point defines a homomorphism

$$\pi_a^\bullet : I_{\bar{v}} \longrightarrow \mathbb{W},$$

where $I_{\bar{v}} = \mathrm{Gal}(\bar{F}_{\bar{v}}^{\mathrm{sep}}/\bar{F}_{\bar{v}})$. Since the characteristic of k does not divide the order of $\mathbb{W}$, $\pi_a^\bullet$ factors through the tame quotient $I_{\bar{v}}^{\mathrm{tame}}$ of $I_{\bar{v}}$. To follow the notations of [**28**], choose a topological generator of $I_{\bar{v}}^{\mathrm{tame}}$ and denote by w_a the image of this generator under $\pi_a^\bullet$.

For every root $\alpha \in \Phi$, define an integer

$$r(\alpha) := \mathrm{val}_{\bar{v}}(\alpha(x)),$$

where $\mathrm{val}_{\bar{v}}$ is the unique extension of the valuation with $\mathrm{val}_{\bar{v}}(\varepsilon_{\bar{v}}) = 1$ on $\bar{F}_{\bar{v}}$ to $\bar{F}_{\bar{v}}^{\mathrm{sep}}$. This yields a function

$$r : \Phi \to \mathbb{Q}_+.$$

The pair (w_a, r) depends on the choice of x, but its $\mathbb{W}$-orbit does not. We denote by $[w_a, r]$ the $\mathbb{W}$-orbit of the pair (w_a, r).

We have the obvious equality

$$\sum_{\alpha\in\Phi} r(\alpha) = \deg_{\bar{v}}(a^* \operatorname{discrim}_G) = d_{\bar{v}}(a).$$

The invariant

$$c_{\bar{v}}(a) = \dim(\mathfrak{t}) - \dim(\mathfrak{t}^{w_a})$$

is the drop in the toric rank of the Néron model of J_a. According to Bezrukavnikov's formula, we have

$$\delta_{\bar{v}}(a) = \frac{d_{\bar{v}}(a) - c_{\bar{v}}(a)}{2}.$$

Let $\mathfrak{c}^\heartsuit(O_{\bar{v}})_{[w,r]}$ be the subset of $a \in \mathfrak{c}^\heartsuit(O_{\bar{v}})$ with the invariant $[w, r]$ as above. According to [**28**], this set is admissible, i.e., there exists an integer N and a locally closed subscheme Z of $\mathfrak{c}(O_{\bar{v}}/\varepsilon_{\bar{v}}^N O_{\bar{v}})$ viewed as a $\bar{k}$-scheme such that $\mathfrak{c}^\heartsuit(O_{\bar{v}})_{[w,r]}$ is the inverse image of $Z(\bar{k})$ under the natural projection

$$\mathfrak{c}(O_{\bar{v}}) \to \mathfrak{c}(O_{\bar{v}}/\varepsilon_{\bar{v}}^N O_{\bar{v}}).$$

We say that $\mathfrak{c}^\heartsuit(O_{\bar{v}})_{[w,r]}$ is admissible of level N.

They then define the *codimension of the stratum* $\mathfrak{c}^\heartsuit(O_{\bar{v}})_{[w,r]}$ as the codimension of Z in $\mathfrak{c}(O_{\bar{v}}/\varepsilon_{\bar{v}}^N O_{\bar{v}})$ (viewed as a $\bar{k}$-scheme). This codimension clearly does not depend on the level N chosen, provided that N is sufficiently large. We denote this codimension by $\mathrm{codim}[w, r]$. According to [**28**, 8.2.2] one has the explicit formula

$$\mathrm{codim}[w, r] = d(w, r) + \frac{d_{\bar{v}}(a) + c_{\bar{v}}(a)}{2},$$

where the integer $d(w,r)$ is the codimension of $\mathfrak{t}_w(O)_r$ in $\mathfrak{t}_w(O)$ (in the notations of loc. cit.). We will be content with a coarser estimate.

Proposition 5.7.1. — *If $\delta_a > 0$, then*

$$\operatorname{codim}[w,r] \geq \delta_a + 1.$$

Proof. — It is clear that

$$\operatorname{codim}[w,r] = \delta_{\bar{v}}(a) + c_{\bar{v}}(a) + d(w,r),$$

where $\delta_{\bar{v}}(a) = \frac{d_{\bar{v}}(a)-c_{\bar{v}}(a)}{2}$. If w is not the trivial element of W, then $c_{\bar{v}}(a) \geq 1$. If $w = 1$, by the definition in [**28**, 8.2.2], $d(w,r)$ is the codimension of $\mathfrak{t}(\bar{O}_{\bar{v}})_r$ in $\mathfrak{t}(\bar{O}_{\bar{v}})$, where $\mathfrak{t}(\bar{O}_{\bar{v}})_r$ is the admissible part of $\mathfrak{t}(\bar{O}_{\bar{v}})$ consisting of elements having radicial valuation r. If $\delta_{\bar{v}}(a) > 0$, then $r \neq 0$ so that this part is of strictly positive codimension. Hence $d(w,r) \geq 1$. In either case, we obtain the desired inequality. □

Proposition 5.7.2. — *If the degree of D is very large compared to δ, then the δ-constant stratum $\mathcal{A}_\delta$ has codimension at least δ.*

Proof. — For any partition $\delta_\bullet$ of δ as a sum of natural numbers $\delta = \delta_1 + \cdots + \delta_n$, consider the subscheme $Z_{\delta_\bullet}$ of $\mathcal{A}^\heartsuit \times X^n$ consisting of tuples $(a; x_1, \ldots, x_n)$ with $a \in \mathcal{A}^\heartsuit(\bar{k})$ and $x_1, \ldots, x_n \in X(\bar{k})$ such that the local invariant $\delta_{x_i}(a)$ equals δ_i. One may stratify $Z_{\delta_\bullet}$ as the union of the strata $Z_{[w_\bullet, r_\bullet]}$ of those $(a; x_1, \ldots, x_n)$ for which the image in $\mathfrak{c}^\heartsuit(\bar{O}_{x_i})$ lies in the radicial valuation stratum $\mathfrak{c}^\heartsuit(\bar{O}_{x_i})_{[w_i, r_i]}$. Suppose that each such stratum is admissible of level N_i. For $\deg(D)$ very large, the linear map

$$\mathcal{A} \longrightarrow \prod_{i=1}^{n} \mathfrak{c}(\bar{O}_{x_i}/\varepsilon^{N_i}\bar{O}_{x_i})$$

is surjective (see 4.7.2). It follows that $Z_{[w_\bullet, r_\bullet]}$ has codimension at least

$$\sum_{i=1}^{n}(\delta_i + 1)$$

in $\mathcal{A} \times X^n$. Hence its image in $\mathcal{A}$ has codimension at least $\delta = \sum_{i=1}^{n} \delta_i$. This proves the proposition. □

We believe that this inequality holds without the hypothesis that $\deg(D)$ is very large relative to δ. A calculation of the infinitesimal action of $\mathcal{P}_a$ on $\mathcal{M}_a$ shows that it is true in characteristic zero. We will present that calculation on another occasion.

6. Cohomology over the Anisotropic Open Set

In this chapter we shall state the geometric stabilization theorems 6.4.1 and 6.4.2. They give an endoscopic description of an isotypic component of the perverse cohomology sheaves of the direct image

$${}^p\mathrm{H}^n(\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)$$

with respect to the action of $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$. Although the statement 6.4.2 implies the fundamental lemma of Langlands and Shelstad 1.11.1, the proof of 1.11.1 forms one step of the proof of 6.4.2.

6.1. The Anisotropic Open Set. — By varying the point ∞, it follows from 5.4.7 and 5.5.4 that there exists an open subset $\mathscr{A}^{\rm ani}$ of $\mathscr{A}^{\heartsuit}$ whose points are the $a \in \mathscr{A}^{\rm ani}$ such that the group of connected components $\pi_0(\mathscr{P}_a)$ is finite. It follows from 4.11.2 that for every $a \in \mathscr{A}^{\rm ani}(\bar{k})$ and $(E,\phi) \in \mathscr{M}_a(\bar{k})$, the group $\mathrm{Aut}(E,\phi)$ is a finite reduced group, under the hypothesis that the order of $\mathbb{W} \rtimes \Theta$ is prime to the characteristic.

***6.1.1.* —** In [**24**, II.4], Faltings proved the semi-stable reduction theorem for Higgs bundles, which states that a semi-stable Higgs bundle on X with coefficients in the fraction field of a discrete valuation ring can be extended to a Higgs bundle over the ring after a finite separable extension, and furthermore, if the Higgs bundle in the special fiber is stable, this extension is unique. We refer to [**24**] for the definition of semi-stable and stable Higgs bundles. Let us just say that it is exactly the same definition as for G-torsors, except that one considers only the parabolic reductions of E compatible with the Higgs field ϕ. This suffices to prove the following lemma.

***Lemma 6.1.2.* —** *Let $a \in \mathscr{A}^{\rm ani}(\bar{k})$ and $(E,\varphi) \in \mathscr{M}^{\rm ani}(\bar{k})$. Then (E,ϕ) is stable.*

Proof. — Since $a \in \mathscr{A}^{\rm ani}(\bar{k})$, E has no parabolic reduction compatible with ϕ. □

We may now apply Faltings' results in [**24**] to obtain the following statement.

***Proposition 6.1.3.* —** *The restriction $\mathscr{P}^{\rm ani}$ of the Picard stack $\mathscr{P}$ to $\mathscr{A}^{\rm ani}$ is a separated smooth Deligne-Mumford stack of finite type over $\mathscr{A}^{\rm ani}$. The open subset $\mathscr{M}^{\rm ani} := \mathscr{M} \times_{\mathscr{A}} \mathscr{A}^{\rm ani}$ of $\mathscr{M}$ is a separated smooth Deligne-Mumford stack of finite type over k.*

Moreover, there exists a coarse moduli space $M^{\rm ani}$ such that the morphism $f^{\rm ani} : \mathscr{M}^{\rm ani} \to \mathscr{A}^{\rm ani}$ factors as a homeomorphism $\mathscr{M}^{\rm ani} \to M^{\rm ani}$ followed by a projective morphism $M^{\rm ani} \to \mathscr{A}^{\rm ani}$.

Proof. — We already know that $\mathscr{P}$ is a smooth Picard stack over $\mathscr{A}^{\heartsuit}$ *cf.* 4.3.5 and that $\mathscr{M}$ is smooth over k *cf.* 4.14.1.

By [**24**, II.4] and 6.1.2, $\mathscr{M}^{\rm ani}$ is separated; in other words the diagonal morphism of $\mathscr{M}^{\rm ani}$ is universally closed. We know that it is quasi-finite with reduced fibers. We deduce that it is finite and unramified. Consequently $\mathscr{M}^{\rm ani}$ is a separated Deligne-Mumford stack. The same holds for $\mathscr{P}^{\rm ani}$, since $\mathscr{P}^{\rm ani}$ is identified with an open subset of $\mathscr{M}^{\rm ani}$.

It remains to show that $\mathscr{P}$ is of finite type and that $\mathscr{M}$ is proper over $\mathscr{A}^{\rm ani}$. Having the valuative criterion of properness at our disposal *cf.* [**24**, II.4], it suffices to show that it is of finite type over $\mathscr{A}^{\rm ani}$. We know that $\mathscr{M}$ is locally of finite type. For every $a \in \mathscr{A}^{\rm ani}(\bar{k})$ one may choose a family of open subsets of finite type of $\mathscr{M}^{\rm ani}$ covering the fiber $\mathscr{M}_a$. By the product formula 4.15.1, the fiber $\mathscr{M}_a$ is noetherian, so that there exists a finite family of open subsets of finite type of $\mathscr{M}^{\rm ani}$ covering $\mathscr{M}_a$. Since $f : \mathscr{M}^{\rm ani} \to \mathscr{A}^{\rm ani}$ is flat, the images of these open subsets are open subsets of $\mathscr{A}^{\rm ani}$ containing a. Writing V_a for their intersection, the open subset $f^{-1}(V_a)$ of $\mathscr{M}^{\rm ani}$ can be covered by a finite family of open subsets of finite type. It remains to note that $\mathscr{A}^{\rm ani}$ is also noetherian, so that there is a finite number of points a such that the open subsets V_a as above cover $\mathscr{A}^{\rm ani}$. The proposition follows.

The assertion on the coarse moduli space now follows from [**24**, II.5] and [**64**]. □

6.2. The κ-decomposition on $\tilde{\mathscr{A}}^{\rm ani}$. — Recall the notations $\tilde{\mathscr{M}}^{\rm ani} = \mathscr{M} \times_{\mathscr{A}} \tilde{\mathscr{A}}^{\rm ani}$, $\tilde{\mathscr{P}}^{\rm ani} = \mathscr{P} \times_{\mathscr{A}} \tilde{\mathscr{A}}^{\rm ani}$ and $\tilde{f}^{\rm ani} : \tilde{\mathscr{M}}^{\rm ani} \to \tilde{\mathscr{A}}^{\rm ani}$. By 6.1.3, this morphism is proper and its source is a smooth Deligne-Mumford stack. By Deligne's purity theorem [**18**], $\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell$ is a pure complex. By [**7**, 5.4.5], over $\tilde{\mathscr{A}}^{\rm ani}$ it is isomorphic to the direct sum of its perverse cohomology sheaves

$$\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_n {}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)[-n],$$

where ${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)$ is a pure perverse sheaf of weight n.

***6.2.1.* —** Since $\tilde{\mathscr{P}}^{\rm ani}$ acts on $\tilde{\mathscr{M}}^{\rm ani}$ over $\tilde{\mathscr{A}}^{\rm ani}$, it acts on the direct image $\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell$. By the homotopy lemma *cf.* [**54**, 3.2.3], the action of $\tilde{\mathscr{P}}^{\rm ani}$ on the perverse cohomology sheaves ${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)$

factors through the sheaf of finite abelian groups $\pi_0(\tilde{\mathscr{P}}^{\rm ani})$. By 5.5.1, we have a surjective homomorphism

$$\mathbb{X}_* \times \tilde{\mathscr{A}}^{\rm ani} \to \pi_0(\tilde{\mathscr{P}}^{\rm ani})$$

which identifies $\pi_0(\tilde{\mathscr{P}}^{\rm ani})$ with an explicit finite quotient of the constant sheaf $\mathbb{X}_*$. In particular, for every $\kappa \in \hat{\mathbb{T}}$ one may define a direct factor ${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)_\kappa$ on which $\mathbb{X}_*$ acts through the character $\kappa : \mathbb{X}_* \to \overline{\mathbb{Q}}_\ell^\times$, so that there is a direct sum decomposition

$${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell) = \bigoplus_{\kappa\in\hat{\mathbb{T}}} {}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)_\kappa$$

having only finitely many non-zero factors.

6.2.2. — Recall that we have a stratification $\tilde{\mathscr{A}}^{\rm ani} = \bigsqcup_{\psi\in\Psi^{\rm ani}} \tilde{\mathscr{A}}_\psi$ *cf.* 5.4.7. For every $\kappa \in \hat{\mathbb{T}}$, the union of the strata $\tilde{\mathscr{A}}_\psi$ with $\psi \in \Psi$ such that $\kappa \in \hat{\mathbb{T}}(I_\psi, W_\psi)$ is a closed subset of $\tilde{\mathscr{A}}$, by 5.5.3 and 5.4.5. We write $\tilde{\mathscr{A}}^{\rm ani}_\kappa = \tilde{\mathscr{A}}_\kappa \cap \tilde{\mathscr{A}}^{\rm ani}$ for the trace of this closed subset on $\tilde{\mathscr{A}}^{\rm ani}$.

Proposition 6.2.3. — *The support of the perverse sheaf ${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)_\kappa$ is contained in $\tilde{\mathscr{A}}^{\rm ani}_\kappa$.*

Proof. — It follows from the explicit description 5.5.4 of $\pi_0(\tilde{\mathscr{P}}^{\rm ani})$ that the restriction of ${}^p\mathrm{H}^n(\tilde{f}^{\rm ani}_*\overline{\mathbb{Q}}_\ell)_\kappa$ to the open subset $\tilde{\mathscr{A}}^{\rm ani} - \tilde{\mathscr{A}}^{\rm ani}_\kappa$ is zero. □

6.3. The Closed Immersion of $\tilde{\mathscr{A}}_H$ into $\tilde{\mathscr{A}}$. — Let $(\kappa, \rho^\bullet_\kappa)$ be a pointed endoscopic datum of G over $\bar{X}$, that is, a continuous homomorphism $\pi_1(\bar{X}, \infty) \to \pi_0(\kappa)$ as in 1.8.2. We then have a $\pi_0(\kappa)$-torsor $\rho_\kappa : \bar{X}_{\rho_\kappa} \to \bar{X}$ with a point ∞_{ρ_κ} above ∞. Recall that one may identify $\mathfrak{c}_D$ with the quotient in the sense of invariants of $\bar{X}_{\rho_\kappa} \times_{\bar{X}} \mathfrak{t}_D$ by the action of $\mathbb{W} \rtimes \pi_0(\kappa)$. We have also constructed a homomorphism $\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \pi_0(\kappa)$ *cf.* 1.9.1 such that the quotient in the sense of invariants of $\bar{X}_{\rho_\kappa} \times_{\bar{X}} \mathfrak{t}_D$ by $\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa)$ is $\mathfrak{c}_{H,D}$. We have thus deduced a morphism $\mathfrak{c}_{H,D} \to \mathfrak{c}_D$ and hence a morphism $\nu : \mathscr{A}_H \to \mathscr{A}$ *cf.* 4.17.

6.3.1. — For every $a \in \mathscr{A}^\infty(\bar{k})$ one forms, as in 4.5.4, the cartesian product

$$\widetilde{X}_{\rho_\kappa,a} = \widetilde{X}_a \times_{\bar{X}} \bar{X}_{\rho_\kappa}.$$

This is a curve traced on $\bar{X}_{\rho_\kappa} \times_{\bar{X}} \mathfrak{t}_D$ whose projection to $\bar{X}$ is generically étale Galois with Galois group $\mathbb{W} \rtimes \pi_0(\kappa)$. The choice of a point $\tilde{\infty} \in \widetilde{X}_a$ above ∞ is equivalent to the choice of a point $\tilde{\infty}_{\rho_\kappa}$ above ∞_{ρ_κ}, so that it is legitimate to write $\tilde{a} = (a, \tilde{\infty}_{\rho_\kappa}) \in \tilde{\mathscr{A}}$ with $a \in \mathscr{A}^\infty$ and $\tilde{\infty}_{\rho_\kappa}$ as above.

By construction of the morphism $\nu : \mathscr{A}_H \to \mathscr{A}$, if $a_H \in \mathscr{A}_H$ and $\nu(a_H) = a$, one has an embedding $\widetilde{X}_{\rho_\kappa,a_H} \to \widetilde{X}_{\rho_\kappa,a}$ which realizes the former as a union of certain irreducible components of the latter. One defines a morphism

$$\tilde{\nu} : \tilde{\mathscr{A}}_H \to \tilde{\mathscr{A}}$$

by the recipe $(a_H, \tilde{\infty}_{\rho_\kappa}) \mapsto (\nu(a_H), \tilde{\infty}_{\rho_\kappa})$. Note that if the point ∞_{ρ_κ} is defined over k, this morphism is also defined over k.

Proposition 6.3.2. — *The morphism $\tilde{\nu} : \tilde{\mathscr{A}}_H \to \tilde{\mathscr{A}}$ is a closed immersion.*

Proof. — We know by [**57**, 10.3] that it is a finite unramified morphism. It now suffices to check that it induces an injection at the level of geometric points.

Suppose that a point $\tilde{a} = (a, \tilde{\infty}_{\rho_\kappa}) \in \tilde{\mathscr{A}}(\bar{k})$ lies in the image of $\tilde{\nu}$. We shall show that it comes from a unique point $\tilde{a}_H = (a_H, \tilde{\infty}_{\rho_\kappa}) \in \tilde{\mathscr{A}}_H$. For this, note that a_H is completely determined by the curve $\widetilde{X}_{\rho_\kappa,a_H}$, and that this curve is the smallest union of irreducible components of $\widetilde{X}_{\rho_\kappa,a}$ which is stable under $\mathbb{W}_\mathbb{H} \rtimes \pi_0(\kappa)$ and which contains the point $\tilde{\infty}_{\rho_\kappa}$. □

Proposition 6.3.3. — *The closed subscheme $\tilde{\mathscr{A}}_\kappa$ of $\tilde{\mathscr{A}}$ defined in 6.2.2 is the disjoint union of the closed subsets $\tilde{\nu}(\tilde{\mathscr{A}}_H)$ associated with the homomorphisms $\rho^\bullet_\kappa : \pi_1(\bar{X}, \infty) \to \pi_0(\kappa)$.*

Proof. — In 5.4.1 we attached to a geometric point $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}(\bar{k})$ a diagram

$$\begin{tikzcd} \pi_1(U,\infty) \arrow[r, "\pi^\bullet_{\tilde{a}}"] \arrow[d] & \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) \arrow[d] \\ \pi_1(\bar{X},x) \arrow[r, "\rho^\bullet_G"'] & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

Here U is the largest open subset of $\bar{X}$ containing ∞ above which the cameral cover $\widetilde{X}_a \to \bar{X}$ is étale. We wrote $W_{\tilde{a}}$ for the image of $\pi^\bullet_{\tilde{a}}$ in $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ and $I_{\tilde{a}}$ for the image of the kernel of the homomorphism $\pi_1(U,\infty) \to \pi_1(\bar{X},\infty)$.

By definition, $(a,\tilde{\infty})$ belongs to $\tilde{\mathcal{A}}_\kappa$ if and only if $W_{\tilde{a}}$ is contained in $(\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ and $I_{\tilde{a}}$ is contained in the Weyl group $\mathbb{W}_{\mathbb{H}}$ of the neutral component $\hat{\mathbb{H}}$ of the centralizer of κ in $\hat{\mathbb{G}}$. Recall that one has an exact sequence

$$1 \to \mathbb{W}_{\mathbb{H}} \to (\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa \to \pi_0(\kappa) \to 1. \tag{6.3.4}$$

By [**57**, Lemma 10.1] there exists a canonical splitting which allows one to identify $(\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ with a semi-direct product $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$.

Let H be the endoscopic group associated with a homomorphism $\rho^\bullet_\kappa : \pi_1(\bar{X},\infty) \to \pi_0(\kappa)$. Let $\tilde{a}_H \in \tilde{\mathcal{A}}_H(\bar{k})$, and let U be the open subset of $\bar{X}$ containing ∞ above which the cameral cover $\widetilde{X}_{a_H} \to \bar{X}$ is étale. The homomorphism

$$\pi^\bullet_{\tilde{a}_H} : \pi_1(U,\infty) \to \mathbb{W}_{\mathbb{H}} \rtimes \mathrm{Out}(\mathbb{H})$$

induces a homomorphism

$$\pi^{\kappa,\bullet}_{\tilde{a}_H} : \pi_1(U,\infty) \to \mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$$

over $\rho^\bullet_\kappa : \pi_1(\bar{X},\infty) \to \pi_0(\kappa)$. By construction of $\tilde{a} = \tilde{\nu}(\tilde{a}_H)$, the homomorphism $\pi^\bullet_{\tilde{a}}$ is obtained by composing $\pi^{\kappa,\bullet}_{\tilde{a}_H}$ with the embedding $\mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ via the identification $(\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa = \mathbb{W}_{\mathbb{H}} \rtimes \pi_0(\kappa)$. We therefore have $W_{\tilde{a}} \subset (\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ and $I_{\tilde{a}} \subset \mathbb{W}_{\mathbb{H}}$, that is, $\tilde{a} \in \tilde{\mathcal{A}}_\kappa(\bar{k})$.

Conversely, let $\tilde{a} \in \tilde{\mathcal{A}}_\kappa(\bar{k})$. Since $W_{\tilde{a}} \subset (\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ and $I_{\tilde{a}} \subset \mathbb{W}_{\mathbb{H}}$, the homomorphism $\pi^\bullet_{\tilde{a}}$ induces a homomorphism

$$\rho^\bullet_\kappa : \pi_1(\bar{X},\infty) \to W_{\tilde{a}}/I_{\tilde{a}} \to \pi_0(\kappa).$$

Let H be the endoscopic group associated with this $\rho^\bullet_\kappa$. It remains to construct a point $\tilde{a}_H \in \tilde{\mathcal{A}}_H(\bar{k})$ such that $\tilde{\nu}(\tilde{a}_H) = \tilde{a}$. This construction is identical to the one appearing in the proof of the preceding proposition. □

Similar arguments allow a natural interpretation of the complement of the open subset $\tilde{\mathcal{A}}^{\mathrm{ani}}$.

Proposition 6.3.5. — *The complement of the open subset $\tilde{\mathcal{A}}^{\mathrm{ani}}$ in $\tilde{\mathcal{A}}$ is the union of the images of the closed immersions*

$$\tilde{\mathcal{A}}_M \to \tilde{\mathcal{A}}$$

over the set of Levi subgroups M containing the maximal torus T.

Proof. — Let $\tilde{a} \in (\tilde{\mathcal{A}} - \tilde{\mathcal{A}}^{\mathrm{ani}})(\bar{k})$. By definition the group $\mathbb{T}^{W_{\tilde{a}}}$ is not finite, and therefore contains a torus S. The centralizer of this torus in G is a Levi subgroup M of G. The group $W_{\tilde{a}}$ is a subgroup of the fixer of S in $\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$. The group $I_{\tilde{a}}$ is contained in the intersection of the fixer of M with $\mathbb{W}$, hence contained in $\mathbb{W}_M$. The same argument as the one used in the proof of 6.3.3 shows that $\tilde{a}$ comes from a point $\tilde{a}_M \in \tilde{\mathcal{A}}_M(\bar{k})$. □

Corollary 6.3.6. — *The codimension of $\mathcal{A}^\heartsuit - \mathcal{A}^{\mathrm{ani}}$ in $\mathcal{A}^\heartsuit$ is greater than or equal to* $\deg(D)$.

Proof. — By the description of the complement of $\tilde{\mathscr{A}}^{\mathrm{ani}}$ and the dimension formula 4.13.1, we have

$$\dim(\mathscr{A}) - \dim(\mathscr{A}_M) = (\sharp\,\Phi - \sharp\,\Phi_M)\deg(D)/2 \geq \deg(D),$$

whence the proposition. □

6.4. Geometric Stabilization. — By 6.2.3, the support of the perverse sheaf ${}^p\mathrm{H}^n(\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_\kappa$ is contained in $\tilde{\mathscr{A}}_\kappa^{\mathrm{ani}}$. By the description of $\tilde{\mathscr{A}}_\kappa^{\mathrm{ani}}$ in terms of endoscopic groups *cf.* 6.3.3, this perverse sheaf decomposes as a direct sum of factors parametrized by the set of pointed endoscopic data $(\kappa, \rho_{\kappa,\xi}^\bullet)$, the direct factor parametrized by ξ being supported on the closed subset $\tilde{\nu}_\xi(\tilde{\mathscr{A}}_{H_\xi}) \cap \tilde{\mathscr{A}}^{\mathrm{ani}}$, where H_ξ is the endoscopic group attached to $(\kappa, \rho_{\kappa,\xi}^\bullet)$ and $\tilde{\nu}_\xi$ is the closed immersion of $\tilde{\mathscr{A}}_{H_\xi}$ into $\tilde{\mathscr{A}}$. Our main result is a description of each of these direct factors.

Theorem 6.4.1. — *There exists an isomorphism between the graded perverse sheaves*

$$\bigoplus_n {}^p\mathrm{H}^n(\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_\kappa[2r_H^G(D)](r_H^G(D))$$

and

$$\bigoplus_n \bigoplus_{(\kappa,\rho_{\kappa,\xi}^\bullet)} \tilde{\nu}_{\xi,*}\,{}^p\mathrm{H}^*(\tilde{f}_{H_\xi,*}^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}},$$

where the last direct sum runs over the set of all pointed endoscopic data $(\kappa, \rho_{\kappa,\xi}^\bullet)$ having the fixed component κ. Here the index st *denotes the direct factor on which $\tilde{\mathscr{P}}_{H_\xi}$ acts trivially, and the integer $r_H^G(D)$ is the codimension of $\tilde{\mathscr{A}}_{H_\xi}$ in $\tilde{\mathscr{A}}$, given by the formula*

$$r_H^G(D) = (\sharp\,\Phi - \sharp\,\Phi_{\mathbb{H}})\deg(D)/2.$$

Although the statement above is over $\bar{k}$, our proof is in part arithmetic. Suppose now that we have a pointed endoscopic datum $(\kappa, \rho_\kappa^\bullet)$ defined over X, that is, $\rho_\kappa^\bullet$ extends to a homomorphism

$$\pi_1(X, \infty) = \pi_1(\bar{X}, \infty) \rtimes \mathrm{Gal}(\bar{k}/k) \to \pi_0(\kappa).$$

The associated endoscopic group H is then defined over X. We have a Hitchin fibration $f^H : \mathscr{M}_H \to \mathscr{A}_H$ for the group H and an étale cover $\tilde{\mathscr{A}}_H$ of $\mathscr{A}_H$. The closed immersion $\tilde{\nu} : \tilde{\mathscr{A}}_H \to \tilde{\mathscr{A}}_G$ of 6.3.2 is then defined over k.

Theorem 6.4.2. — *There exists an isomorphism defined over k between the semi-simplifications of the graded perverse sheaves*

$$\bigoplus_n \tilde{\nu}^*\,{}^p\mathrm{H}^n(\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_\kappa[2r_H^G(D)](r_H^G(D)) \simeq \bigoplus_n {}^p\mathrm{H}^*(\tilde{f}_{H,*}^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}.$$

Note that, both sides of the equality being graded pure perverse sheaves, they are geometrically semisimple *cf.* [**7**]. It follows that the graded perverse sheaves above are isomorphic over $\tilde{\mathscr{A}}^{\mathrm{ani}} \otimes_k \bar{k}$. Moreover, since every endoscopic datum over $\bar{X}$ is defined over $X \otimes_k k'$ for some finite extension k' of k, Theorem 6.4.1 is a consequence of 6.4.2. The proof of the latter is not completed until paragraph 8.7. We shall obtain the Langlands-Shelstad conjecture 1.11.1 in particular in the course of its proof.

6.5. Ordinary Cohomology of Maximal Degree. — In this paragraph we prove an elementary variant of the geometric stabilization theorem 6.4.1, in which, instead of the perverse cohomology sheaves, one considers the ordinary cohomology sheaf of maximal degree.

Write d for the relative dimension of the morphism $\tilde{f}^{\mathrm{ani}} : \tilde{\mathscr{M}}^{\mathrm{ani}} \to \tilde{\mathscr{A}}^{\mathrm{ani}}$. By the homotopy lemma, the action of $\tilde{\mathscr{P}}^{\mathrm{ani}}$ on $\mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell$ factors through the finite sheaf $\pi_0(\tilde{\mathscr{P}}^{\mathrm{ani}})$. Recall also that the latter is a quotient of the constant sheaf $\mathbb{X}_*$. We write $(\mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ for the largest direct factor on which $\mathbb{X}_*$ acts trivially, and $(\mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_\kappa$ for the one on which it acts through the character $\kappa : \mathbb{X}_* \to \overline{\mathbb{Q}}_\ell^\times$ that we have fixed.

Proposition 6.5.1. — *There is an isomorphism between the stable part* $(\mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ *and the constant sheaf* $\overline{\mathbb{Q}}_\ell$*. There is also an isomorphism between* $(\mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_\kappa$ *and the direct sum*

$$\bigoplus_{(\kappa,\rho^\bullet_{\kappa,\xi})} \tilde{\nu}_{\xi,*}\overline{\mathbb{Q}}_\ell$$

extended over the set of pointed endoscopic data $(\kappa,\rho^\bullet_{\kappa,\xi})$ *having the fixed component* κ*, where* $\tilde{\nu}_\xi$ *denotes the closed immersion of* $\tilde{\mathcal{A}}^{\mathrm{ani}}_{H_\xi}$ *into* $\tilde{\mathcal{A}}^{\mathrm{ani}}$*, with* H_ξ *the endoscopic group attached to* $(\kappa,\rho^\bullet_{\kappa,\xi})$*.*

Proof. — The Hitchin-Kostant section defines an open immersion $\tilde{\mathcal{P}}^{\mathrm{ani}} \to \tilde{\mathcal{M}}^{\mathrm{ani}}$ whose closed complement has relative dimension $\leq d-1$ *cf.* 4.16.1. There follows an isomorphism

$$\mathrm{R}^{2d}g_!\overline{\mathbb{Q}}_\ell \to \mathrm{R}^{2d}\tilde{f}_*^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell$$

compatible with the action of $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$. Here g denotes the morphism $\tilde{\mathcal{P}}^{\mathrm{ani}} \to \tilde{\mathcal{A}}^{\mathrm{ani}}$.

The trace morphism allows one to identify $\mathrm{R}^{2d}g_!\overline{\mathbb{Q}}_\ell$ with the sheaf associated with the presheaf

$$U \mapsto \overline{\mathbb{Q}}_\ell^{\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})(U)}.$$

The proposition therefore follows from the explicit description of the sheaf $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$ given in 5.5.4. Indeed, $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$ is a quotient of the constant sheaf $\mathbb{X}_*$. One may replace $\mathbb{X}_*$ by a finite quotient $\mathbb{X}$. Thus $\mathrm{R}^{2d}g_!\overline{\mathbb{Q}}_\ell$ is a quotient of the constant sheaf $\overline{\mathbb{Q}}_\ell^{\mathbb{X}}$. The isotypic parts $(\mathrm{R}^{2d}g_!\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ and $(\mathrm{R}^{2d}g_!\overline{\mathbb{Q}}_\ell)_\kappa$ are quotients of the corresponding isotypic parts of $\overline{\mathbb{Q}}_\ell^{\mathbb{X}}$, which are constant sheaves of rank one. The assertions to be proved are now reduced to a fiber-by-fiber verification, which is easy. □

7. The Support Theorem

Let S be a k-scheme of finite type and $f : M \to S$ a proper morphism whose source is smooth. By the purity [**18**] and decomposition [**7**] theorems, the complex $f_*\overline{\mathbb{Q}}_\ell$ is pure and is isomorphic, over $S \otimes_k \bar{k}$, to a direct sum of geometrically simple perverse sheaves with shifts. Their supports constitute an important topological invariant of f. In general, it is difficult to determine this invariant explicitly.

We introduce a notion of *δ-regular abelian fibration cf.* 7.1.5 for which it is possible to determine the supports *cf.* 7.2.1. Since we do not know how to prove the δ-regularity of the Hitchin fibration in positive characteristic, we formulate the statements 7.2.2 and 7.2.3, which give sufficient control for the proof of the theorems 6.4.1 and 6.4.2.. With the exception of the last paragraph, this chapter is written in the general situation of weak abelian fibrations.

7.1. Abelian Fibration. — We are going to formalize the properties observed on the Hitchin fibration in order to axiomatize a notion of algebraic abelian fibration. We shall in fact introduce, on the one hand, the notion of *weak abelian fibration*, which gathers together those properties that are stable under arbitrary base change, and on the other hand that of *δ-regular abelian fibration*, which is preserved only under flat base change. The right notion of abelian fibration ought to lie somewhere between the two.

7.1.1. — A *weak abelian fibration* over a k-scheme S consists of a proper morphism $f : M \to S$ together with a smooth commutative group scheme $g : P \to S$ equipped with an action

$$\mathrm{act} : P \times_S M \to M$$

satisfying the three properties 7.1.2, 7.1.3 and 7.1.4 below.

7.1.2. — The morphisms f and g have the same relative dimension d.(1)

7.1.3. — The action of P on M has only affine stabilizers; that is, for every geometric point $m \in M$ lying over $s \in S$, the stabilizer of m in P_s is an affine subgroup.

7.1.4. — Let P^0 be the open subgroup scheme of neutral components of the fibers of P, and write $g^0 : P^0 \to S$. Consider the sheaf of Tate modules

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0) = \mathrm{H}^{2d-1}(g^0_!\overline{\mathbb{Q}}_\ell)(d)$$

whose fiber over each geometric point s of S is the $\overline{\mathbb{Q}}_\ell$-Tate module $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0_s)$. For every geometric point s of S, the canonical Chevalley dévissage of P^0_s

$$1 \to R_s \to P^0_s \to A_s \to 1,$$

in which A_s is an abelian variety and R_s is a connected commutative affine algebraic group, induces a dévissage of Tate modules *cf.* [**29**]

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0_s) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s) \to 0.$$

We shall say that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0)$ is *polarizable* if, locally for the étale topology on S, there exists an alternating bilinear form

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0) \times \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0) \to \overline{\mathbb{Q}}_\ell$$

whose fiber at each geometric point s has $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s)$ as its kernel, that is, which annihilates $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s)$ and induces a perfect pairing on $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s)$.

The three properties 7.1.2, 7.1.3 and 7.1.4 are preserved under arbitrary base change. In particular, the generic fiber of a weak abelian fibration need not be an abelian variety. We now introduce a strong restriction, called δ-regularity, which will guarantee this property among others.

7.1.5. — For every geometric point $s \in S$, write $\delta_s = \dim(R_s)$ for the dimension of the affine part of P_s. If $s \in S$ is an arbitrary point, the Chevalley dévissage of P_s exists and is unique after a radicial base change, so that the integer δ_s is well defined. The function δ, defined on the topological space underlying S with values in the natural numbers, is semi-continuous *cf.* 5.6.3. Assuming this function to be constructible, there exists a stratification of S into locally closed subschemes S_δ such that $\delta_s = \delta$ for every geometric point $s \in S_\delta$.

We shall say that the smooth commutative S-group scheme P is *δ-regular* if for every $\delta \in \mathbb{N}$ we have

$$\mathrm{codim}_S(S_\delta) \geq \delta.$$

If S_δ is empty, our convention assigns the value infinity to the codimension.

A weak abelian fibration whose component P enjoys δ-regularity will be called a *δ-regular abelian fibration*.

7.1.6. — There is another formulation of the notion of δ-regularity. Let Z be an irreducible closed subscheme of S and let δ_Z be the minimal value taken by the function δ on Z. Then P is δ-regular if and only if, for every irreducible closed subscheme Z of S, we have $\mathrm{codim}(Z) \geq \delta_Z$.

Indeed, since the value δ_Z is attained on a dense open subset of Z, the subscheme Z is contained in the closure of the stratum S_{δ_Z}. If P is δ-regular, then $\mathrm{codim}(S_{\delta_Z}) \geq \delta_Z$ and a fortiori $\mathrm{codim}(Z) \geq \delta_Z$. The other implication is equally clear.

Note that δ-regularity is preserved under flat base change. Note also that δ-regularity implies

$$\mathrm{codim}(S_1) \geq 1,$$

so that $S_0 \neq \emptyset$. It follows that P^0 is generically an abelian variety.

(1) This hypothesis is in no way necessary, but it simplifies the numerology.

7.2. Statement of the Support Theorem. — Let S be a k-scheme of finite type and $f : M \to S$ a proper morphism. Suppose that the constant sheaf $\overline{\mathbb{Q}}_\ell$ on M is self-dual, and hence pure; this is in particular the case if M is a smooth k-scheme. By Deligne's purity theorem, the direct image complex $f_*\overline{\mathbb{Q}}_\ell$ is pure. Applying the decomposition theorem [**7**], we know that over $S \otimes_k \bar{k}$ the complex $f_*\overline{\mathbb{Q}}_\ell$ is isomorphic to the direct sum of its perverse cohomology sheaves with the evident shifts

$$f_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)[-n],$$

and moreover the perverse sheaves ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ are semisimple. By [**7**], for every geometrically simple perverse sheaf K on $S \otimes_k \bar{k}$ there exist an irreducible reduced closed subscheme $i : Z \hookrightarrow S \otimes_k \bar{k}$, a dense open subset $j : U \hookrightarrow Z$ and an irreducible local system $\mathcal{K}$ on U such that

$$K = i_* j_{!*}\mathcal{K}[\dim(Z)].$$

The closed subscheme Z is completely determined by K and will be called the *support* of K. In general, the problem of determining the supports of the simple perverse sheaves occurring in the decomposition of $f_*\overline{\mathbb{Q}}_\ell$ is a very difficult one. It can however be solved in the case of a δ-regular abelian fibration.

Theorem 7.2.1. — *Let S be a geometrically irreducible k-scheme of finite type. Let $f : M \to S$ be a projective morphism, purely of relative dimension d, equipped with an action of a smooth group scheme $g : P \to S$ forming a δ-regular abelian fibration. Suppose that the constant sheaf $\overline{\mathbb{Q}}_\ell$ on M is self-dual, and hence pure.*

Let K be a geometrically simple perverse sheaf occurring in the decomposition of one of the perverse cohomology sheaves ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$, and let Z be its support. Then there exist an open subset U of $S \otimes_k \bar{k}$ with $U \cap Z$ non-empty, and a non-trivial local system L on $U \cap Z$, such that i_*L, where i denotes the closed immersion $U \cap Z \to U$, is a direct factor of the restriction to U of the ordinary cohomology sheaf of maximal degree $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)$.

Despite its complicated appearance, this statement is in fact effective for determining the supports, because the ordinary cohomology sheaf of maximal degree is in general known *cf.* 6.5, as are its local direct factors. Note also that K and L have nothing in common except their support. The proof of the theorem passes through the following statement.

Proposition 7.2.2. — *Let S be a geometrically irreducible k-scheme of finite type, and let $f : M \to S$ be a projective morphism, purely of relative dimension d, equipped with an action of a smooth group scheme $g : P \to S$ forming a weak abelian fibration. Suppose that the constant sheaf $\overline{\mathbb{Q}}_\ell$ on M is self-dual, and hence pure.*

Let K be a geometrically simple perverse sheaf occurring in the decomposition of one of the perverse cohomology sheaves ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$, and let Z be its support. Let δ_Z be the minimal value taken on Z by the function δ attached to P. Then one has the inequality

$$\mathrm{codim}(Z) \le \delta_Z.$$

If equality is attained, then there exist an open subset U of $S \otimes_k \bar{k}$ with $U \cap Z$ non-empty and a non-trivial local system L on $U \cap Z$ such that i_*L, where i denotes the closed immersion $U \cap Z \to U$, is a direct factor of the restriction to U of the ordinary cohomology sheaf of maximal degree $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)$.

It is clear that 7.2.2 implies 7.2.1, since the hypothesis of δ-regularity yields the opposite inequality $\mathrm{codim}(Z) \ge \delta_Z$ *cf.* 7.1.6.

Consider a variant of the delta inequality 7.2.2 which takes the groups of connected components of the fibers of P into account as well. Let $\pi_0(P)$ be the sheaf of groups of connected components of the fibers of P. Suppose that it is a quotient of a constant sheaf with fiber the finite abelian group $\mathbb{X}$, as for instance in 5.5.4.

The group scheme P acts on the perverse cohomology sheaf ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ through $\pi_0(P)$. The group $\mathbb{X}$ therefore acts on ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$. For every character $\kappa : \mathbb{X} \to \overline{\mathbb{Q}}_\ell^\times$ we have the largest direct factor ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)_\kappa$ of ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ on which $\mathbb{X}$ acts through κ. Likewise, we have the direct factor $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)_\kappa$ of the ordinary cohomology sheaf $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)$.

In fact, there exist a natural number N and a decomposition of the bounded constructible complex $f_*\overline{\mathbb{Q}}_\ell$

$$f_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\kappa \in \mathbb{X}^*} (f_*\overline{\mathbb{Q}}_\ell)_\kappa$$

such that for every $\alpha \in \mathbb{X}$ the endomorphism $(\alpha - \kappa(\alpha)\mathrm{id})^N$ vanishes on the direct factor $(f_*\overline{\mathbb{Q}}_\ell)_\kappa$ *cf.* [**57**, 3.2.5]. If one replaces $f_*\overline{\mathbb{Q}}_\ell$ by $(f_*\overline{\mathbb{Q}}_\ell)_\kappa$ in the proof of 7.2.2, one obtains the following variant.

Proposition 7.2.3. — *In 7.2.2 one may replace the perverse sheaf* ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ *by its isotypic factor* ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)_\kappa$*, and the ordinary sheaf* $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)$ *by its isotypic factor* $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)_\kappa$.

The reader will note that, just as in 7.2.2, we do not assume here that P is δ-regular.

In the last paragraph of this chapter we shall apply 7.2.3 to the case of the Hitchin fibration, more precisely to the morphism $\tilde{f}^{\mathrm{ani}} : \tilde{\mathcal{M}}^{\mathrm{ani}} \to \tilde{\mathcal{A}}^{\mathrm{ani}}$. The rest of the chapter is devoted to proving the delta inequality 7.2.2.

7.3. The Goresky-MacPherson Inequality. — Goresky and MacPherson observed that Poincaré duality imposes a constraint on the codimension of the support of the simple perverse sheaves occurring in the decomposition theorem. This is a crucial observation, of which our delta inequality 7.2.2 is a variant. We begin by recalling their original inequality.

Theorem 7.3.1. — *Let S be a geometrically irreducible k-scheme of finite type. Let $f : M \to S$ be a proper morphism, purely of relative dimension d. Suppose that the constant sheaf $\overline{\mathbb{Q}}_\ell$ on M is self-dual, and hence pure.*

Let K be an irreducible perverse sheaf on $S \otimes_k \bar{k}$ occurring in the decomposition of one of the perverse cohomology sheaves ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$, and let Z be the support of K. Then one has the inequality

$$\mathrm{codim}(Z) \leq d.$$

Suppose that equality holds. Then there exist an open subset U of $S \otimes_k \bar{k}$ with $U \cap Z$ non-empty and a non-trivial local system L on $U \cap Z$ such that i_*L, where i denotes the closed immersion $U \cap Z \to U$, is a direct factor of the restriction to U of the ordinary cohomology sheaf of maximal degree $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)$.

Proof. — Let Z be an irreducible closed subscheme of $S \otimes_k \bar{k}$. Define $\mathrm{occ}(Z)$ to be the set of integers n such that there exists an irreducible direct factor of the perverse cohomology sheaf ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ with support Z. By Poincaré duality, ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ is the dual of ${}^p\mathrm{H}^{2\dim(M)-n}(f_*\overline{\mathbb{Q}}_\ell)$, so that the set $\mathrm{occ}(Z)$ is symmetric about the integer $\dim(M)$.

If $\mathrm{occ}(Z) \neq \emptyset$, there exists an integer $n \geq \dim(M)$ belonging to $\mathrm{occ}(Z)$. There exist an open subset U of $S \otimes_k \bar{k}$ and an irreducible local system L on $U \cap Z$ such that $i_*L[\dim(Z)]$ is a direct factor of ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)|_U$. This implies that $i_*L[\dim(Z) - n]$ is a direct factor of the complex $f_*\overline{\mathbb{Q}}_\ell|_U$, since the latter is pure. Now passing to the ordinary cohomology sheaf, i_*L becomes a direct factor of $\mathrm{H}^{n-\dim(Z)}(f_*\overline{\mathbb{Q}}_\ell)$. Since the fibers of the morphism $f : M \to S$ are purely of dimension d, we have $\dim(M) = d + \dim(S)$. On the other hand, the non-vanishing of $\mathrm{H}^{n-\dim(Z)}(f_*\overline{\mathbb{Q}}_\ell)$ implies that

$$\dim(M) - \dim(Z) \leq n - \dim(Z) \leq 2d$$

by the theorem on cohomological amplitude. Combining the two pieces of information, we obtain the desired inequality

$$\mathrm{codim}(Z) \leq d,$$

together with the necessary condition for equality to hold. □

In the more specific situation of a weak abelian fibration, it is in fact possible to improve the Goresky-MacPherson inequality

$$\mathrm{codim}(Z) \leq d$$

to the delta inequality 7.2.2

$$\mathrm{codim}(Z) \leq \delta_Z.$$

Although the proof of the latter is appreciably more difficult, the underlying topological idea is very simple.

The delta inequality can be reduced to the Goresky-MacPherson inequality if one grants the existence of certain local liftings. Let s be a geometric point of Z such that $\delta_s = \delta_Z$. Denote by A_s the maximal abelian quotient of P_s^0. Suppose that there exists an étale neighbourhood S' of s in S such that, over S', the abelian variety A_s extends to an abelian scheme $A_{S'}$, and that moreover there exists a homomorphism $A_{S'} \to P_{S'}^0$ such that at the point s the composite $A_s \to P_s^0 \to A_s$ is an isogeny of A_s. Over S', the abelian scheme $A_{S'}$ acts on $M_{S'}$ with finite stabilizers, in view of hypothesis 7.1.3. Forming the quotient $[M_{S'}/A_{S'}]$, one factors the morphism $M_{S'} \to S'$ into a proper and smooth morphism $M_{S'} \to [M_{S'}/A_{S'}]$ followed by a morphism $[M_{S'}/A_{S'}] \to S'$ which is purely of relative dimension δ_s. One may then apply the Goresky-MacPherson inequality to the second morphism.

In general these liftings do not exist. The proof of 7.2.2 consists precisely in transposing the geometric argument above to the cohomological level. Let Z be an irreducible closed subscheme of S. In the proof of 7.3.1 we introduced the set $\mathrm{occ}(Z)$ of occurrences of Z as the support of the irreducible factors of the perverse cohomology sheaves ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$. When this set is non-empty, we define

$$\mathrm{amp}(Z) = \max(\mathrm{occ}(Z)) - \min(\mathrm{occ}(Z)).$$

Proposition 7.3.2. — *Place ourselves under the hypotheses of 7.2.2; in particular* $\mathrm{occ}(Z) \neq \emptyset$. *Then one has the inequality*

$$\mathrm{amp}(Z) \geq 2(d - \delta_Z).$$

We now prove 7.2.2, granting 7.3.2. Poincaré duality implies that the set $\mathrm{occ}(Z)$ is symmetric about $\dim(M)$. The additional constraint $\mathrm{amp}(Z) \geq 2(d-\delta_Z)$ forces the existence of an integer $n \geq \dim(M) + d - \delta_Z$ belonging to $\mathrm{occ}(Z)$. The remainder of the proof of 7.2.2 is now exactly the same as the end of the proof of 7.3.1.

The rest of the chapter is devoted to the proof of the amplitude inequality 7.3.2.

7.4. Cap-Product and Freeness. — We shall develop the construction of the cap-product and state a freeness property which implies the amplitude inequality 7.3.2.

7.4.1. — We place ourselves in the following general situation. Let S be an arbitrary scheme, and let $g : P \to S$ be a smooth commutative S-group scheme of finite type with connected fibers of dimension d. Consider the homology complex of P over S, defined by the formula

$$\Lambda_P = g_!\overline{\mathbb{Q}}_\ell[2d](d).$$

This complex is concentrated in non-positive degrees. In degree 0 we have $\mathrm{H}^0(\Lambda_P) = \overline{\mathbb{Q}}_\ell$. The most important part of Λ_P is

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) := \mathrm{H}^{-1}(\Lambda_P),$$

where $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ is a sheaf whose fiber at each geometric point $s \in S$ is the $\overline{\mathbb{Q}}_\ell$-Tate module $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)$ of the fiber of P at s. More generally, the base-change theorem provides an isomorphism

$$\mathrm{H}^{-i}(\Lambda_P)_s = \mathrm{H}_c^{2d-i}(P_s)(d)$$

between the fiber at s of $\mathrm{H}^{-i}(\Lambda_P)$ and the $(2d-i)$-th compactly supported cohomology group

$$\mathrm{H}_i(P_s) = \mathrm{H}_c^{2d-i}(P_s)(d)$$

of the fiber of P at s.

7.4.2. — Let $f : M \to S$ be a morphism of finite type equipped with an action of P relative to the base S

$$\mathrm{act} : P \times_S M \to M.$$

Since P is smooth of relative dimension d over S, the morphism act is also smooth of the same relative dimension. We therefore have a trace morphism

$$\mathrm{act}_! \overline{\mathbb{Q}}_\ell[2d](d) \to \overline{\mathbb{Q}}_\ell$$

over M. Pushing this trace morphism forward by $f_!$, we obtain a morphism

$$(g \times_S f)_! \overline{\mathbb{Q}}_\ell[2d](d) \to f_! \overline{\mathbb{Q}}_\ell.$$

Now using the Künneth isomorphism, we obtain a cap-product morphism

$$\Lambda_P \otimes f_! \overline{\mathbb{Q}}_\ell \to f_! \overline{\mathbb{Q}}_\ell.$$

7.4.3. — This construction applies in particular to $f = g$. It then defines a morphism of complexes

$$\Lambda_P \otimes \Lambda_P \to \Lambda_P.$$

From it we deduce a graded algebra structure on the cohomology sheaves of Λ_P

$$\mathrm{H}^{-i}(\Lambda_P) \otimes \mathrm{H}^{-j}(\Lambda_P) \to \mathrm{H}^{-i-j}(\Lambda_P),$$

which is commutative in the graded sense. In particular we deduce a morphism of sheaves

$$\wedge^i \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \to \mathrm{H}^{-i}(\Lambda_P),$$

which is in fact an isomorphism. To check this it suffices to do so fiber by fiber, where one recovers the homology groups of P_s equipped with the Pontryagin product.

7.4.4. — Multiplication by an integer $N \neq 0$ in P induces multiplication by N on the Tate module $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ of P, and hence induces multiplication by N^i on $\mathrm{H}^{-i}(\Lambda_P) = \wedge^i \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$. Lieberman's trick [**37**, 2A11], which consists in constructing projectors from linear combinations of these endomorphisms of Λ_P associated with different values of N, allows one to decompose this complex canonically as

$$\Lambda_P = \bigoplus_{i \geq 0} \wedge^i \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)[i]$$

in a way compatible with the multiplication.

7.4.5. — We shall now study the cap-product action of Λ_P on $f_! \overline{\mathbb{Q}}_\ell$ in the situation of 7.2.2. Recall that since f is assumed projective, we have $f_! = f_*$. Moreover, the constant sheaf $\overline{\mathbb{Q}}_\ell$ on M being assumed pure, the direct image $f_* \overline{\mathbb{Q}}_\ell$ is also pure and decomposes over $S \otimes_k \bar{k}$ into a direct sum of irreducible perverse sheaves with shifts.

For every irreducible closed subset Z of $S \otimes_k \bar{k}$ we introduced the set $\mathrm{occ}(Z)$ in the proof of 7.3.1. Since $f_* \overline{\mathbb{Q}}_\ell$ is a bounded constructible complex, the set $\mathrm{occ}(Z)$ is empty except for finitely many irreducible closed subschemes Z of $S \otimes_k \bar{k}$. We index these by a finite set $\mathfrak{B}$, and for every $\alpha \in \mathfrak{B}$ we write Z_α for the corresponding irreducible closed subset. For every n we then have the canonical decomposition

$$^p\mathrm{H}^n(f_* \overline{\mathbb{Q}}_\ell) = \bigoplus_{\alpha \in \mathfrak{B}} K_\alpha^n,$$

where K_α^n is the direct sum of the irreducible perverse factors of ${}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)$ having the irreducible closed subset Z_α as support. We write

$$K_\alpha = \bigoplus_{n\in\mathbb{Z}} K_\alpha^n[-n].$$

We assume that K_α is non-zero for every $\alpha \in \mathfrak{B}$.

7.4.6. — In view of the decomposition 7.4.4, we have a cap-product by the Tate module 7.4.2

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes f_!\overline{\mathbb{Q}}_\ell \to f_!\overline{\mathbb{Q}}_\ell[-1].$$

Applying the functor ${}^p\tau^{\leq n}$ to $f_!\overline{\mathbb{Q}}_\ell$, we have a morphism

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell) \to f_!\overline{\mathbb{Q}}_\ell[-1]$$

for every n. Applying the functor ${}^p\mathrm{H}^n$, we now obtain an arrow

$${}^p\mathrm{H}^n\big(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell)\big) \to {}^p\mathrm{H}^{n-1}(f_!\overline{\mathbb{Q}}_\ell).$$

Note now that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\tau^{\leq n-1}(f_!\overline{\mathbb{Q}}_\ell) \in {}^p\mathrm{D}_c^{\leq n-1}(S, \overline{\mathbb{Q}}_\ell)$, so that the arrow

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)[-n]$$

induces an isomorphism at the level of the n-th perverse cohomology sheaves

$${}^p\mathrm{H}^n\big(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell)\big) \to {}^p\mathrm{H}^0\big(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)\big).$$

From this we deduce an arrow

$${}^p\mathrm{H}^0\big(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)\big) \to {}^p\mathrm{H}^{n-1}(f_!\overline{\mathbb{Q}}_\ell).$$

Again, since $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell) \in {}^p\mathrm{D}_c^{\leq 0}(S, \overline{\mathbb{Q}}_\ell)$, we have an arrow from this object into its ${}^p\mathrm{H}^0$, which therefore induces a canonical arrow

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell) \to {}^p\mathrm{H}^{n-1}(f_!\overline{\mathbb{Q}}_\ell).$$

7.4.7. — Consider the decomposition by supports

$$\bigoplus_{\alpha\in\mathfrak{B}} \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes K_\alpha^n \to \bigoplus_{\alpha\in\mathfrak{B}} K_\alpha^{n-1}.$$

For every $\alpha \in \mathfrak{B}$ we therefore have a diagonal arrow

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes K_\alpha^n \to K_\alpha^{n-1}.$$

Note that it is not excluded that the non-diagonal arrows

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes K_\alpha^n \to K_{\alpha'}^{n-1}$$

be non-zero.

7.4.8. — For every $\alpha \in \mathfrak{B}$ there exists a dense open subset V_α of Z_α such that the restriction of the perverse sheaf K_α^n to V_α is of the form $\mathcal{K}_\alpha^n[\dim(V_\alpha)]$, where $\mathcal{K}_\alpha^n$ is a pure local system of weight n.

After possibly shrinking the open subset V_α, there exists a finite radicial base change $V'_\alpha \to V_\alpha$ such that the group scheme $P|_{V'_\alpha}$ admits a dévissage

$$1 \to R_\alpha \to P|_{V'_\alpha} \to A_\alpha \to 1,$$

where A_α is a V'_α-abelian scheme and R_α is a smooth commutative affine V'_α-group scheme with connected fibers. From this we deduce an exact sequence of Tate modules

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V'_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha) \to 0.$$

Since the morphism $V'_\alpha \to V_\alpha$ is a homeomorphism, it is legitimate to regard $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha)$ and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$, as well as the exact sequence above, as objects existing over V_α. As A_α is an abelian scheme, the Tate module $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$ is a pure local system of weight -1. After shrinking V_α further if necessary, we may assume that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha)$, which is the Tate module of the toric part of R_α, is a pure local system of weight -2. Here the weights in question are relative to a sufficiently large extension of the finite base field k over which our objects are defined.

After shrinking V_α if necessary, we may assume that unless Z_α is entirely contained in $Z_{\alpha'}$, we have $V_\alpha \cap Z_{\alpha'} = \emptyset$.

7.4.9. **—** Let V_α be a dense open subset of Z_α as in 7.4.8. For every $\alpha \in \mathfrak{B}$, choose a Zariski open subset U_α of $S \otimes_k \bar{k}$ containing V_α as a closed subscheme. Write $i_\alpha : V_\alpha \to U_\alpha$ for the closed immersion. Restricting the diagonal arrow constructed in 7.4.7 to the open subset U_α, we obtain an arrow

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes i_{\alpha *}\mathcal{K}_\alpha^n[\dim(V_\alpha)] \to i_{\alpha *}\mathcal{K}_\alpha^{n-1}[\dim(V_\alpha)].$$

By the projection formula we have

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes i_{\alpha *}\mathcal{K}_\alpha^n = i_{\alpha *}\big(i_\alpha^*\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha) \otimes \mathcal{K}_\alpha^n\big).$$

Applying the functor i_α^* to

$$i_{\alpha *}\big(i_\alpha^*\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha) \otimes \mathcal{K}_\alpha^n\big) \to i_{\alpha *}\mathcal{K}_\alpha^{n-1},$$

we obtain a morphism

$$i_\alpha^*\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha) \otimes \mathcal{K}_\alpha^n \to \mathcal{K}_\alpha^{n-1}$$

over V_α.

By 7.4.8 we have a dévissage of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha)$

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha) \to 0,$$

where $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$ is a pure local system of weight -1 and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha)$ is a pure local system of weight -2. Since $\mathcal{K}_\alpha^n$ is pure of weight n and $\mathcal{K}_\alpha^{n-1}$ is pure of weight $n-1$, the action of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_\alpha)$ factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha) \otimes \mathcal{K}_\alpha^n \to \mathcal{K}_\alpha^{n-1}.$$

We have thus endowed the direct sum of local systems $\mathcal{K}_\alpha = \bigoplus_n \mathcal{K}_\alpha^n[-n]$ with the structure of a graded module over the graded algebra of local systems Λ_{A_α}

$$\Lambda_{A_\alpha} \otimes \mathcal{K}_\alpha \to \mathcal{K}_\alpha.$$

In particular, for every geometric point u_α of V_α, the fiber $\mathcal{K}_{\alpha,u_\alpha}$ is a graded module over the graded algebra $\Lambda_{A_{\alpha,u_\alpha}}$.

Proposition 7.4.10. **—** *Place ourselves under the hypotheses of 7.2.2 and with the notations of 7.4.5, 7.4.8 and 7.4.9. For every geometric point u_α of V_α, the fiber $\mathcal{K}_{\alpha,u_\alpha}$ of $\mathcal{K}_\alpha$ is a free graded module over the graded algebra $\Lambda_{A_{\alpha,u_\alpha}}$.*

The amplitude inequality 7.3.2 is an immediate consequence of this freeness property. Indeed, $\mathrm{amp}(Z_\alpha)$ is then at least equal to

$$2\dim(A_\alpha) = 2(d - \delta_\alpha).$$

The rest of the chapter is devoted to the proof of the freeness property 7.4.10. But before embarking on the proof proper of 7.4.10, we observe that the property that $\mathcal{K}_{\alpha,u_\alpha}$ be a free module over $\Lambda_{A_{\alpha,u_\alpha}}$ is independent of the choice of the geometric point $u_\alpha \in V_\alpha$. We shall prove the following statement, which allows one to formulate this freeness property independently of the base point. Observe that under the purity hypotheses of 7.4.10, the local systems $\mathcal{K}_\alpha^n$ are geometrically semisimple.

Lemma 7.4.11. — *Let U be a connected $\bar{k}$-scheme. Let Λ be a graded local system in finitely many non-positive degrees with $\Lambda^0 = \overline{\mathbb{Q}}_\ell$, equipped with a graded algebra structure $\Lambda \otimes \Lambda \to \Lambda$. Let L be a graded local system equipped with a graded module structure*

$$\Lambda \otimes L \to L.$$

Suppose that there exists a geometric point u of U such that the fiber L_u of L at u is a free module over the fiber Λ_u of Λ at u. Suppose that L is semisimple as a graded local system. Then there exist a graded local system E on U and an isomorphism

$$L = \Lambda \otimes E$$

compatible with the Λ-module structures.

Proof. — Consider the augmentation ideal of Λ

$$\Lambda^+ = \bigoplus_{i>0} \Lambda^{-i}[i].$$

Write E for the cokernel of the arrow

$$\Lambda^+ \otimes L \to L.$$

Since L is semisimple, the surjective homomorphism $L \to E$ admits a lifting $E \to L$. We shall show that the induced arrow

$$\Lambda \otimes E \to L$$

is an isomorphism. Since this is an arrow between local systems, in order to check that it is an isomorphism it suffices to check that its fiber at u is one. In the vector space L_u, the subspace E_u is a complement of $\Lambda^+ L_u$. The arrow

$$\Lambda_u \otimes E_u \to L_u$$

is therefore surjective by Nakayama's lemma. It is bijective because, L_u being a free Λ_u-module, one has the equality of dimensions

$$\dim_{\overline{\mathbb{Q}}_\ell}(\Lambda_u) \times \dim_{\overline{\mathbb{Q}}_\ell}(E_u) = \dim_{\overline{\mathbb{Q}}_\ell}(L_u),$$

whence the lemma. □

7.5. The Freeness Property in the Punctual Case. — We begin by considering statements analogous to 7.4.10 in the case where the base is a point.

Proposition 7.5.1. — *Let M be a projective scheme over an algebraically closed field $\bar{k}$ equipped with an action of a $\bar{k}$-abelian variety A with finite stabilizers. Then*

$$\bigoplus_n \mathrm{H}_c^n(M \otimes_k \bar{k})[-n]$$

is a free module over the graded algebra $\Lambda_A = \bigoplus_i \wedge^i \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A)[i]$.

Proof. — Since the stabilizer in A of every point of M is a finite subgroup, the quotient $N = [M/A]$ is a proper algebraic stack over k with finite inertia. Since M is projective, the morphism $m : M \to N$ is a projective and smooth morphism. The cup-product with the hyperplane class induces, by Deligne [**19**], an isomorphism

$$m_* \overline{\mathbb{Q}}_\ell \simeq \bigoplus_i \mathrm{R}^i m_* \overline{\mathbb{Q}}_\ell[-i],$$

where the $\mathrm{R}^i(m_*\overline{\mathbb{Q}}_\ell)$ are local systems on N. Since $M \times_N M = A \times M$, the inverse image $m^*\mathrm{R}^i(m_*\overline{\mathbb{Q}}_\ell)$ is the constant sheaf on M with value $\mathrm{H}^i(A)$. Using [**7**, 4.2.5], we obtain a canonical isomorphism between $\mathrm{R}^i(m_*\overline{\mathbb{Q}}_\ell)$ and the constant sheaf on N with value $\mathrm{H}^i(A)$.

The direct sum decomposition above implies the degeneration of the spectral sequence

$$\mathrm{H}^j_c(N, \mathrm{R}^i m_* \overline{\mathbb{Q}}_\ell) \Rightarrow \mathrm{H}^{i+j}_c(M).$$

From it we deduce, in $\bigoplus_n \mathrm{H}^n_c(M)$, a Λ_A-stable filtration whose j-th graded piece is

$$\bigoplus_i \mathrm{H}^j_c(N, \mathrm{R}^i m_* \overline{\mathbb{Q}}_\ell) = \mathrm{H}^j_c(N) \otimes \bigoplus_i \mathrm{H}^i(A).$$

These graded pieces are free Λ_A-modules. It follows that the direct sum $\bigoplus_n \mathrm{H}^n_c(M)[-n]$ is a free Λ_A-module. □

7.5.2. **—** Let P be a smooth connected commutative algebraic group over an algebraically closed field $\bar{k}$. By Chevalley's theorem *cf.* [**66**], P admits a dévissage as an extension of an abelian variety by an affine group

$$1 \to R \to P \to A \to 1.$$

This exact sequence also exists if P is defined over a perfect field, in particular over a finite field.

From the exact sequence above we deduce an exact sequence of Tate modules

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \to 0.$$

We call a *homological lifting* a linear map

$$\lambda : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$$

splitting this exact sequence. A homological lifting induces an algebra homomorphism $\lambda : \Lambda_A \to \Lambda_P$. The set of homological liftings of the abelian part forms an affine space, a torsor under the $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{Hom}(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A), \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R))$.

Proposition 7.5.3. **—** *Let P be a smooth connected commutative algebraic group defined over a finite field k. Let*

$$1 \to R \to P \to A \to 1$$

be the canonical exact sequence realizing P as an extension of an abelian variety A by an affine group R. Then there exist an integer $N > 0$ and a homomorphism $a : A \to P$ such that composing it with $P \to A$ yields multiplication by N in A. We then say that a is a quasi-lifting.

Moreover, if $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a) : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ is the map induced on Tate modules, then $N^{-1}\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)$ is the unique homological lifting compatible with the action of $\mathrm{Gal}(\bar{k}/k)$ *on* $\mathrm{T}(P)$ *and* $\mathrm{T}(A)$. *We say that $N^{-1}\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)$ is the* canonical lifting.

Proof. — By [**69**, p. 184], the group of extensions of an abelian variety A by $\mathbb{G}_m$ is identified with the group of k-points of the dual abelian variety. The group of extensions of an abelian variety by $\mathbb{G}_a$ is the finite-dimensional k-vector space $\mathrm{H}^1(A, \mathcal{O}_A)$. A smooth connected commutative affine group is isomorphic to a product of copies of $\mathbb{G}_m$ and $\mathbb{G}_a$ after passing to a finite extension of the base field; it follows that the group of extensions of an abelian variety A by a smooth commutative affine group R defined over a finite field is a finite group. The first assertion follows.

The homological lifting $N^{-1}\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)$ is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$, since the lifting a is defined over k. The Frobenius element $\sigma \in \mathrm{Gal}(\bar{k}/k)$ acts on $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A)$ with eigenvalues of absolute value $|k|^{-1/2}$, and on $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R)$ with eigenvalues of absolute value $|k|^{-1}$. Hence $N^{-1}\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)$ is the unique homological lifting compatible with the action of $\mathrm{Gal}(\bar{k}/k)$. □

Corollary 7.5.4. **—** *Let $\bar{k}$ be an algebraically closed field. Let M be a $\bar{k}$-scheme of finite type. Let P be a smooth connected commutative $\bar{k}$-group scheme acting on M with affine stabilizers. Let A be the maximal abelian quotient of P. Suppose that P is defined over a finite field, so* *that we have a quasi-lifting $a : A \to P$. Then the Λ_A-module deduced from the Λ_P-module $\bigoplus_n \mathrm{H}^n_c(M)$ is a free module.*

Proof. — The quasi-lifting $a : A \to P$ defines an action of A on M with finite stabilizers. We are thus immediately reduced to Proposition 7.5.1. □

Here is a more general and more satisfactory statement. The proof that follows is due to Deligne *cf.* [**20**]. Note that this statement will not be used in what follows; in fact, Lemma 7.4.11 will play the role of the independence of the lifting in the proof of 7.4.10.

Proposition 7.5.5. — *Let $\bar{k}$ be an algebraically closed field. Let M be a projective $\bar{k}$-scheme. Let P be a smooth connected commutative $\bar{k}$-group acting on M with affine stabilizers. Let A be the maximal abelian quotient of P and let $\lambda : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ be an arbitrary homological lifting. Then the Λ_A-module deduced from the Λ_P-module $\bigoplus_n \mathrm{H}^n_c(M)$ by means of λ is a free module.*

Proof. — By the general process of passing to the inductive limit as in [**7**, 6.1.7], we reduce to the case where M, P and the action of P on M are defined over a finite field k. Write $\bar{k}$ as an inductive limit $(A_i)_{i\in I}$ of rings of finite type over $\mathbb{Z}$. The category of $\bar{k}$-schemes of finite type is a 2-inductive limit of the categories of A_i-schemes of finite type; that is, for every $X/\bar{k}$ there exist $i \in I$ and X_i/A_i of finite type such that $X = X_i \otimes_{A_i} \bar{k}$ *cf.* [**30**, 8.9.1], and for all $X_i, Y_i/A_i$ one has *cf.* [**30**, 8.8.2]

$$\mathrm{Hom}_{\bar{k}}(X_i \otimes_{A_i} \bar{k}, Y_i \otimes_{A_i} \bar{k}) = \varinjlim_{j\geq i} \mathrm{Hom}_{A_j}(X_i \otimes_{A_i} A_j, Y_i \otimes_{A_i} A_j).$$

The same statement therefore holds for the category of algebraic groups and for that of triples consisting of an algebraic group G, a scheme X of finite type and an action of G on X. Let $f : X \to Y$ be a morphism in the category of $\bar{k}$-schemes of finite type coming from a morphism $f_i : X_i \to Y_i$ in the category of A_i-schemes. In order for f to have one of the following properties — flat, smooth, affine or projective — it is necessary and sufficient that f_i acquire this property after an extension of scalars $A_i \to A_j$ *cf.* [**30**, 11.2.6 and 8.10.5]. There therefore exist a ring A_i of finite type over $\mathbb{Z}$ contained in $\bar{k}$, a projective A_i-scheme M_i, and a smooth A_i-group scheme P_i, an extension of an abelian scheme by an affine group scheme, acting on M_i with affine stabilizers, such that after the extension of scalars $A_i \to \bar{k}$ we recover the initial situation. We may also assume that the $\mathrm{H}^n(f_{i!}\overline{\mathbb{Q}}_\ell)$ are local systems. We have thus reduced the statement to be proved to the case where the base is the spectrum of a finite field.

We may therefore assume that M, P and the action of P on M are defined over a finite field k. Let A be the abelian quotient of P and $a : A \to P$ the quasi-lifting, which defines a canonical lifting $\lambda_0 : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ of Tate modules. We already know, by the preceding corollary 7.5.4, that the Λ_A-module deduced from the Λ_P-module $\bigoplus_n \mathrm{H}^n_c(M)$ by means of λ_0 is a free module.

Let us now prove the same statement for an arbitrary homological lifting. The space of all homological liftings, pointed by λ_0, is identified with the $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{Hom}(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s), \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s))$, on which the Frobenius element $\sigma \in \mathrm{Gal}(\bar{k}/k)$ acts with eigenvalues of absolute value $|k|^{1/2}$. On this space, the set of those λ for which the Λ_{A_s}-module structure on $\bigoplus_n \mathrm{H}^n_c(M)$ deduced from λ is free forms a Zariski open subset of $\mathrm{Hom}(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s), \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s))$. This open subset is stable under σ and contains the origin λ_0. Its complement is a closed subset, stable under σ, which does not contain λ_0. Since σ acts with eigenvalues of absolute value $|k|^{1/2}$, a closed subset of $\mathrm{Hom}(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s), \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s))$ stable under σ is necessarily stable under the action of $\mathbb{G}_m$. The fact that this closed subset does not contain the origin λ_0 implies that it is empty. □

7.6. Study over a Henselian Base. — In this paragraph we shall study the cap-product over a henselian base and analyse the spectral sequence abutting to the cohomology of the special fiber.

***7.6.1.* —** Let now S be a strictly henselian scheme with a morphism $\epsilon : S \to \mathrm{Spec}(\bar{k})$, where $\bar{k}$ is the residue field of S. Write s for the closed point of S. The fiber $\Lambda_{P,s}$ is then identified with $\epsilon^*\Lambda_P$. By adjunction we deduce a morphism

$$\epsilon^*\Lambda_{P,s} \to \Lambda_P.$$

Restricting the cap-product 7.4.2 to $\epsilon^*\Lambda_{P,s}$, we obtain an arrow

$$\Lambda_{P,s} \otimes f_!\overline{\mathbb{Q}}_\ell \to f_!\overline{\mathbb{Q}}_\ell,$$

which defines an action of the graded algebra $\Lambda_{P,s} = \bigoplus_i \wedge^i \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)[i]$ on the complex $f_!\overline{\mathbb{Q}}_\ell$.

In particular we have a morphism

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \otimes f_!\overline{\mathbb{Q}}_\ell \to f_!\overline{\mathbb{Q}}_\ell[-1].$$

From it we deduce a morphism between the truncations for the perverse t-structure

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \otimes {}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell) \to {}^p\tau^{\leq n-1}(f_!\overline{\mathbb{Q}}_\ell)$$

for every $n \in \mathbb{Z}$. This induces a morphism between the perverse cohomology sheaves in degrees n and $n-1$

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \otimes {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell) \to {}^p\mathrm{H}^{n-1}(f_!\overline{\mathbb{Q}}_\ell),$$

whence an action of $\Lambda_{P,s}$ on the direct sum $\bigoplus_n {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)$.

***7.6.2.* —** This arrow is compatible with the one constructed at the end of 7.4.6. With respect to the decomposition by the support

$${}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell) = \bigoplus_{\alpha\in\mathfrak{B}} K_\alpha^n,$$

it is expressed in terms of a matrix whose entry indexed by a pair (α, α') is an element of

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)^* \otimes \mathrm{Hom}(K_\alpha^n, K_{\alpha'}^{n-1}).$$

The diagonal arrows are compatible with those of 7.4.7, whereas the non-diagonal arrows are here clearly zero, since $\mathrm{Hom}(K_\alpha^n, K_{\alpha'}^{n-1}) = 0$ if $\alpha \neq \alpha'$.

***7.6.3.* —** The filtration ${}^p\tau^{\leq n}(f_!\overline{\mathbb{Q}}_\ell)$ of $f_!\overline{\mathbb{Q}}_\ell$ defines a spectral sequence

$$E_{m,n}^2 = \mathrm{H}^m({}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)_s) \Rightarrow \mathrm{H}_c^{m+n}(M_s)$$

which is equivariant for the action of $\Lambda_{P,s}$. This spectral sequence is convergent because the complex $f_!\overline{\mathbb{Q}}_\ell$ is bounded. We therefore have, on the direct sum

$$\mathrm{H}_c^\bullet(M_s) = \bigoplus_r \mathrm{H}_c^r(M_s)[-r],$$

a decreasing filtration $\mathrm{F}^m\mathrm{H}_c(M_s)$ such that

$$\mathrm{F}^m\mathrm{H}_c(M_s)/\mathrm{F}^{m+1}\mathrm{H}_c(M_s) = \bigoplus_n E_{m,n}^\infty[-m-n].$$

The action of $\Lambda_{P,s}$ on $\mathrm{H}_c^\bullet(M_s)$ respects this filtration. The induced action on the graded pieces associated with the filtration $\mathrm{F}^m\mathrm{H}_c(M_s)$

$$\bigoplus_n E_{m,n}^\infty[-m-n]$$

is deduced from its action on the $E_{m,n}^2$, which in turn is deduced from the graded action of $\Lambda_{P,s}$ on $\bigoplus_n {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)[-n]$.

7.6.4. — Let us now place ourselves under the hypotheses of 7.2.2. In particular we have $f_! = f_*$. Having purity at our disposal as well, there exists a non-canonical isomorphism over $S \otimes_k \bar{k}$

$$f_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{n\in\mathbb{Z}} {}^p\mathrm{H}^n(f_!\overline{\mathbb{Q}}_\ell)[-n].$$

This isomorphism persists when restricted to the strict henselization S_s of a geometric point s of S, so that the spectral sequences 7.6.3 degenerate at E^2, that is, $E^\infty_{m,n} = E^2_{m,n}$.

7.6.5. — At the level of the perverse cohomology sheaves we have a canonical decomposition by the support

$${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell) = \bigoplus_{\alpha\in\mathfrak{B}} K^n_\alpha.$$

We observed in 7.6.2 that over a strictly henselian base the action of the graded algebra $\Lambda_{P,s}$ on the direct sum $\bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ respects the decomposition according to the support. It is very tempting to look for an isomorphism

$$f_*\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$$

better than the others, which would in particular be compatible with the action of $\Lambda_{P,s}$ defined in 7.6.1. In that case one could use the freeness of the abutment of the spectral sequence to conclude the proof of 7.4.10. However, a direct sum decomposition of $f_*\overline{\mathbb{Q}}_\ell$ compatible with the action of $\Lambda_{P,s}$ does not appear to exist.

One can nevertheless prove 7.4.10 by relying on the degeneration of the spectral sequence and proceeding by induction on the strata. This is what we shall do in the following paragraph.

7.7. Freeness by Induction. — In this paragraph we shall complete the proof of the freeness property 7.4.10, and hence also those of the amplitude inequality 7.3.2 and of the delta inequality 7.2.2.

7.7.1. — We prove Proposition 7.4.10 by a descending induction on the dimension of Z_α. Let $\alpha_0 \in \mathfrak{B}$ be the maximal element, so that Z_{α_0} is the whole of $S \otimes_k \bar{k}$. Let V_{α_0} be a dense open subset of $S \otimes_k \bar{k}$ small enough in the sense of 7.4.8. For $\alpha \neq \alpha_0$, the other strata Z_α having no intersection with V_{α_0}, the restriction of the perverse cohomology sheaf ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ to V_{α_0} is a local system with a shift

$${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)|_{U_{\alpha_0}} = \mathcal{K}^n_{\alpha_0}[\dim(S)].$$

Here $\mathcal{K}^n_{\alpha_0}$ is a semisimple local system on U_{α_0}, pure of weight n by the purity hypothesis of 7.2.2.

As in 7.4.8, over the open subset V_{α_0} the sheaf of Tate modules $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ is devissed into an exact sequence

$$0 \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_{\alpha_0}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\alpha_0}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\alpha_0}) \to 0,$$

where $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\alpha_0})$ is a pure local system of weight -1 and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_{\alpha_0})$ is a pure local system of weight -2. As explained in 7.4.9, the action of the Tate module

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\alpha_0}) \otimes \mathcal{K}^n_{\alpha_0} \to \mathcal{K}^{n-1}_{\alpha_0}$$

factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\alpha_0})$ for reasons of weight.

The first step of the induction consists in showing that for every geometric point u_{α_0} of V_{α_0}, the resulting $\Lambda_{A_{\alpha_0},u_{\alpha_0}}$-module structure on the direct sum

$$\mathrm{H}^\bullet(M_{u_{\alpha_0}}) := \bigoplus_n \mathrm{H}^n(M_{u_{\alpha_0}})[-n] = \bigoplus_n \mathcal{K}^n_{\alpha_0,u_{\alpha_0}}[-n+\dim(S)]$$

makes it a free module.

By 7.4.11, this assertion is independent of the choice of u_{α_0}. We may in fact assume that the geometric point u_{α_0} is defined over a finite field. As in 7.5.3, there then exists a quasi-lifting $A_{\alpha_0} \to P_{\alpha_0}$, so that the action of $\Lambda_{A_{\alpha_0},u_{\alpha_0}}$ on $\mathrm{H}^\bullet(M_{u_{\alpha_0}})$ comes from a geometric action of $A_{\alpha_0,u_{\alpha_0}}$ on $M_{u_{\alpha_0}}$. We now use the hypothesis that the fiber $M_{u_{\alpha_0}}$ is projective to conclude that $\mathrm{H}^\bullet(M_{u_{\alpha_0}})$ is a free module over $\Lambda_{A_{\alpha_0},u_{\alpha_0}}$, as in Lemma 7.5.1.

7.7.2. — We shall use the freeness property of the cohomology of the fiber M_{u_α} to deduce the freeness of the graded local system $\mathcal{K}_\alpha$ as a Λ_{A_α}-module. The difficulty is to control the noise caused by the $\mathcal{K}_{\alpha'}$ with $\alpha' \in \mathfrak{B}$ such that $Z_{\alpha'}$ is strictly contained in Z_α. Note that by induction we may assume that for such α', $\mathcal{K}_{\alpha'}$ is a free module over $\Lambda_{A_{\alpha'}}$.

Take a geometric point u_α of V_α lying over a point with values in a finite field. Let S_{u_α} be the strict henselization of S at u_α. The construction of 7.6.1 applies to S_{u_α}. We therefore have an action of Λ_{P,u_α} on the restriction of $f_*\overline{\mathbb{Q}}_\ell$ to S_{u_α}

$$\Lambda_{P,u_\alpha} \otimes (f_*\overline{\mathbb{Q}}_\ell|_{S_{u_\alpha}}) \to (f_*\overline{\mathbb{Q}}_\ell|_{S_{u_\alpha}}). \tag{7.7.3}$$

As in 7.6.1, this induces a graded action of Λ_{P,u_α} on the direct sum of perverse cohomology sheaves, whose degree -1 part reads

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha}) \otimes {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)|_{S_{u_\alpha}} \to {}^p\mathrm{H}^{n-1}(f_*\overline{\mathbb{Q}}_\ell)|_{S_{u_\alpha}}.$$

By 7.6.2, this arrow is expressed in terms of a diagonal matrix with respect to the canonical decomposition of ${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)$ and ${}^p\mathrm{H}^{n-1}(f_*\overline{\mathbb{Q}}_\ell)$ by the support

$${}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell) = \bigoplus_{\alpha\in\mathfrak{B}} K_\alpha^n.$$

In other words, the direct sum decomposition

$$\bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell) = \bigoplus_{\alpha\in\mathfrak{B}} K_\alpha$$

respects the structure of graded Λ_{P,u_α}-modules.

The geometric point u_α being defined over a finite field, by 7.5.3 there exists a quasi-lifting $A_{u_\alpha} \to P_{u_\alpha}$ which induces a canonical direct sum decomposition

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha}) = \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_{u_\alpha}) \oplus \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}).$$

From it we deduce an arrow

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \otimes {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)|_{S_{u_\alpha}} \to {}^p\mathrm{H}^{n-1}(f_*\overline{\mathbb{Q}}_\ell)|_{S_{u_\alpha}}$$

which decomposes, by what precedes, into a direct sum of arrows

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \otimes K_{\alpha'}^n|_{S_{u_\alpha}} \to K_{\alpha'}^{n-1}|_{S_{u_\alpha}}$$

over the set of indices $\alpha' \in \mathfrak{B}$ such that $Z_{\alpha'}$ is contained in Z_α.

Proposition 7.7.4. — *For every $\alpha' \in \mathfrak{B}$ such that $Z_{\alpha'}$ is strictly contained in Z_α, and for every integer m, the graded $\overline{\mathbb{Q}}_\ell$-vector space*

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{H}^m(K_{\alpha',u_\alpha}^n)[-n]$$

is a free $\Lambda_{A_{u_\alpha}}$-module.

Proof. — Over $V_{\alpha'}$ we have a canonical arrow between graded local systems

$$\Lambda_{A_{\alpha'}} \otimes \mathcal{K}_{\alpha'} \to \mathcal{K}_{\alpha'}$$

defined in 7.4.9. By the induction hypothesis, for every geometric point $u_{\alpha'}$ of $V_{\alpha'}$ the fiber of $\mathcal{K}_{\alpha'}$ at $u_{\alpha'}$ is a free $\Lambda_{A_{\alpha'},u_{\alpha'}}$-module. By the decomposition theorem, $\mathcal{K}_{\alpha'}$ is a semisimple graded local system. Applying Lemma 7.4.11, we know that there exist a graded local system $E_{\alpha'}$ on $V_{\alpha'}$ and an isomorphism

$$\mathcal{K}_{\alpha'} \simeq \Lambda_{A_{\alpha'}} \otimes E_{\alpha'}$$

as graded $\Lambda_{A_{\alpha'}}$-modules.

This factorization continues to exist on the intersection $V_{\alpha'} \cap S_{u_\alpha}$. Write $y_{\alpha'}$ for the generic point of $V_{\alpha'} \cap S_{u_\alpha}$ and $\bar{y}_{\alpha'}$ for a geometric point above $y_{\alpha'}$. We then have an isomorphism of representations of $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$

$$L_{\alpha',\bar{y}_{\alpha'}} = \Lambda_{A_{\alpha',\bar{y}_{\alpha'}}} \otimes E_{\alpha',\bar{y}_{\alpha'}}.$$

Lemma 7.7.5. — *Under the hypothesis that* $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ *is polarizable as in 7.1.4, for every homological lifting* $\beta : \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha})$, *the map*

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$$

obtained by composing β *with the specialization arrow* $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\bar{y}_{\alpha'}})$ *followed by the canonical projection* $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\bar{y}_{\alpha'}}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$ *is injective. Moreover, there exists a complement of* $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha})$ *in* $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$ *which is* $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$*-stable.*

Proof. — The specialization arrow $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\bar{y}_{\alpha'}})$ is compatible with the alternating polarization form. Any homological lifting $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha})$ is compatible with the alternating form, which vanishes on the affine part $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_{u_\alpha})$, so that the map $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{\bar{y}_{\alpha'}})$ deduced from it is too. It follows that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$ is injective and that the orthogonal of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha})$ in $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$ is a $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$-stable complement. □

Let us continue the proof of 7.7.4. The direct sum decomposition 7.7.5

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}}) = \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \oplus \mathrm{U}$$

of representations of $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$ induces an isomorphism of representations of $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$

$$\Lambda_{A_{\bar{y}_{\alpha'}}} = \Lambda_{A_{u_\alpha}} \otimes \Lambda(\mathrm{U}),$$

where $\Lambda(\mathrm{U}) = \bigoplus_i \wedge^i(\mathrm{U})[i]$. This implies a factorization as a tensor product of representations of $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$

$$\mathcal{K}_{\alpha',\bar{y}_{\alpha'}} = \Lambda_{A_{u_\alpha}} \otimes \Lambda(\mathrm{U}) \otimes E_{\alpha',\bar{y}_{\alpha'}}.$$

There therefore exists an isomorphism

$$\mathcal{K}_{\alpha'}|_{V_{\alpha'} \cap S_{u_\alpha}} = \Lambda_{A_{u_\alpha}} \boxtimes E'_{\alpha'},$$

where $E'_{\alpha'}$ is a local system on $V_{\alpha'} \cap S_{u_\alpha}$. Since the external tensor product with $\Lambda_{A_{u_\alpha}}$ commutes with the intermediate extension from $V_{\alpha'} \cap S_{u_\alpha}$ to $Z_{\alpha'} \cap S_{u_\alpha}$ and with the fiber functor at u_{α_0}, Proposition 7.7.4 follows. □

Consider now the spectral sequence 7.6.3

$$E_2^{m,n} = \mathrm{H}^m({}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)_{u_\alpha}) \Rightarrow \mathrm{H}^{m+n}(M_{u_\alpha}),$$

which degenerates at E_2 by 7.6.4. We thus obtain a filtration of

$$\mathrm{H} = \bigoplus_j \mathrm{H}^j(M_{u_\alpha})[-j]$$

whose m-th graded piece is

$$\mathrm{H}^m\Big(\bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)_{s_0}[-n]\Big)[-m].$$

This filtration is stable under the action of $\Lambda_{A_{u_\alpha}}$. Its action on the m-th graded piece is deduced from the action of $\Lambda_{A_{u_\alpha}}$ on the direct sum

$$\bigoplus_n {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell)[-n]_{S_{u_\alpha}}$$

and hence from that on the $K_{\alpha'}|_{S_{u_\alpha}}$. This m-th graded piece therefore decomposes into a direct sum of graded $\Lambda_{A_{u_\alpha}}$-modules

$$\mathrm{H}^m\Big(\bigoplus_{\alpha'}\bigoplus_{n\in\mathbb{Z}} K^n_{\alpha',u_\alpha}[-n]\Big)[-m].$$

For $\alpha'\neq\alpha$, we already know that $\mathrm{H}^m(\bigoplus_{n\in\mathbb{Z}} K^n_{\alpha',u_\alpha}[-n])$ is a free $\Lambda_{A_{u_\alpha}}$-module *cf.* 7.7.4. For $\alpha'=\alpha$, we have $\mathrm{H}^m(K_{\alpha,u_\alpha})=0$ except for $m=-\dim(Z_\alpha)$. We thus obtain a filtration of H by $\Lambda_{A_{u_\alpha}}$-submodules

$$0\subset \mathrm{H}'\subset\mathrm{H}''\subset\mathrm{H}=\bigoplus_j \mathrm{H}^j(M_{u_\alpha})$$

such that H' and H/H'' are free $\Lambda_{A_{u_\alpha}}$-modules and such that

$$\mathrm{H}''/\mathrm{H}'=L_{\alpha,u_\alpha}.$$

By 7.5.4, we know that H is also a free $\Lambda_{A_{u_\alpha}}$-module. We shall deduce from this that L_{α,u_α} is also a free $\Lambda_{A_{u_\alpha}}$-module, by means of a particular property of the ring $\Lambda_{A_{u_\alpha}}$.

Since $\Lambda_{A_{u_\alpha}}$ is a local algebra, every projective $\Lambda_{A_{u_\alpha}}$-module is free. The exact sequence

$$0\to\mathrm{H}''\to\mathrm{H}\to\mathrm{H}/\mathrm{H}''\to 0,$$

with H and H/H'' free, implies that H'' is also free.

Note also that $\Lambda_{A_{u_\alpha}}$ is a local $\overline{\mathbb{Q}}_\ell$-algebra of finite dimension having a socle of dimension one. From this we deduce that the dual $(\Lambda_{A_{u_\alpha}})^*$ of $\Lambda_{A_{u_\alpha}}$ as a $\overline{\mathbb{Q}}_\ell$-vector space is a free $\Lambda_{A_{u_\alpha}}$-module. Consider the dual exact sequence

$$0\to(\mathrm{H}''/\mathrm{H}')^*\to(\mathrm{H}'')^*\to(\mathrm{H}')^*\to 0,$$

where $(\mathrm{H}'')^*$ and $(\mathrm{H}')^*$ are free $\Lambda_{A_{u_\alpha}}$-modules. It follows that $(\mathrm{H}''/\mathrm{H}')^*$ is a free $\Lambda_{A_{u_\alpha}}$-module, so that H''/H' is one too.

7.8. The Case of the Hitchin Fibration. — Let us apply the result of this section to the case of the Hitchin fibration, more precisely to the morphism $\tilde f^{\mathrm{ani}}:\tilde{\mathcal{M}}^{\mathrm{ani}}\to\tilde{\mathcal{A}}^{\mathrm{ani}}$ equipped with the action of $\tilde g^{\mathrm{ani}}:\tilde{\mathcal{P}}^{\mathrm{ani}}\to\tilde{\mathcal{A}}^{\mathrm{ani}}$. Here $\tilde{\mathcal{M}}^{\mathrm{ani}}$ and $\tilde{\mathcal{P}}^{\mathrm{ani}}$ are not schemes but Deligne-Mumford stacks of finite type *cf.* 6.1.3. We know that $\tilde{\mathcal{P}}^{\mathrm{ani}}$ is smooth over $\tilde{\mathcal{A}}^{\mathrm{ani}}$ by 4.3.5 and that $\tilde{\mathcal{M}}^{\mathrm{ani}}$ is smooth over k by 4.14.1. By 4.15.2, the action of $\tilde{\mathcal{P}}^{\mathrm{ani}}$ on $\tilde{\mathcal{M}}^{\mathrm{ani}}$ has affine stabilizers. By 4.16.1, the morphism $\tilde f^{\mathrm{ani}}$ is flat of relative dimension d, and the same holds for $\tilde g^{\mathrm{ani}}$. By 5.5.4, the sheaf $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$ is a finite quotient of the constant sheaf $\mathbb{X}_*$. The Tate module is polarizable by 4.12.1. The morphism $\tilde f^{\mathrm{ani}}$ is proper without being projective; nevertheless $\tilde{\mathcal{M}}^{\mathrm{ani}}$ is homeomorphic to a projective scheme $\bar{\mathcal{M}}^{\mathrm{ani}}$ over $\tilde{\mathcal{A}}^{\mathrm{ani}}$ by 6.1.3.

There are two ways of applying the inequality 7.2.3 to our situation. The first is to replace $\tilde{\mathcal{M}}^{\mathrm{ani}}$ by $\bar{\mathcal{M}}^{\mathrm{ani}}$, which is projective over $\tilde{\mathcal{A}}^{\mathrm{ani}}$. The constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\bar{\mathcal{M}}^{\mathrm{ani}}$ is pure, because the stack $\tilde{\mathcal{M}}^{\mathrm{ani}}$, which is homeomorphic to it, is smooth. One may replace the Deligne-Mumford Picard stack $\tilde{\mathcal{P}}^{\mathrm{ani}}$ by the group scheme $\mathcal{P}^{\infty,\mathrm{ani}}$ defined by means of a rigidifier 4.5.6. Here $\mathcal{P}^{\infty,\mathrm{ani}}$ has relative dimension larger than d, but adapting 7.2.3 to this situation is merely a matter of numerology.

In fact, there is a much more economical way to proceed. It consists in observing that the entire proof of 7.2.3 adapts as it stands to Deligne-Mumford stacks, with the exception of Lemma 7.5.1, where we assumed that the fiber $\mathcal{M}_a$ is projective. But by 4.15.2 we know that $\mathcal{M}_a$ is homeomorphic to a projective scheme, so that 7.2.3 applies. Note also that instead of using

4.15.2 and 7.5.1 in combination, it is even more economical to apply the product formula 4.15.1 directly in order to obtain the conclusion of 7.5.1. In the situation of the Hitchin fibration, where a product formula is available, paragraph 7.5 is in fact not necessary.

7.8.1. — We have a stratification *cf.* 5.6.6

$$\tilde{\mathscr{A}}^{\mathrm{ani}} = \bigsqcup_{\delta\in\mathbb{N}} \tilde{\mathscr{A}}^{\mathrm{ani}}_{\delta}$$

such that for every $a \in \mathscr{A}^{\mathrm{ani}}_{\delta}(\bar{k})$ the dimension of the affine part of $\mathscr{P}_a$ equals δ. We conjecture that for every δ we have

$$\mathrm{codim}(\tilde{\mathscr{A}}^{\mathrm{ani}}_{\delta}) \geq \delta.$$

The strongest evidence in favour of this conjecture is the case of characteristic zero, a proof of which is sketched in [**60**, p. 4]. Failing a proof of this conjecture in characteristic p, we must resort to a stratagem.

7.8.2. — Suppose that there exist integers δ such that $\mathrm{codim}(\tilde{\mathscr{A}}^{\mathrm{ani}}_{\delta}) < \delta$; we shall write $\delta^{\mathrm{bad}}_G(D)$ for the smallest of them. It is important to note the dependence of $\delta^{\mathrm{bad}}_G(D)$, if it exists, on G and on the invertible bundle D. By 5.7.2, we know that for a fixed group G the integer $\delta^{\mathrm{bad}}_G(D)$ tends to infinity with $\deg(D)$. We write

$$\tilde{\mathscr{A}}^{\mathrm{bad}} = \bigsqcup_{\delta\geq\delta^{\mathrm{bad}}_G(D)} \tilde{\mathscr{A}}^{\mathrm{ani}}_{\delta}$$

and

$$\tilde{\mathscr{A}}^{\mathrm{good}} = \tilde{\mathscr{A}}^{\mathrm{ani}} - \tilde{\mathscr{A}}^{\mathrm{bad}}.$$

By construction, over $\tilde{\mathscr{A}}^{\mathrm{good}}$ the morphisms $\tilde{f}^{\mathrm{ani}}$ and $\tilde{g}^{\mathrm{ani}}$ form a δ-regular abelian fibration.

Theorem 7.8.3. — *Let ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ be the largest direct factor of ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)$ on which $\mathbb{X}_*$ acts trivially. Let K be a geometrically irreducible perverse factor of ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ and let Z be the support of K. Suppose that $Z \cap \tilde{\mathscr{A}}^{\mathrm{good}} \neq \emptyset$. Then $Z = \tilde{\mathscr{A}}^{\mathrm{ani}}$.*

Proof. — Since $\tilde{\mathscr{P}}$ is δ-regular over $\tilde{\mathscr{A}}^{\mathrm{good}}$, we have the inequality $\mathrm{codim}(Z) \geq \delta_Z$. Applying 7.2.3 with trivial κ, we obtain on the one hand the equality $\mathrm{codim}(Z) = \delta_Z$, and on the other hand the existence of an open subset U of $\tilde{\mathscr{A}}^{\mathrm{ani}}$ and a local system L on $U \cap Z$ such that i_*L, where i is the closed immersion $Z \cap U \to U$, is a direct factor of $\mathrm{H}^{2d}(f_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}|_U$. But we know by 6.5.1 that the latter is a constant sheaf on $\tilde{\mathscr{A}}^{\mathrm{ani}}$, so that $U \cap Z = U$ and therefore $Z = \tilde{\mathscr{A}}^{\mathrm{ani}}$. □

7.8.4. — The application of 7.2.3 to the κ-part is more subtle, because a priori it is possible that $\tilde{\mathscr{A}}^{\mathrm{good}} \cap \tilde{\mathscr{A}}_\kappa = \emptyset$. Indeed, $\tilde{\mathscr{A}}_\kappa$ is the union of the $\tilde{\mathscr{A}}_H$, where H runs over the endoscopic groups associated with the pointed endoscopic data $(\kappa, \rho^\bullet_\kappa)$, and the codimension of $\tilde{\mathscr{A}}_H$ in $\tilde{\mathscr{A}}$ is a multiple of $\deg(D)$. Nevertheless, it is not necessary to know that the abelian fibration is δ-regular in order to apply the inequalities 7.2.2 and 7.2.3; it in fact suffices to know the inequality $\mathrm{codim}(Z) \geq \delta_Z$ for the possible supports Z.

Replacing G by an endoscopic group H, we have an integer $\delta^{\mathrm{bad}}_H(D)$, a closed subscheme

$$\tilde{\mathscr{A}}^{\mathrm{bad}}_H = \bigsqcup_{\delta_H\geq\delta^{\mathrm{bad}}_H(D)} \tilde{\mathscr{A}}^{\mathrm{ani}}_{\delta_H}$$

and its complementary open subset $\tilde{\mathscr{A}}^{\mathrm{good}}_H = \tilde{\mathscr{A}}_H - \tilde{\mathscr{A}}^{\mathrm{bad}}_H$.

Theorem 7.8.5. — *Let ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_\kappa$ be the largest direct factor of ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)$ on which $\mathbb{X}_*$ acts through the character κ. Let K be a geometrically irreducible perverse factor of ${}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_\kappa$ and let Z be the support of K. Then there exists a pointed endoscopic datum $(\kappa, \rho^\bullet_\kappa)$ such that Z is contained in $\tilde{\nu}(\tilde{\mathscr{A}}_H)$, where H is the endoscopic group attached to $(\kappa, \rho^\bullet_\kappa)$ and $\tilde{\nu} : \tilde{\mathscr{A}}_H \to \tilde{\mathscr{A}}$ is the closed immersion deduced from it.*

If moreover $Z \cap \tilde{\nu}(\tilde{\mathscr{A}}^{\mathrm{good}}_H) \neq \emptyset$, then $Z = \tilde{\nu}(\tilde{\mathscr{A}}^{\mathrm{ani}}_H)$.

Proof. — We know that $Z \subset \tilde{\mathscr{A}}_\kappa$, where *cf.* 6.2.2

$$\tilde{\mathscr{A}}_\kappa = \bigcup_{(\kappa,\rho^\bullet_{\kappa,\xi})} \tilde{\nu}(\tilde{\mathscr{A}}_{H_\xi}),$$

so that the irreducible closed subset Z is contained in one of the irreducible components $\tilde{\nu}(\tilde{\mathscr{A}}_{H_\xi})$. We shall simply write H for the endoscopic group attached to the endoscopic datum $(\kappa, \rho^\bullet_\kappa)$ such that $Z \subset \tilde{\nu}(\tilde{\mathscr{A}}_H)$. Let Z_H be the closed subset of $\tilde{\mathscr{A}}_H$ such that $Z = \tilde{\nu}(Z_H)$.

Suppose that $Z_H \cap \tilde{\mathscr{A}}_H^{\mathrm{good}} \neq \emptyset$; then we have the inequality

$$\mathrm{codim}(Z_H) \geq \delta_{H,Z_H},$$

where δ_H is the delta function for the Picard stack $\tilde{\mathscr{P}}_H$ over $\tilde{\mathscr{A}}_H$. By 4.17.3 and 4.17.1, we know that for every $a_H \in \tilde{\mathscr{A}}_H^{\mathrm{ani}}(\bar{k})$ the difference

$$\delta(\tilde{\nu}(a_H)) - \delta_H(a_H)$$

is independent of a_H and equals exactly the codimension of $\tilde{\nu}(\tilde{\mathscr{A}}_H)$ in $\tilde{\mathscr{A}}$. We therefore have

$$\mathrm{codim}(Z) \geq \delta_H(a_H) + \mathrm{codim}(\tilde{\nu}(\tilde{\mathscr{A}}_H)) = \delta_Z.$$

Applying 7.2.3, there exist an open subset U of $\tilde{\mathscr{A}}^{\mathrm{ani}}$ and a non-trivial local system L on $Z \cap U$ such that $i_* L$, where i is the closed immersion $i : Z \cap U \to U$, is a direct factor of $\mathrm{H}^{2d}(\tilde{f}^{\mathrm{ani}}_* \overline{\mathbb{Q}}_\ell)_\kappa$. By 6.5.1, we then know that $Z \cap U = \tilde{\nu}(\tilde{\mathscr{A}}_H) \cap U$, so that finally $Z = \tilde{\nu}(\tilde{\mathscr{A}}_H)$. □

8. Point Counting

In this last chapter, we complete the proof of the geometric stabilization theorem 6.4.2 as well as that of the Langlands-Shelstad conjectures 1.11.1 and Waldspurger 1.12.7 by relying on theorem 7.8.5.

The proof is based on the following principle. Let K_1, K_2 be two pure complexes on an irreducible k-scheme of finite type S. Suppose that every geometrically simple perverse sheaf present in K_1 or K_2 has support S entirely, then to demonstrate that K_1 and K_2 have the same class in the Grothendieck group, it suffices to demonstrate that there exists a dense open set U of S such that for any finite extension k' of k, for any $u \in U(k')$, the traces of the Frobenius $\sigma_{k'}$ of k' on $K_{1,u}$ and $K_{2,u}$ are equal. The equality in the Grothendieck group implies that we have the same equality but for every point $s \in S(k')$. Point counting in a fiber above a point of a small open set U should be much more agreeable than in any fiber, which is the strength of the support theorem.

We will therefore count the number $\sharp\mathscr{M}_a(k)$ of k-points in an anisotropic Hitchin fiber. More precisely, we want a formula for the stable part $\sharp\mathscr{M}_a(k)_{\mathrm{st}}$. The product formula 4.15.1 allows us to express this number as a product of the number $\sharp\mathscr{P}^0_a(k)$ of k-points of the neutral component of $\mathscr{P}_a$ and the numbers of points in quotients of affine Springer fibers *cf.* 8.4. For these quotients of affine Springer fibers, a more or less direct counting yields a stable orbital integral *cf.* 8.2. We also have the counting variant with a κ-weighting which gives rise to κ-orbital integrals. Each time, it is a matter of counting points of an Artin stack of the form $[M/P]$ where M is a k-scheme and P is a k-algebraic group acting on M. This formalism is recalled in 8.1 along with an ad hoc fixed-point formula which was proved in appendix A.3 of [**54**].

For $a \in \mathscr{A}^\diamond(k)$, the quotients of affine Springer fibers are all trivial such that $\sharp\mathscr{M}_a(k)_{\mathrm{st}}$ is equal to the number $\sharp\mathscr{P}^0_a(k)$ where $\mathscr{P}^0_a$ is essentially an abelian variety. This allows us to prove the equality of the numbers $\sharp\mathscr{M}_{1,a}(k)_{\mathrm{st}}$ and $\sharp\mathscr{M}_{2,a}(k)_{\mathrm{st}}$ for $a \in \mathscr{A}^\diamond(k)$ associated with two groups G_1 and G_2 having isogenous root data. The support theorem 7.8.3 allows extending the identity to $a \in (\mathscr{A}^{\mathrm{ani}} - \mathscr{A}^{\mathrm{bad}})(k)$. We thus obtain enough global points to obtain all local stable orbital integrals by letting $\deg(D)$ tend to infinity. We thus prove the non-standard fundamental lemma conjectured by Waldspurger *cf.* 8.8. By reversing the global-local argument, we can extend the identity $\sharp\mathscr{M}_{1,a}(k)_{\rm st} = \sharp\mathscr{M}_{2,a}(k)_{\rm st}$ to all $a\in\mathscr{A}^{\rm ani}(k)$.

The proof of the Langlands-Shelstad conjecture essentially follows the same strategy with a few more technical difficulties. For $a_H\in\tilde{\mathscr{A}}_H^{\diamondsuit}(k)$, the local stable orbital integrals of a_H are trivial but the local κ-orbital integrals in G of the corresponding point a are not necessarily so. However, by shrinking $\tilde{\mathscr{A}}_H^{\diamondsuit}$ even further, we can assume that these non-trivial local κ-orbital integrals are as simple as possible. The calculation of these simple local integrals is well known and reduces to the SL(2) case treated by Labesse and Langlands. We review it in 8.3. By combining this calculation with the support theorem 7.8.5, we obtain the part of the geometric stabilization theorem on the open set $\tilde{\mathscr{A}}_H^{\rm ani}-\tilde{\mathscr{A}}_H^{\rm bad}$ *cf.* 8.5. Again, by letting $\deg(D)$ tend to infinity, we obtain all local orbital integrals and thus prove the fundamental lemma *cf.* 8.6. By reversing the global-local argument, we deduce the entire geometric stabilization theorem.

8.1. General remarks on counting. — In this paragraph, we will fix a framework for the various counting problems that we will have to solve later. We will also clarify the abuses of notation that we will systematically practice in this chapter. In this paragraph and unlike the rest of the article, the letter X does not necessarily designate a curve.

***8.1.1.* —** If M is a k-scheme of finite type. According to the Grothendieck-Lefschetz trace formula, the number of k-points of M can be calculated as an alternating sum of traces of the Frobenius element $\sigma\in\mathrm{Gal}(\bar k/k)$

$$\sharp\, M(k)=\sum_n(-1)^n\mathrm{tr}(\sigma,\mathrm{H}^n_c(M))$$

where $\mathrm{H}^n_c(M)$ denotes the n-th compactly supported cohomology group $\mathrm{H}^n_c(M\otimes_k\bar k,\overline{\mathbb{Q}}_\ell)$ of $M\otimes_k\bar k$.

We will need a variant of this trace formula in the following context. Let M be a k-scheme of finite type or more generally a Deligne-Mumford stack of finite type, equipped with an action of a commutative k-algebraic group of finite type P. We want to relate the trace of σ on a part of the compactly supported cohomology of M with the number of points of the quotient $X=[M/P]$ and the number of points of the neutral component of P. This is the content of appendix A.3 of [**54**] which we will now recall and slightly generalize.

***8.1.2.* —** Let $X=[M/P]$ be as above. We will write X for the groupoid $X(\bar k)$, M for the set $M(\bar k)$ and P for the group $P(\bar k)$. By definition, the set of objects of $X=[M/P]$ is the set M. Let $m_1,m_2\in M$. Then the set of morphisms $\mathrm{Hom}_X(m_1,m_2)$ is the transporter

$$\mathrm{Hom}_X(m_1,m_2)=\{p\in P|pm_1=m_2\}.$$

The composition rule for morphisms is deduced from multiplication in the group P.

***8.1.3.* —** The action of the Frobenius element $\sigma\in\mathrm{Gal}(\bar k/k)$ on M and P induces an action on the groupoid $X=[M/P]$. The groupoid $X(k)$ of fixed points under the action of σ is by definition the category whose

– objects are pairs (m,p) where $m\in M$ and $p\in P$ such that $p\sigma(m)=m$;

– a morphism $h:(m,p)\to(m',p')$ in $X(k)$ is an element $h\in P$ such that $hm=m'$ and $p'=hp\sigma(h)^{-1}$.

The category $X(k)$ has a finite number of isomorphism classes of objects and each object has a finite number of automorphisms. We are interested in the sum

$$\sharp\, X(k)=\sum_{x\in X(k)/\sim}\frac{1}{\sharp\,\mathrm{Aut}_{X(k)}(x)}$$

where x ranges over a set of representatives of the isomorphism classes of objects in $X(k)$.

8.1.4. — Let $X = [M/P]$ as above and let $x = (m, p)$ be an object of $X(k)$. By definition of the morphisms in $X(k)$, the σ-conjugacy class of p depends only on the isomorphism class of x. Since P is a k-group of finite type, the group of σ-conjugacy classes of P is canonically identified with $\mathrm{H}^1(k, P)$. Let $\mathrm{cl}(x) \in \mathrm{H}^1(k, P)$ denote the σ-conjugacy class of p which depends only on the isomorphism class of x. According to a theorem of Lang, $\mathrm{H}^1(k, P)$ is identified with $\mathrm{H}^1(k, \pi_0(P))$ where $\pi_0(P)$ denotes the group of connected components of $P \otimes_k \bar{k}$ which is thus a finite commutative group equipped with an action of σ. Thus, for any σ-invariant character $\kappa : \pi_0(P) \to \overline{\mathbb{Q}}_\ell^\times$, for any object $x \in X(k)$, we can define a pairing

$$\langle \mathrm{cl}(x), \kappa \rangle = \kappa(\mathrm{cl}(x)) \in \overline{\mathbb{Q}}_\ell^\times$$

which depends only on the isomorphism class of x. We can therefore define the number of points of $X(k)$ with κ-weighting

$$\sharp X(k)_\kappa = \sum_{x \in X(k)/\sim} \frac{\langle \mathrm{cl}(x), \kappa \rangle}{\sharp \mathrm{Aut}_{X(k)}(x)}$$

where x ranges over a set of representatives of the isomorphism classes of objects in $X(k)$.
8.1.5. — The group P acts on the cohomology groups $\mathrm{H}^n_c(M)$ through its group of connected components $\pi_0(P)$ according to the homotopy lemma. We can form the eigenspace $\mathrm{H}^n_c(M)_\kappa$ of $\mathrm{H}^n_c(M)$ with eigenvalue κ. Since κ is σ-invariant, σ acts on $\mathrm{H}^n_c(M)_\kappa$.

We have the following variant of the Grothendieck-Lefschetz trace formula proved in appendix A.3 of [**54**].

Proposition 8.1.6. — *Let M be a k-scheme of finite type, P a commutative k-group of finite type acting on M and $X = [M/P]$. Then the category $X(k)$ has a finite number of isomorphism classes of objects and each object has a finite number of automorphisms. For any $\kappa : \pi_0(P) \to \overline{\mathbb{Q}}_\ell^\times$ a σ-invariant character, the number $\sharp X(k)_\kappa$ has the following cohomological interpretation:*

$$\sharp P^0(k) \sharp X(k)_\kappa = \sum_n (-1)^n \mathrm{tr}(\sigma, \mathrm{H}^n_c(M)_\kappa)$$

where P^0 is the neutral component of P.

We refer to [**54**, A.3.1] for the proof. We can consider two instructive examples that explain why we need to separate the role of P^0 from $\pi_0(P)$.

Example 8.1.7. — Suppose $M = \mathrm{Spec}(k)$ and P is a finite group on which σ acts. Let $X = [M/P]$ be the classifying stack of P. Then the objects of the category $X(k)$ are the elements $p \in P$ while the morphisms $p \to p'$ are the elements $h \in P$ such that $p' = hp\sigma(h)^{-1}$. We then have

$$\sharp X(k) = \frac{\sharp P}{\sharp P} = 1.$$

Furthermore, for any σ-invariant character $\kappa : P \to \overline{\mathbb{Q}}_\ell^\times$ which is non-trivial, we have $\sharp X(k)_\kappa = 0$.

Example 8.1.8. — If $M = \mathrm{Spec}(k)$ and $P = \mathbb{G}_m$. Let $X = [M/P]$ be the classifying stack of $\mathbb{G}_m$. The objects of $X(k)$ are again the elements $p \in P$ while the morphisms $p \to p'$ are the elements $h \in P$ such that $p' = hp\sigma(h)^{-1}$. According to Lang, all elements of $\bar{k}^\times$ are σ-conjugate such that there is only one isomorphism class of objects in $X(k)$. The automorphism group of each object in $X(k)$ is $k^\times$ and thus has $q - 1$ elements. We have therefore $\sharp X(k) = (q-1)^{-1}$.

To study point counting in affine Springer fibers, we need a variant of the previous discussion for the case where M and P are locally of finite type and where $[M/P]$ has a reasonable finiteness property.

Let M be a locally finite type k-scheme and P a locally finite type commutative k-group acting on M. For the quotient $[M/P]$ to be reasonably finite, we also make the following hypotheses.

Hypothesis 8.1.9. — (1) The group of connected components $\pi_0(P)$ is a finitely generated abelian group.

(2) The stabilizer in P of each point of M is a subgroup of finite type.

(3) There exists a discrete torsion-free subgroup $\Lambda \subset P$ such that P/Λ and M/Λ are of finite type.

The last hypothesis needs to be commented on. Since the group Λ is torsion-free, it acts without fixed points on M since the stabilizers in P of any point of M is of finite type and in particular have a trivial intersection with Λ. Moreover, if the hypothesis is verified, it is verified for any subgroup $\Lambda' \subset \Lambda$ of finite index and σ-invariant.

8.1.10. — We also need to explain how to construct a discrete torsion-free subgroup Λ of P such that P/Λ is of finite type. The group P admits a canonical short exact sequence

$$1 \to P^{\mathrm{tf}} \to P \to \pi_0(P)^{\mathrm{lib}} \to 0$$

where $\pi_0(P)^{\mathrm{lib}}$ is the maximal free quotient of $\pi_0(P)$ and where P^{tf} is the maximal finite type subgroup of P. Since $\pi_0(P)^{\mathrm{lib}}$ is a free abelian group, there exists a lifting

$$\gamma : \pi_0(P)^{\mathrm{lib}} \to P$$

which is not necessarily σ-equivariant. Since P^{tf} is a k-group of finite type, the restriction of γ to $\Lambda = N\pi_0(P)^{\mathrm{lib}}$ for N sufficiently divisible is σ-equivariant. This gives a discrete torsion-free subgroup P. Hypothesis (3) is equivalent to the quotient of M by this discrete torsion-free subgroup being a k-scheme of finite type.

8.1.11. — Consider the quotient $X = [M/P]$. The groupoid $X(k)$ of fixed points under σ has as objects pairs $x = (m, p)$ with $m \in M$ and $p \in P$ such that $p\sigma(m) = m$. A morphism $h : (m, p) \to (m', p')$ is an element $h \in P$ such that $hm = m'$ and $p' = hp\sigma(h)^{-1}$. Let P_σ be the group of σ-conjugacy classes in P. The σ-conjugacy class of p defines an element $\mathrm{cl}(x) \in P_\sigma$ which depends only on the isomorphism class of x. Since m is defined over a finite extension of k, $\mathrm{cl}(x)$ belongs to the subgroup

$$\mathrm{cl}(x) \in \mathrm{H}^1(k, P)$$

which is the torsion part of P_σ.

Lemma 8.1.12. — *Any character $\kappa : \mathrm{H}^1(k, P) \to \overline{\mathbb{Q}}_\ell^\times$ can be extended to a character of finite order $\tilde{\kappa} : P_\sigma \to \overline{\mathbb{Q}}_\ell^\times$.*

Proof. — Let P^0 be the group of neutral components of P. According to Lang's theorem, every element of P^0 is σ-conjugate to the identity. It follows that the map $P_\sigma \to \pi_0(P)_\sigma$ from P_σ to the group of σ-conjugacy classes of $\pi_0(P)$ is an isomorphism. The characters $\tilde{\kappa} : P_\sigma \to \overline{\mathbb{Q}}_\ell^\times$ are therefore the $\overline{\mathbb{Q}}_\ell$-points of the $\overline{\mathbb{Q}}_\ell$-group diagonalizable of finite type $(\pi_0(P)_\sigma)^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(P)_\sigma])$. The characters $\kappa : \mathrm{H}^1(k, P) \to \overline{\mathbb{Q}}_\ell^\times$ form the group $\pi_0((\pi_0(P)_\sigma)^*)$ of connected components of $(\pi_0(P)_\sigma)^*$. Any element $\kappa \in \pi_0((\pi_0(P)_\sigma)^*)$ can be lifted to a torsion element $\tilde{\kappa} \in (\pi_0(P)_\sigma)^*$. □

The quotient $X = [M/P]$ is clearly equivalent to $[(M/\Lambda)/(P/\Lambda)]$ where M/Λ and P/Λ are of finite type. In particular, the set of isomorphism classes of objects of $X(k)$ is finite and the group of automorphisms of each object is finite. For any character $\kappa : \mathrm{H}^1(k, P) \to \overline{\mathbb{Q}}_\ell^\times$, we can therefore form the finite sum

$$\sharp X(k)_\kappa = \sum_{x \in X(k)/\sim} \frac{\langle \mathrm{cl}(x), \kappa \rangle}{\sharp \mathrm{Aut}(x)}.$$

Let $\tilde{\kappa} : P \to \overline{\mathbb{Q}}_\ell^\times$ be a σ-invariant character of finite order that represents κ. We can then choose a discrete subgroup $\Lambda \subset P$ such that condition (3) of 8.1.9 is satisfied and such that the restriction κ to Λ is trivial. We denote by $\tilde{\kappa}$ the character of P/Λ which is thus deduced. Let

$\mathrm{H}^n_c(M/\Lambda)_{\tilde\kappa}$ be the eigenspace of $\mathrm{H}^n_c(M/\Lambda)$ with eigenvalue $\tilde\kappa$. The following statement is an immediate corollary of 8.1.6.

Proposition 8.1.13. — *We have the formula*

$$\sharp P^0(k)\sharp X(k)_\kappa = \sum_n (-1)^n \mathrm{tr}(\sigma, \mathrm{H}^n_c(M/\Lambda)_{\tilde\kappa}).$$

Furthermore, if $\Lambda' \subset \Lambda$ is a subgroup of maximal rank and σ-invariant then we have a canonical isomorphism

$$\mathrm{H}^n_c(M/\Lambda')_{\tilde\kappa} \to \mathrm{H}^n_c(M/\Lambda)_{\tilde\kappa}$$

for each integer n.

Proof. — Since Λ is a torsion-free subgroup of P, we have that the intersection of Λ with the neutral component P^0 is trivial. Consequently, the homomorphism $P^0 \to P/\Lambda$ induces an isomorphism of P^0 onto the neutral component of P/Λ. By comparing with 8.1.6, we obtain the desired formula. □

In practice, this proposition is useful for comparing and controlling the variation of the sum $\sharp X(k')_\kappa$ when we take points with values in a variable finite extension k' of k. Let $m = \deg(k'/k)$. For any isomorphism class x' of $X(k')$, we have a σ^m-conjugacy class in P. Since $\kappa : P \to \overline{\mathbb{Q}}_\ell^\times$ is σ-invariant, it is a fortiori σ^m-invariant, so that we can define the pairing $\langle \mathrm{cl}(x), \kappa\rangle \in \overline{\mathbb{Q}}_\ell^\times$. We then have the formula

$$\sharp P^0(k')\sharp X(k')_\kappa = \sum_n (-1)^n \mathrm{tr}(\sigma^m, \mathrm{H}^n_c(M/\Lambda)_{\tilde\kappa}).$$

Corollary 8.1.14. — *Let $X = [M/P]$ and $X' = [M'/P']$ be as above. Let $\kappa : P \to \overline{\mathbb{Q}}_\ell^\times$ and $\kappa' : P' \to \overline{\mathbb{Q}}_\ell^\times$ be two σ-invariant characters of finite order. Suppose there exists an integer m such that for any extension k'/k of degree $m' \geq m$, we have*

$$\sharp P^0(k')\sharp X(k')_\kappa = \sharp P'^0(k')\sharp X'(k')_{\kappa'}.$$

Then, we have the equality

$$\sharp P^0(k)\sharp X(k)_\kappa = \sharp P'^0(k)\sharp X'(k)_{\kappa'}.$$

We will now study two twin examples that are among the simplest affine Springer fibers. In these cases, we can directly calculate the numbers $\sharp X(k)_\kappa$.

Example 8.1.15. — Let A be a one-dimensional split torus over k and $\mathbb{X}_*(A)$ its group of cocharacters. Consider an extension

$$1 \to A \to P \to \mathbb{X}_*(A) \to 1.$$

The sheaf $\underline{\mathrm{RHom}}(\mathbb{X}_*(A), A)$ is concentrated in degree 0 with

$$\mathrm{Hom}(\mathbb{X}_*(A), A) = \mathbb{G}_m.$$

Since $\mathrm{H}^1(k, \mathbb{G}_m) = 0$, we can split the exact sequence above and in particular, there exists an isomorphism

$$P \simeq A \times \mathbb{X}_*(A).$$

In both cases, we have an isomorphism $P = \mathbb{G}_m \times \mathbb{Z}$ over $\bar k$.

Consider the infinite chain of $\mathbb{P}^1$ obtained from the disjoint union $\bigsqcup_i \mathbb{P}^n_i$ where $\mathbb{P}^n_i$ denotes the i-th copy of $\mathbb{P}^1$, by gluing the infinite point ∞_i of $\mathbb{P}^1_i$ with the zero point 0_{i+1} of $\mathbb{P}^1_{i+1}$. The group $\mathbb{G}_m \times \mathbb{Z}$ acts on $\bigsqcup_i \mathbb{P}^n_i$ in a compatible way with the gluing so that it still acts on the chain.

Consider a k-scheme M equipped with an action of P such that after base change to $\bar k$, M is isomorphic to the infinite chain of $\mathbb{P}^1$ equipped with the action of $\mathbb{G}_m \times \mathbb{Z}$ as above. We have in particular a stratification $M = M_0 \sqcup M_1$ where M_0 is a principal homogeneous space under P and where M_1 is a principal homogeneous space under the discrete group $\mathbb{X}_*(A)$. The groupoid

$X = [M/P]$ has essentially two objects x_1 and x_0 with $\mathrm{Aut}(x_1) = A$ and $\mathrm{Aut}(x_0)$ trivial. We also have two cohomology classes

$$\mathrm{cl}_1, \mathrm{cl}_0 \in \mathrm{H}^1(k, P) = \mathrm{H}^1(k, \mathbb{X}_*(A)).$$

There are exactly three possibilities for these classes.

(1) If $A = \mathbb{G}_m$ then $\mathbb{X}_*(A) = \mathbb{Z}$. In this case, we have the vanishing of $\mathrm{H}^1(k, \mathbb{X}_*(A))$. We then have the formula

$$\sharp X(k) = 1 + \frac{1}{q-1} = \frac{q}{q-1}$$

which is more agreeable to remember in the form

$$\sharp A(k) \sharp X(k) = q.$$

(2) If A is the one-dimensional non-split torus over k thus having $q + 1$ points with values in k. In this case $\mathbb{X}_*(A)$ is the group $\mathbb{Z}$ equipped with the action of σ given by $m \mapsto -m$. In this case, the group $\mathrm{H}^1(k, \mathbb{X}_*(A))$ has two elements and its elements can be described explicitly as follows. An $\mathbb{X}_*(A)$-torsor E is a principal homogeneous space E under $\mathbb{Z}$ equipped with a map $\sigma : E \to E$ compatible with the action of σ on $\mathbb{Z}$ given above. For any $e \in E$, the difference $\sigma(e) - e$ is an integer whose parity is independent of the choice of e. If $\sigma(e) - e$ is even, $\mathrm{cl}(E)$ is the trivial element of $\mathrm{H}^1(k, \mathbb{X}_*(A))$. If $\sigma(e) - e$ is odd, $\mathrm{cl}(E)$ is the non-trivial element of $\mathrm{H}^1(k, \mathbb{X}_*(A))$.

Suppose now that $\mathrm{cl}_0 = 0$. Then, there exists exactly one copy $\mathbb{P}^1_i$ in the infinite chain stable under σ. Since A is the non-split torus, σ necessarily exchanges 0_i and ∞_i. It follows that $\mathrm{cl}_1 \neq 0$.

If $\kappa : \mathrm{H}^1(k, \mathbb{X}_*(A)) \to \overline{\mathbb{Q}}_\ell^\times$ is the non-trivial character, we then have the formula

$$\sharp X(k)_\kappa = 1 - \frac{1}{q+1} = \frac{q}{q+1}$$

which is more agreeable to remember in the form

$$\sharp A(k) \sharp X(k)_\kappa = q.$$

(3) Suppose always that A is the non-split torus but consider the case where $\mathrm{cl}_0 \neq 0$. There exist then two copies $\mathbb{P}^1_i$ and $\mathbb{P}^1_{i+1}$ exchanged by σ such that in the infinite chain M, the common point $\infty_i = 0_{i+1}$ is σ-invariant. It follows that $\mathrm{cl}_1 = 0$. We have then the formula

$$\sharp A(k) \sharp X(k)_\kappa = -q$$

where $\kappa : \mathrm{H}^1(k, \mathbb{X}_*(A)) \to \overline{\mathbb{Q}}_\ell^\times$ is the non-trivial character.

Assuming that M_0 has at least one k-point, we have $\mathrm{cl}_0 = 0$ so that the third possibility is excluded. With this hypothesis, we observe that the formula

$$\sharp A(k) \sharp X(k)_\kappa = q$$

is then valid.

8.2. Counting in an affine Springer fiber. — In this paragraph, we study point counting in an affine Springer fiber, essentially following [**26**, §15] but with the language developed in the previous paragraph. As in the previous paragraph, the notation $x \in X$ will mean $x \in X(\bar{k})$. When referring to coefficients other than $\bar{k}$, we will specify.

Let $v \in |X|$ be a closed point of X. Let F_v be the completion of the field of rational functions F by the v-adic topology and $\mathcal{O}_v$ the ring of integers of F_v, k_v the residue field. We denote $X_v = \mathrm{Spec}(\mathcal{O}_v)$ and $X_v^\bullet = \mathrm{Spec}(F_v)$. Let $\bar{F}_v$ be the v-adic completion of $\bar{F} = F \otimes_k \bar{k}$. Let $\bar{\mathcal{O}}_v$ be the ring of integers of $\bar{F}_v$.

8.2.1. — Let $a \in \mathfrak{c}^\heartsuit(\mathcal{O}_v)$ be an X_v-point of $\mathfrak{c}$ whose generic fiber is semi-simple and regular. The reduced affine Springer fiber $\mathcal{M}_v^{\mathrm{red}}(a)$ is a locally finite type k-scheme whose set of $\bar{k}$-points is

$$\mathcal{M}_v^{\mathrm{red}}(a) = \{g \in G(\bar{F}_v)/G(\bar{\mathcal{O}}_v) | \mathrm{ad}(g)^{-1}\gamma_0 \in \mathfrak{g}(\bar{\mathcal{O}}_v)\}$$

where $\gamma_0 = \epsilon(a)$ is the Kostant section at point a. Since nilpotents play no role in the following discussion, we will simply write $\mathcal{M}_v(a)$ instead of $\mathcal{M}_v^{\mathrm{red}}(a)$.

8.2.2. — Let $J_a = a^*J$ be the inverse image of the regular centralizer. Let J'_a be a smooth X_v-group scheme with connected fibers and equipped with a homomorphism $J'_a \to J_a$ which induces an isomorphism on the generic fiber. In particular, J'_a can be the group scheme J_a^0 of the neutral components of J_a, but it will be more flexible for applications to consider general J'_a.

8.2.3. — Consider the locally finite type k-group scheme $\mathcal{P}_v^{\mathrm{red}}(J'_a)$ whose set of $\bar{k}$-points is

$$\mathcal{P}_v^{\mathrm{red}}(J'_a) = J_a(\bar{F}_v)/J'_a(\bar{\mathcal{O}}_v).$$

Again, we will simply write $\mathcal{P}_v(J'_a)$ for $\mathcal{P}_v^{\mathrm{red}}(J'_a)$ because nilpotents play no role in the following discussion. The homomorphism $J'_a \to J_a$ induces a homomorphism

$$\mathcal{P}_v(J'_a) \to \mathcal{P}_v(J_a)$$

which induces an action of $\mathcal{P}_v(J'_a)$ on $\mathcal{M}_v(a)$. According to 3.4.1, this action satisfies hypothesis 8.1.9 so that we can consider the number of k-points of the quotient $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]$ as well as the variant with a κ-weighting. According to 8.1.13, we know that these are finite numbers that have a cohomological interpretation. In this paragraph, we are interested in the question of expressing these numbers in terms of stable orbital integrals and κ-orbital integrals.

Let's begin by comparing the κs that appear in the counting problem 8.1 and those that appear in the definition of κ-orbital integrals 1.7.

Lemma 8.2.4. — *Assume that the special fiber of J'_a is connected. There exists a canonical isomorphism*

$$\mathrm{H}^1(F_v, J_a) = \mathrm{H}^1(k, \mathcal{P}_v(J'_a)).$$

Proof. — According to a theorem of Steinberg, $\mathrm{H}^1(\bar{F}_v, J_a) = 0$. It follows that

$$\mathrm{H}^1(F_v, J_a) = \mathrm{H}^1(\mathrm{Gal}(\bar{k}/k), J_a(\bar{F}_v)).$$

According to a theorem of Lang, we have the vanishing $\mathrm{H}^1(k, J'_a(\bar{\mathcal{O}}_v)) = 0$ because J'_a is a smooth group scheme with connected fibers. The lemma follows. □

Proposition 8.2.5. — *Assume the special fiber of J'_a is connected. Let $\kappa : \mathrm{H}^1(F_v, J_a) \to \overline{\mathbb{Q}}_\ell^\times$ be a character and $\kappa : \mathrm{H}^1(k, \mathcal{P}_v(J'_a)) \to \overline{\mathbb{Q}}_\ell^\times$ the resulting character. The number of k-points with κ-weighting of $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]$ can then be expressed as follows*

$$\sharp\, [\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v}, \mathrm{d}t_v).$$

where $1_{\mathfrak{g}_v}$ is the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$ and where $\mathrm{d}t_v$ is any Haar measure of $J_a(F_v)$.

Proof. — Recall the description of the category

$$[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k).$$

An object of this category consists of a pair $x = (m, p)$ formed by an element $m \in \mathcal{M}_v(a)$ and an element $p \in \mathcal{P}_v(J'_a)$ such that the equality $p\sigma(m) = m$ holds. A morphism from (m, p) to (m', p') in this category consists of an element $h \in \mathcal{P}_v(J'_a)$ such that $m' = hm$ and $p' = hp\sigma(h)^{-1}$.

Let $\mathcal{P}_v(J'_a)_\sigma$ be the group of σ-conjugacy classes of $\mathcal{P}_v(J'_a)$ and $\mathrm{H}^1(k, \mathcal{P}_v(J'_a))$ the subgroup of continuous cocycles. For each object $x = (m, p)$ as above, we define $\mathrm{cl}(x) \in \mathcal{P}_v(J'_a)_\sigma$ as the σ-conjugacy class of p which actually belongs to the subgroup

$$\mathrm{cl}(x) \in \mathrm{H}^1(k, \mathcal{P}_v(J'_a)) = \mathrm{H}^1(F_v, J_a).$$

Let γ_0 be the image of a in $\mathfrak{g}$ by the Kostant section. We have a canonical isomorphism $J_a = I_{\gamma_0}$ *cf.* 1.4.3 so that $\mathrm{cl}(x)$ can also be seen as an element of $\mathrm{H}^1(F_v, I_{\gamma_0})$.

Lemma 8.2.6. — *For any object x of $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)$, the class $\mathrm{cl}(x)$ belongs to the kernel of the homomorphism*

$$\mathrm{H}^1(F_v, I_{\gamma_0}) \to \mathrm{H}^1(F_v, G).$$

Proof. — Let $x = (p, m)$ be as above. Let $g \in G(\bar F_v)$ be a representative of $m \in G(\bar F_v)/G(\bar{\mathcal{O}}_v)$ and $j \in I_{\gamma_0}(\bar F_v)$ a representative of p. The equality $p\sigma(m) = m$ implies that

$$g^{-1} j \sigma(g) \in G(\bar{\mathcal{O}}_v)$$

so that this element is σ-conjugate to the neutral element in $G(\bar F_v)$. Thus $\mathrm{cl}(x) \in \mathrm{H}^1(F_v, I_{\gamma_0})$ has a trivial image in $\mathrm{H}^1(F_v, G)$. □

Continue the proof of 8.2.5. For any element ξ in the kernel of $\mathrm{H}^1(F_v, I_{\gamma_0}) \to \mathrm{H}^1(F_v, G)$, consider the subcategory

$$[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]_\xi(k)$$

of objects in $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)$ such that $\mathrm{cl}(x) = \xi$. We propose to write the number of isomorphism classes of $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]_\xi(k)$ as an orbital integral.

Fix $j_\xi \in I_{\gamma_0}(\bar F_v)$ in the σ-conjugacy class ξ. Let (m, p) be an object of $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]_\xi(k)$. Since p is σ-conjugate to j_ξ in $\mathcal{P}_v(J'_a)$, there exists $h \in \mathcal{P}_v(J'_a)$ such that $j_\xi = h^{-1} p \sigma(h)$. Then (m, p) is isomorphic to $(h^{-1}m, j_\xi)$. Consequently, the category $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]_\xi(k)$ is equivalent to the full subcategory formed by objects of the form (m, j_ξ). A morphism $(m, j_\xi) \to (m', j_\xi)$ is an element $h \in \mathcal{P}_v(J'_a)(k)$ such that $hm = m'$. Since J'_a has connected fibers, we have

$$\mathcal{P}_v(J'_a)(k) = J_a(F_v)/J'_a(\mathcal{O}_v).$$

Let (m, j_ξ) be as above. Choose a representative $g \in G(\bar F_v)$ of m. Then $g^{-1} j_\xi \sigma(g) \in G(\bar{\mathcal{O}}_v)$. As all elements of $G(\bar{\mathcal{O}}_v)$ are σ-conjugate to the neutral element, we can choose g with $m = gG(\bar{\mathcal{O}}_v)$ such that

$$g^{-1} j_\xi \sigma(g) = 1.$$

Let $g, g' \in G(\bar F_v)$ be two elements satisfying the above equation and representing the same class $m \in G(\bar F_v)/G(\bar{\mathcal{O}}_v)$. Then $g' = gk$ with $k \in G(\mathcal{O}_v)$ so that the image of g in $G(\bar F_v)/G(\mathcal{O}_v)$ is well-determined by (m, j_ξ).

Thus the category $[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]_\xi(k)$ is equivalent to the category O_ξ whose objects are the elements $g \in G(\bar F_v)/G(\mathcal{O}_v)$ satisfying two equations

(1) $g^{-1} j_\xi \sigma(g) = 1$

(2) $\mathrm{ad}(g)^{-1}\gamma_0 \in \mathfrak{g}(\bar{\mathcal{O}}_v)$

and whose morphisms $g \to g_1$ are the elements $h \in J_a(F_v)/J'_a(\mathcal{O}_v)$ such that $g = hg_1$. Here, to make h act on the left, we used the canonical isomorphism $J_a = I_{\gamma_0}$.

Since the image of ξ in $\mathrm{H}^1(F_v, G)$ is trivial, there exists $g_\xi \in G(\bar F_v)$ such that $g_\xi^{-1} j_\xi \sigma(g_\xi) = 1$. Fix such a G_ξ and set

$$\gamma_\xi = \mathrm{ad}(g_\xi)^{-1}\gamma_0.$$

The equation $g_\xi^{-1} j_\xi \sigma(g_\xi) = 1$ implies that $\gamma_\xi \in G(F_v)$. The $G(F_v)$-conjugacy class of γ_ξ does not depend on the choices of j_ξ and g_ξ but only on the class $\xi \in \mathrm{H}^1(F_v, I_{\gamma_0})$.

By setting $g' = g_\xi^{-1} g$, the category O_ξ can be described as follows. Its objects are the elements $g' \in G(F_v)/G(\mathcal{O}_v)$ satisfying

$$\mathrm{ad}(g')^{-1}(\gamma_\xi) \in \mathfrak{g}(\mathcal{O}_v).$$

Its morphisms $g' \to g'_1$ are the elements $h \in J_a(F_v)/J'_a(\mathcal{O}_v)$ such that $g' = hg'_1$. Here, to make h act on the left, we used the canonical isomorphism $J_a = I_{\gamma_\xi}$ where I_{γ_ξ} is the centralizer of γ_ξ.

The set of isomorphism classes of O_ξ is therefore the set of double-classes

$$g' \in I_{\gamma_\xi}(F_v)\backslash G(F_v)/G(\mathcal{O}_v)$$

such that $\mathrm{ad}(g')^{-1}\gamma_\xi \in \mathfrak{g}(\mathcal{O}_v)$. The automorphism group of g' is the group

$$(I_{\gamma_\xi}(F_v) \cap g'G(\mathcal{O}_v)g'^{-1})/J'_a(\mathcal{O}_v)$$

whose cardinality can be expressed in terms of volumes as follows

$$\frac{\mathrm{vol}(I_{\gamma_\xi}(F_v) \cap g'G(\mathcal{O}_v)g'^{-1}, \mathrm{d}t_v)}{\mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)}.$$

We thus have

$$\sharp\, O_\xi = \sum \frac{\mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)}{\mathrm{vol}(J_a(F_v) \cap g'G(\mathcal{O}_v)g'^{-1}, \mathrm{d}t_v)}$$

the summation being extended over the set of double classes

$$g' \in I_{\gamma_\xi}(F_v)\backslash G(F_v)/G(\mathcal{O}_v)$$

such that $\mathrm{ad}(g')^{-1}\gamma_\xi \in \mathfrak{g}(\mathcal{O}_v)$. We thus obtain the formula

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)]_\xi(k) = \sharp\, O_\xi = \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t)\mathbf{O}_{\gamma_\xi}(1_{\mathfrak{g}_v}, \mathrm{d}t_v).$$

By summing over the kernel of $\mathrm{H}^1(F, I_{\gamma_0}) \to \mathrm{H}^1(F, G)$, we obtain the formula

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{O}^\kappa_a(1_{\mathfrak{g}_v}, \mathrm{d}t_v)$$

which was desired. □

We will eventually want the same type of statement for group schemes J'_a not necessarily having a connected special fiber and in particular for J_a itself. The direct counting of the number of k-points of the quotient $[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)]$ proves to be very tedious. It is more economical to go through the comparison with the connected case. We will state the result only in the case of J_a, although it is generally valid.

Let $\kappa : \mathrm{H}^1(k, \mathcal{P}_v(J_a)) \to \overline{\mathbb{Q}}_\ell^\times$. By using the homomorphism

$$\mathrm{H}^1(k, P_v(J^0_a)) \to \mathrm{H}^1(k, \mathcal{P}_v(J_a)),$$

we obtain a character of $\mathrm{H}^1(k, \mathcal{P}_v(J^0_a))$ and consequently a character of $\mathrm{H}^1(F_v, J_a)$ which we will all denote by κ.

Proposition 8.2.7. — *Consider κ a character of $\mathrm{H}^1(k, \mathcal{P}_v(J_a))$ and let $\kappa : \mathrm{H}^1(F_v, J_a) \to \overline{\mathbb{Q}}_\ell^\times$ be the character of $\mathrm{H}^1(F_v, J_a)$ that results from it. Then, the number of k-points with the κ-weighting of $[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)]$ can be expressed as follows*

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)](k)_\kappa = \mathrm{vol}(J^0_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{O}^\kappa_a(1_{\mathfrak{g}_v}, \mathrm{d}t_v).$$

where $1_{\mathfrak{g}_v}$ is the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$ and where $\mathrm{d}t_v$ is any Haar measure of $J_a(F_v)$.

Moreover, if $\kappa : \mathrm{H}^1(F_v, J_a) \to \overline{\mathbb{Q}}_\ell^\times$ is a character that does not come from a character of $\mathrm{H}^1(k, \mathcal{P}_v(J_a))$, then the κ-orbital integral $\mathbf{O}^\kappa_a(1_{\mathfrak{g}_v}, \mathrm{d}t_v)$ is zero.

Proof. — This is a comparison between $\mathcal{P}_v(J_a)$ and the known case $\mathcal{P}_v(J^0_a)$. It suffices to prove the equality

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)](k)_\kappa = \sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J^0_a)](k)_\kappa$$

in the first case and the vanishing of $[\mathcal{M}_v(a)/\mathcal{P}_v(J^0_a)](k)_\kappa$ in the second case. The proof will be based on the same principle as 8.1.7 although the details are more complicated in the present case.

We have an exact sequence

$$1 \to \pi_0(J_{a,v}) \to \mathcal{P}_v(J^0_a) \to \mathcal{P}_v(J_a) \to 1$$

where $\pi_0(J_{a,v})$ is the group of connected components of the fiber of J_a at v. This implies a long exact sequence

$$1 \to \pi_0(J_{a,v})^\sigma \to \mathscr{P}_v(J_a^0)^\sigma \to \mathscr{P}_v(J_a)^\sigma \to \tag{8.2.8}$$

$$\pi_0(J_{a,v})_\sigma \to \mathscr{P}_v(J_a^0)_\sigma \to \mathscr{P}_v(J_a)_\sigma \to 1 \tag{8.2.9}$$

where the exponent σ denotes a group of σ-invariants and the index σ denotes a group of σ-coinvariants.

Consider the functor

$$\pi : [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k) \to [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a)](k).$$

It maps an object $x' = (m, p')$ with $m \in \mathscr{M}_v(a)$ and $p' \in \mathscr{P}_v(J_a^0)$ such that $p'\sigma(m) = m$ to the object $x = (m, p)$ where p is the image of p' in $\mathscr{P}_v(J_a)$. Choose a set of representatives $\{x_\psi \mid \psi \in \Psi\}$ of the isomorphism classes of $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a)](k)$.

Since the homomorphism $\mathscr{P}_v(J_a^0) \to \mathscr{P}_v(J_a)$ is surjective, this subcategory of $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$ is equivalent to $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$ so that to calculate the numbers $\sharp\, [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_\kappa$, it is permissible to restrict to this subcategory.

For each $\psi \in \Psi$, consider the full subcategory

$$[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_\psi}$$

of $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$ formed by objects of the form $x'_\psi = (m_\psi, p'_\psi)$ with $p'_\psi \mapsto p_\psi$. If $\psi \neq \psi'$, two objects $x'_\psi \in [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_\psi}$ and $x'_{\psi'} \in [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_{\psi'}}$ are not isomorphic. Moreover, for any $x' \in [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$ there exists $\psi \in \Psi$ and $x'_\psi \in [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_\psi}$ such that x' and x'_ψ are isomorphic. Thus, the disjoint union of the categories

$$\bigsqcup_\psi [\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_\psi}$$

forms a full subcategory of $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$ which is equivalent to it. Consequently, to count the objects of $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)$, we can count in each $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_{x_\psi}$ and then sum over the $\psi \in \Psi$.

Fix an x_ψ and simply denote it as $x = (m, p)$. The category

$$[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)](k)_x$$

is not difficult to describe. By fixing an object $x_1 = (m, p_1)$ with $p_1 \mapsto p$, we identify the set of objects of this category with the group $\pi_0(J_{a,v})$ which is the kernel of $\mathscr{P}_v(J_a^0) \to \mathscr{P}_v(J_a)$. A morphism in this category is given by an element of the group

$$H_1 = \{h_1 \in \mathscr{P}_v(J_a^0) \mid hm = m \text{ and } h_1\sigma(h_1)^{-1} \in \pi_0(J_{a,v})\}.$$

The category $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)]_x$ is equivalent to the quotient category of the set $\pi_0(J_{a,v})$ by the action of H_1 with H_1 acting via the homomorphism $\alpha : H_1 \to \pi_0(J_{a,v})$ defined by $\alpha(h_1) = h_1\sigma(h_1)^{-1}$. The set of isomorphism classes of this category is identified with the cokernel $\mathrm{cok}(\alpha)$ of α and the automorphism group of each object is identified with the kernel $\ker(\alpha)$. This shows in particular that H_1 is a finite group.

Let κ be a character of $\mathrm{H}^1(k, \mathscr{P}_v(J_a^0))$. This group is canonically identified with the torsion part of $\mathscr{P}_v(J_a^0)_\sigma$. Consider the restriction of κ to $\pi_0(J_{a,v})_\sigma$ and also denote κ as the character $\pi_0(J_{a,v}) \to \overline{\mathbb{Q}}_\ell^\times$ that results from it. This character is certainly trivial on the image of $\alpha : H_1 \to \pi_0(J_{a,v})_\sigma$ so that it defines a character on the cokernel $\kappa : \mathrm{cok}(\alpha) \to \overline{\mathbb{Q}}_\ell^\times$. The sum

$$\sum_{x'} \frac{\langle \mathrm{cl}(x'), \kappa\rangle}{\sharp\,\mathrm{Aut}(x')} = 0$$

over the set of isomorphism classes of the full subcategory $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a^0)]_x$ is then equal to

$$\sum_{z\in\mathrm{cok}(\alpha)} \frac{\langle z,\kappa\rangle}{\sharp\ker(\alpha)}\langle \mathrm{cl}(x_1),\kappa\rangle.$$

If the restriction of κ to $\pi_0(J_{a,v})$ is non-trivial then this sum is zero, which implies the vanishing of $[\mathscr{M}_v(a)/\mathscr{P}_v(J_a^0)](k)_\kappa = 0$.

Now assume that the restriction of κ to $\pi_0(J_{a,v})$ is trivial. In this case, κ factors through $\mathrm{H}^1(k,\mathscr{P}_v(J_a))$ and the sum above is equal to

$$\frac{\sharp\,\mathrm{cok}(\alpha)}{\sharp\ker(\alpha)}\langle \mathrm{cl}(x),\kappa\rangle.$$

It remains to calculate $\sharp\,\mathrm{cok}(\alpha)/\sharp\ker(\alpha)$. The exact sequence

$$1\to \ker(\alpha)\to H_1\to \pi_0(J_{a,v})\to \mathrm{coker}(\alpha)\to 1$$

implies the equality

$$\frac{\sharp\,\mathrm{cok}(\alpha)}{\sharp\ker(\alpha)} = \frac{\sharp\,\pi_0(J_{a,v})}{\sharp H_1}.$$

Let $h_1\in H_1$ and h its image in $\mathscr{P}_v(J_a)$. Then $h\sigma(h)^{-1}=1$ so that $h\in\mathscr{P}_v(J_a)^\sigma$. Let $\mathrm{stab}(m)$ be the subgroup of elements of $\mathscr{P}_v(J_a)$ that stabilize m. We then have the exact sequence

$$1\to \pi_0(J_{a,v})\to H_1\to \mathscr{P}_v(J_a)^\sigma\cap \mathrm{stab}(m)\to 1$$

where $\mathscr{P}_v(J_a)^\sigma\cap\mathrm{stab}(m)$ is exactly the group of automorphisms of $\mathrm{Aur}(x)$ in the category $[\mathscr{M}_v(J_a)/\mathscr{P}_v(J_a)]$. We thus have the equality

$$\frac{\sharp\,\pi_0(J_{a,v})}{\sharp H_1} = \frac{1}{\sharp\,\mathrm{Aut}(x)}$$

which implies the equality

$$\sharp\,[\mathscr{M}_v(a)/\mathscr{P}_v(J_a)](k)_\kappa = \sharp\,[\mathscr{M}_v(a)/\mathscr{P}_v(J_a^0)](k)_\kappa$$

which was desired. □

By applying the general results of 8.1 to the present situation, we obtain a cohomological interpretation of stable orbital integrals and κ-orbital integrals. Take a torsion-free σ-stable subgroup Λ of $\mathscr{P}_v(J_a^0)$ as in 8.1.9. This subgroup exists by virtue of the discussions following 8.1.9 and the result of Kazhdan-Lusztig *cf.* 3.4.1. We obtain the following corollary of 8.1.13 and 8.2.5.

Corollary 8.2.10. — *Let κ be a character of $\mathrm{H}^1(F_v,J_a)$. Let $J_a^{\flat,0}$ be the neutral component of the Néron model of J_a. For any Λ as above, we have the equality*

$$\sum_n(-1)^n\mathrm{tr}(\sigma,\mathrm{H}^n([\mathscr{M}_v(a)/\Lambda])_\kappa) = \mathrm{vol}(J_a^{\flat,0}(\mathscr{O}_v),\mathrm{d}t_v)\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v},\mathrm{d}t_v).$$

Proof. — By combining 8.1.13 and 8.2.5 we obtain the formula

$$\begin{aligned}&\textstyle\sum_n(-1)^n\mathrm{tr}(\sigma,\mathrm{H}^n([\mathscr{M}_v(a)/\Lambda])_\kappa)\\ =\ &(\sharp\,\mathscr{P}_v^0(J_a^0)(k))\mathrm{vol}(J_a^0(\mathscr{O}_v),\mathrm{d}t_v)\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v},\mathrm{d}t_v).\end{aligned}$$

where $\sharp\,\mathscr{P}_v^0(J_a^0)(k)$ is the number of k-points of the neutral component of $\mathscr{P}_v(J_a^0)$. If $J_a^{\flat,0}$ is the neutral component of the Néron model, we have a homomorphism $J_a^0\to J_a^{\flat,0}$ which induces an exact sequence

$$1\to J_a^{\flat,0}(\bar{\mathscr{O}}_v)/J_a^0(\bar{\mathscr{O}}_v)\to \mathscr{P}_v(J_a^0)\to\mathscr{P}_v(J_a^{\flat,0})\to 1$$

which allows identifying the neutral component $\mathscr{P}_v^0(J_a^0)$ of $\mathscr{P}_v(J_a^0)$ with the connected affine k-group whose $\bar{k}$-points are $J_a^{\flat,0}(\bar{\mathcal{O}}_v)/J_a^0(\bar{\mathcal{O}}_v)$. Since J_a^0 and $J_a^{\flat,0}$ are group schemes with connected fibers, we deduce an exact sequence

$$1 \to J_a^0(\mathcal{O}_v) \to J_a^{\flat,0}(\mathcal{O}_v) \to \mathscr{P}_v^0(J_a^0)(k) \to 1$$

from which the equality

$$(\sharp\,\mathscr{P}_v^0(J_a^0)(k))\mathrm{vol}(J_a^0(\mathcal{O}_v), \mathrm{d}t_v) = \mathrm{vol}(J_a^{\flat,0}(\mathcal{O}_v), \mathrm{d}t_v)$$

for any Haar measure $\mathrm{d}t_v$ on $J(F_v)$. The corollary follows. □

8.3. A very simple case. — Let v be a closed point of X. Let k_v be the residue field of v. Choose a geometric point $\bar{v}$ above v. Let $X_v = \mathrm{Spec}(\mathcal{O}_v)$ be the completion of X at v and $X_v^\bullet = \mathrm{Spec}(F_v)$ where F_v is the fraction field of $\mathcal{O}_v$. Choose a uniformizer ϵ_v. In this paragraph, we denote G as the restriction of G to X_v and $G^\bullet$ as the restriction of G to $X_v^\bullet$.

***8.3.1.* —** Let $a \in \mathfrak{c}(\mathcal{O}_v)$ whose image in $\mathfrak{c}(F_v)$ is regular and semi-simple. Let $d_v(a) = \deg_v(a^*\mathfrak{D}_G) \in \mathbb{N}$. We will assume that

$$d_v(a) = 2 \text{ and } c_v(a) = 0.$$

According to Bezrukavnikov's formula *cf.* 3.7.5, the dimension of the affine Springer fiber $\mathscr{M}_v(a)$ is then equal to one. We will show that over $\bar{k}$, this affine Springer fiber is a disjoint union of copies of the infinite chain of projective lines.

***8.3.2.* —** Let $\gamma_0 = \epsilon(a) \in \mathfrak{g}(F_v)$ be the Kostant section applied to a. Its centralizer $T^\bullet = I_{\gamma_0}$ is a maximal torus of $G^\bullet$. The hypothesis $c_v(a) = 0$ implies that $T^\bullet$ is an unramified maximal torus, meaning it extends to a maximal torus T of G. Let Φ be the set of roots of $\bar{T} = T \otimes_{\mathcal{O}_v} \bar{\mathcal{O}}_{\bar{v}}$. We then have the root valuation function of Goresky, Kottwitz and MacPherson *cf.* [**28**]

$$r_a : \Phi \to \mathbb{N}$$

defined by $r_a(\alpha) = \mathrm{val}(\alpha(\gamma_0))$ where we have taken the valuation on $\bar{F}_{\bar{v}}$ which extends the valuation on F_v. We then have

$$d_v(a) = \sum_{\alpha\in\Phi} r_\alpha(\gamma_0).$$

Thus, there exists a unique pair of roots $\pm\alpha$ such that $r_{\pm\alpha}(\gamma_0) = 1$ and $r_{\alpha'}(\gamma_0) = 0$ for any root $\alpha' \notin \{\pm\alpha\}$.

***8.3.3.* —** The Galois group $\mathrm{Gal}(\bar{k}/k_v)$ acts on Φ by leaving the function $r_a(\alpha)$ invariant, so that it stabilizes the pair $\{\pm\alpha\}$. Let $\bar{G}_{\pm\alpha}$ be the subgroup of $\bar{G} = G \otimes_{\mathcal{O}_v} \bar{\mathcal{O}}_{\bar{v}}$ generated by $\bar{T}$ and by the root subgroups U_α and $U_{-\alpha}$. The action of the Galois group $\mathrm{Gal}(\bar{k}/k_v)$ on $\bar{T}$ stabilizing $\{\pm\alpha\}$ allows descending $\bar{G}_{\pm\alpha}$ to a reductive group sub-scheme $G_{\pm\alpha}$ of $G_{\mathcal{O}_v}$. The center $Z_{\pm\alpha}$ of $G_{\pm\alpha}$ which is identified with the kernel of $\alpha : T \to \mathbb{G}_m$ is also defined over $\mathcal{O}_v$. Let $A_{\pm\alpha} = T/Z_{\pm\alpha}$; this is a one-dimensional torus over X_v.

***Proposition 8.3.4.* —** *We have a canonical homomorphism $J_a \to T$ whose image at the level of $\bar{\mathcal{O}}_{\bar{v}}$-points is the kernel of the composite homomorphism of the reduction modulo the maximal ideal $T(\bar{\mathcal{O}}_{\bar{v}}) \to T(\bar{k})$ and the root $\alpha : T(\bar{k}) \to \mathbb{G}_m(\bar{k})$.*

We deduce the following description of $\mathscr{P}_v(J_a)$.

***Corollary 8.3.5.* —** *We have an exact sequence of abelian groups with the action of the Frobenius endomorphism σ*

$$1 \to A_{\pm\alpha}(\bar{k}) \to \mathscr{P}_{\bar{v}}(J_a)(\bar{k}) \to \mathbb{X}_*(T) \to 1$$

where $\mathbb{X}_*(T)$ is the group of cocharacters of the torus T above $\bar{\mathcal{O}}_{\bar{v}}$.

Proof. — From the exact sequence

$$1 \to J_a(\bar{\mathscr{O}}_{\bar{v}}) \to T(\bar{\mathscr{O}}_{\bar{v}}) \to A_{\pm\alpha}(\bar{k}) \to 1$$

we deduce the exact sequence

$$1 \to A_{\pm\alpha}(\bar{k}) \to J_a(\bar{F}_{\bar{v}})/J_a(\bar{\mathscr{O}}_{\bar{v}}) \to T(\bar{F}_v)/T(\bar{\mathscr{O}}_v) \to 1.$$

We also have an isomorphism

$$T(\bar{F}_v)/T(\bar{\mathscr{O}}_v) = \mathbb{X}_*(T)$$

which is deduced from the homomorphism $\mathbb{X}_*(T) \to T(\bar{F}_v)$ defined by $\lambda \mapsto \epsilon_v^\lambda$, whence the corollary. □

Lemma 8.3.6. — *The $\bar{k}$-points of the affine Springer fiber*

$$\mathscr{M}_{\bar{v}}(a)(\bar{k}) = \{g \in G(\bar{F}_{\bar{v}})/G(\bar{\mathscr{O}}_{\bar{v}}) | \mathrm{ad}(g)^{-1}(\gamma_0) \in \mathfrak{g}(\bar{\mathscr{O}}_{\bar{v}})\}$$

can be uniquely written in the form

$$g = \epsilon_v^\lambda U_\alpha(x\epsilon_v^{-1})$$

with $\lambda \in \mathbb{X}_*(T)$ and $x \in \bar{k}$.

Proof. — With the Iwasawa decomposition, for any element $g \in G(\bar{F}_{\bar{v}})/G(\bar{\mathscr{O}}_{\bar{v}})$, there exist unique $\lambda \in \mathbb{X}_*(T)$ and $u \in U(\bar{F}_{\bar{v}})/U(\bar{\mathscr{O}}_{\bar{v}})$ such that

$$g = \epsilon_v^\lambda u.$$

As $T_{\bar{v}}$ commutes with γ_0, the embedding

$$\mathscr{M}_{\bar{v}}(a) \to \mathscr{G}_{\bar{v}} = G(\bar{F}_{\bar{v}})/G(\bar{\mathscr{O}}_{\bar{v}})$$

is $T_{\bar{v}}$-equivariant. The action of $T_{\bar{v}}$ on $\mathscr{M}_v(a)$ factors through

$$T_{\bar{v}} \to T(\bar{F}_{\bar{v}}) \to \mathscr{P}_v(J_a) = T(\bar{F}_{\bar{v}})/J_a(\bar{\mathscr{O}}_{\bar{v}})$$

so that it factors through the one-dimensional torus

$$T_{\bar{v}} \to A_{\pm\alpha,\bar{v}}.$$

It follows that if we write $g \in \mathscr{M}_v(a)(\bar{k})$ in the form $g = \epsilon_v^\lambda u$, u must be of the form $u = U_\alpha(y)$ with $y \in \bar{F}_{\bar{v}}/\bar{\mathscr{O}}_{\bar{v}}$ uniquely determined. A calculation in SL_2 *cf.* [**26**, lemma 6.2] then shows that with the hypothesis $r_{\pm\alpha}(a) = 1$, y is uniquely written in the form $y = x\epsilon_v^{-1}$ with $x \in \bar{k}$ and conversely, the elements g of the form $g = \epsilon_v^\lambda U_\alpha(x\epsilon_v^{-1})$ belong to $\mathscr{M}_{\bar{v}}(a)(\bar{k})$. □

Lemma 8.3.7. — *The fixed points of the one-dimensional torus $A_{\pm\alpha,\bar{v}}$ in $\mathscr{M}_{\bar{v}}$ are ϵ_v^λ. For fixed $\lambda \in \mathbb{X}_*(T)$, $A_{\pm\alpha,\bar{v}}$ acts simply transitively on*

$$O_\lambda = \{\epsilon_v^\lambda U_\alpha(x\epsilon_v^{-1}) \in \mathscr{M}_v(a)(\bar{k}) | x \in \bar{k}^\times\}.$$

Furthermore, the boundary of the closure of this orbit consists of ϵ_v^λ and $\epsilon_v^{\lambda-\alpha^\vee}$ where $\alpha^\vee$ is the coroot associated with the root α.

Proof. — A direct calculation shows that the ϵ_v^λ are fixed under the action of $A_{\pm\alpha,\bar{v}}$ and that $A_{\pm\alpha,\bar{v}}$ acts simply transitively on O_λ. This shows that the set of fixed points of $A_{\pm\alpha,\bar{v}}$ is exactly $\{\epsilon_v^\lambda | \lambda \in \mathbb{X}_*(T)\}$. When $x \to 0$, $\epsilon_v^\lambda U_\alpha(x\epsilon_v^{-1})$ tends towards ϵ_v^λ so that ϵ_v^λ belongs to the closure of O_λ.

It remains to prove that when $x \to \infty$, $\epsilon_v^\lambda U_\alpha(x\epsilon_v^{-1})$ tends towards $\epsilon_v^{\lambda-\alpha^\vee}$. It is equivalent to prove that $U_\alpha(x\epsilon_v^{-1})$ tends towards $\epsilon_v^{-\alpha^\vee}$ when $x \to \infty$. This is a well-known calculation in the affine Grassmannian which follows from Steinberg's relation in $G(\bar{F}_{\bar{v}})$ *cf.* [**76**, chap. 3, lemma 19]

$$y^{-\alpha^\vee} w_\alpha = U_{-\alpha}(y) U_\alpha(-y^{-1}) U_\alpha(y)$$

which holds for any $y \in \bar{F}_{\bar{v}}$, for any root α and for a representative w_α of the reflection $s_\alpha \in W$ attached to the root α independent of y and which in particular belongs to $G(\bar{k})$. By taking $y = -x^{-1}\epsilon_v$ with $x \in \bar{k}^\times$, we obtain the following relation in $G(\bar{F}_{\bar{v}})/G(\bar{\mathscr{O}}_{\bar{v}})$

$$U_\alpha(x\epsilon_v^{-1}) = \epsilon_v^{-\alpha^\vee} U_{-\alpha}(-x^{-1}\epsilon_v^{-1}).$$

As x tends to ∞, we observe that $U_\alpha(x\epsilon_v^{-1})$ tends towards $\epsilon_v^{-\alpha^\vee}$. □

Proposition 8.3.8. — *Let $\kappa \in \hat{T}^\sigma$ be a torsion element such that*

$$\kappa(\alpha^\vee) \neq 1.$$

Then, we have the formula

$$[\mathscr{M}_v(a)/\mathscr{P}_v(J_a)](k)_\kappa = \sharp A_{\pm\alpha}(k_v)^{-1} q^{\deg(v)}.$$

Proof. — The affine Springer fiber $\mathscr{M}_v(a)$ is obtained as the restriction of scalars k_v/k of an affine Springer fiber defined over k_v. The same applies to $\mathscr{P}_v(J_a)$ and the quotient $[\mathscr{M}_v(a)/\mathscr{P}_v(J_a)]$. We can therefore assume that $k_v = k$.

Let $A = A_{\pm\alpha,v}$ which is a one-dimensional torus over k. Consider the σ-invariant subgroup of $\mathbb{X}_*(T)$ generated by $\alpha^\vee$ and consider the inverse image of the exact sequence

$$1 \to A \to \mathscr{P}_v(J_a) \to \mathbb{X}_*(T) \to 1$$

by the homomorphism $\mathbb{Z}\alpha^\vee \to \mathbb{X}_*(T)$. This is an algebraic group P defined over k equipped with an exact sequence

$$1 \to A \to P \to \mathbb{Z}\alpha^\vee \to 1$$

and an injective homomorphism $P \to \mathscr{P}_v(J_a)$ which induces the identity on the neutral component A.

In the affine Springer fiber $\mathscr{M}_v(a)$, we have a k-point m given by the Kostant section. According to the description above of $\mathscr{M}_v(a) \otimes_k \bar{k}$, the connected component M of $\mathscr{M}_v(a) \otimes_k \bar{k}$ is an infinite chain of projective lines equipped with an action of $P \otimes_k \bar{k}$ which is simply transitive on M^{reg}. Since M contains the k-point m, it is defined over k. We recover $\mathscr{M}_v(J_a)$ from M by the induction of P into $\mathscr{P}_v(J_a)$

$$\mathscr{M}_v(a) = M \wedge^P \mathscr{P}_v(J_a).$$

In particular, we have an equivalence of categories

$$[\mathscr{M}_v(a)/\mathscr{P}_v(J_a)] = [M/P].$$

We are thus reduced to the point counting exercise already solved in example 8.1.15. □

8.4. Counting in an anisotropic Hitchin fiber. — Recall the point counting in a Hitchin fiber $\mathscr{M}_a$ with $a \in \mathscr{M}_a^{\mathrm{ani}}(k)$ following paragraph 9 of [**57**]. In *loc. cit.*, we considered the quotient of $\mathscr{M}_a$ by $\mathscr{P}_a$. It is in fact more convenient to consider a more general quotient.

8.4.1. — Let J'_a be a smooth commutative finite type X-group scheme equipped with a homomorphism $J'_a \to J_a$ which is an isomorphism on a non-empty open set U of X. The data of J'_a is equivalent to the data of open compact subgroups $J'_a(\mathscr{O}_v) \subset J_a(\mathscr{O}_v)$ for points $v \in |X - U|$. Let $\mathscr{P}'_a = \mathscr{P}(J'_a)$ be the classifying stack of J'_a-torsors over X. We then have a homomorphism $\mathscr{P}'_a \to \mathscr{P}_a$ which induces an action of $\mathscr{P}'_a$ on $\mathscr{M}_a$. For J'_a sufficiently small in the sense that the open compact subgroups $J'_a(\mathscr{O}_v)$ are sufficiently small, $\mathrm{H}^0(\bar{X}, J'_a)$ is trivial and so $\mathscr{P}'_a$ is representable by a locally finite type algebraic group over k. For counting, it will be convenient to assume that the fibers of J'_a are all connected. Consider the quotient category $[\mathscr{M}_a/\mathscr{P}'_a]$.

The counting of k-points of $\mathscr{M}_a$ is based on the following form of the product formula *cf.* 4.15.1 and [**57**, theorem 4.6].

Proposition 8.4.2. — *Let $U = a^{-1}(\mathfrak{c}_D^{\mathrm{rs}})$ be the inverse image of the regular semi-simple locus of $\mathfrak{c}_D$. We have an equivalence of categories*

$$[\mathcal{M}_a/\mathcal{P}'_a] = \prod_{v\in X-U} [\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]$$

compatible with the action of $\sigma \in \mathrm{Gal}(\bar{k}/k)$. In particular, we have an equivalence between the categories of k-points

$$[\mathcal{M}_a/\mathcal{P}'_a](k) = \prod_{v\in|X-U|} [\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k).$$

Let $(\mathcal{P}'_a)^0$ be the neutral component of $\mathcal{P}'_a$. Since $a \in \mathcal{A}^{\mathrm{ani}}(k)$, the group of connected components $\pi_0(\mathcal{P}_a)$ is a finite group equipped with an action of the Frobenius element $\sigma \in \mathrm{Gal}(\bar{k}/k)$.

Proposition 8.4.3. — *Assume that the fibers of J'_a are connected. For any σ-invariant character of $\pi_0(\mathcal{P}_a)$*

$$\kappa : \pi_0(\mathcal{P}_a)_\sigma \to \overline{\mathbb{Q}}_\ell^\times,$$

we have

$$\sharp\,[\mathcal{M}_a/\mathcal{P}'_a](k)_\kappa = \prod_{v\in|X-U|} \sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa.$$

Proof. — For any closed point v of X in the complement of the open set $U = a^{-a}(\mathfrak{c}_D^{\mathrm{rs}})$, the composite homomorphism

$$\mathcal{P}_v(J'_a) \to \mathcal{P}'_a \to \pi_0(\mathcal{P}'_a)$$

factors through $\pi_0(\mathcal{P}_v(J'_a))$. The homomorphism $\kappa : \pi_0(\mathcal{P}'_a)_\sigma \to \overline{\mathbb{Q}}_\ell^\times$ therefore defines a homomorphism $\kappa : \pi_0(\mathcal{P}_v(J'_a))_\sigma \to \overline{\mathbb{Q}}_\ell^\times$.

In 8.4.2, if a point $y \in [\mathcal{M}_a/\mathcal{P}'_a](k)$ corresponds to a collection of points $y_v \in [\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)$ for $v \in |X-U|$, then we have the formula

$$\langle \mathrm{cl}(y), \kappa\rangle = \prod_{v\in|X-U|} \langle \mathrm{cl}(y_v), \kappa\rangle.$$

Consequently, we have the factorization

$$\sharp\,[\mathcal{M}_a/\mathcal{P}'_a](k)_\kappa = \prod_{v\in|X-U|} \sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa$$

whence the proposition. □

The conjunction with 8.1.6 gives the following corollary.

Corollary 8.4.4. — *For any character $\kappa : \pi_0(\mathcal{P}_a)_\sigma \to \overline{\mathbb{Q}}_\ell^\times$, we have*

$$\sum_n (-1)^n \mathrm{tr}(\sigma, \mathrm{H}^n(\mathcal{M}_a)_\kappa) = \sharp\,(\mathcal{P}'_a)^0(k) \prod_{v\in|X-U|} \sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa$$

where $\mathrm{H}^n(\mathcal{M}_a)_\kappa$ is the eigenspace of $\mathrm{H}^n(\mathcal{M}_a)$ where the group $\pi_0(\mathcal{P}_a)$ acts via the character κ.

8.4.5. — To express the numbers $\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa$ in terms of local orbital integrals, make the following choices:

- at every place $v \in |X-U|$, choose a trivialization of the invertible bundle $D'|_{X_v}$,
- at every place $v \in |X-U|$, choose a Haar measure $\mathrm{d}t_v$ of the torus $J_a(F_v)$.

These choices allow us to:

- identify the restriction of a to X_v with an element $a_v \in \mathfrak{c}(\mathcal{O}_v) \cap \mathfrak{c}^{\mathrm{rs}}(F_v)$,
- identify the group schemes J_a and J_{a_v} which equips $J_{a_v}(F_v)$ with a Haar measure $\mathrm{d}t_v$,
- identify the affine Springer fiber $\mathcal{M}_v(a)$ equipped with the action of $\mathcal{P}_v(J_a)$ with the affine Springer fiber $\mathcal{M}_v(a_v)$ equipped with the action of $\mathcal{P}_v(J_{a_v})$.

With these choices made, we can express the numbers $\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa$ in terms of κ-orbital integrals *cf.* 8.2.5

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{O}^\kappa_{a_v}(1_{\mathfrak{g}_v}, \mathrm{d}t_v).$$

We then obtain the formula

$$\sum_n (-1)^n \mathrm{tr}(\sigma, \mathrm{H}^n(\mathcal{M}_a)_\kappa) = (\mathcal{P}'_a)^0(k) \prod_{v\in|X-U|} \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{O}^\kappa_{a_v}(1_{\mathfrak{g}_v}, \mathrm{d}t_v).$$

8.5. Stabilization over $\tilde{\mathcal{A}}_H^{\mathrm{ani}} - \tilde{\mathcal{A}}_H^{\mathrm{bad}}$. — We are in a position to prove Theorem 6.4.2 on the open set $\tilde{\mathcal{A}}_H^{\mathrm{good}} = \tilde{\mathcal{A}}_H^{\mathrm{ani}} - \tilde{\mathcal{A}}_H^{\mathrm{bad}}$. Here $\tilde{\mathcal{A}}_H^{\mathrm{bad}}$ is the closed sub-scheme of $\tilde{\mathcal{A}}_H^{\mathrm{ani}}$ defined in 7.8.2.

Proof. — According to the support theorem 7.8.5, on $\tilde{\mathcal{A}}_H^{\mathrm{good}}$, the pure perverse sheaves

$$K^n_\kappa = \tilde{\nu}^*\,{}^p\mathrm{H}^n(\tilde{f}^{\mathrm{ani}}_*\overline{\mathbb{Q}}_\ell)_\kappa \text{ and } K^n_{H,\mathrm{st}} = {}^p\mathrm{H}^{n+2r^G_H(D)}(\tilde{f}^{\mathrm{ani}}_{H,*}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}(-r^G_H(D))$$

are the intermediate extensions of their restrictions to any non-empty open set $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{A}}_H^{\mathrm{good}}$. To show that K^n and K^n_H are isomorphic on $\tilde{\mathcal{A}}_H^{\mathrm{good}}\otimes_k \bar{k}$ and are isomorphic after semi-simplification on $\tilde{\mathcal{A}}_H^{\mathrm{good}}$, it suffices to do so on any dense open set $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{A}}_H^{\mathrm{good}}$. We will construct a good open set $\tilde{\mathcal{U}}$ as follows.

Recall that we have an equality of divisors *cf.* 1.10.3

$$\nu^*\mathfrak{D}_{G,D} = \mathfrak{D}_{H,D} + 2\mathfrak{R}^G_{H,D}$$

on $\mathfrak{c}_{H,D}$. Moreover, we know that $\mathfrak{D}_{H,D}$ and $\mathfrak{R}^G_{H,D}$ are reduced divisors of $\mathfrak{c}_{H,D}$ which are disjoint, so that the union

$$\mathfrak{D}_{H,D} + \mathfrak{R}^G_{H,D}$$

is also a reduced divisor.

Lemma 8.5.1. — *Assume that* $\deg(D) > 2g$. *Then, the set of $a_H \in \mathcal{A}_H(\bar{k})$ such that $a_H(\bar{X})$ intersects transversally $\mathfrak{D}_{H,D} + \mathfrak{R}^G_{H,D}$ forms a non-empty open set $\mathcal{U}$ of $\mathcal{A}_H$.*

Proof. — The proof is identical to the proof of 4.7.1. □

8.5.2. — Consider the inverse image $\tilde{\mathcal{U}}$ of $\mathcal{U}$ in $\tilde{\mathcal{A}}$. By shrinking this open set if necessary, we can assume that $\tilde{\mathcal{U}} \subset \tilde{\mathcal{A}}^{\mathrm{good}}$. By shrinking it further if necessary, we can assume that for all $n \in \mathbb{Z}$, the restrictions

$$L^n_\kappa = K^n_\kappa|_{\tilde{\mathcal{U}}} \text{ and } L^n_{H,\mathrm{st}} = K^n_{H,\mathrm{st}}|_{\tilde{\mathcal{U}}}$$

are pure local systems of weight n on $\tilde{\mathcal{U}}$.

8.5.3. — It suffices in fact to show that for any finite extension k' of k, for any k'-point $\tilde{a}_H \in \tilde{\mathcal{U}}(k')$ with image $\tilde{a} \in \tilde{\mathcal{A}}(k')$, we have the equality of traces

$$\sum_n (-1)^n \mathrm{tr}(\sigma_{k'}, (L^n_\kappa)_{\tilde{a}}) = \sum_n (-1)^n \mathrm{tr}(\sigma_{k'}, (L^n_{H,\mathrm{st}})_{\tilde{a}_H}). \tag{8.5.4}$$

Indeed, we then have the equality

$$\sum_n (-1)^n \mathrm{tr}(\sigma^j_{k'}, (L^n_\kappa)_{\tilde{a}}) = \sum_n (-1)^n \mathrm{tr}(\sigma^j_{k'}, (L^n_{H,\mathrm{st}})_{\tilde{a}_H}).$$

for any natural integer j by considering $\tilde{a}_H$ as a point with values in the extension of degree j of k'. Moreover, as the eigenvalues of $\sigma_{k'}$ in $(L^n_\kappa)_{\tilde{a}}$ and $(L^n_{H,\mathrm{st}})_{\tilde{a}_H}$ are all of absolute value $q^{n\deg(k'/k)/2}$, these equalities imply that for any finite extension k' of k, for any $\tilde{a}_H \in \tilde{\mathcal{U}}(k')$ with image $\tilde{a} \in \tilde{\mathcal{A}}$ and for any n, we have

$$\mathrm{tr}(\sigma_{k'}, (L^n_\kappa)_{\tilde{a}}) = \mathrm{tr}(\sigma_{k'}, (L^n_{H,\mathrm{st}})_{\tilde{a}_H}).$$

According to Chebotarev's theorem, this implies that L^n_κ and $L^n_{H,\mathrm{st}}$ are isomorphic after semi-simplification.

8.5.5. — Let's now prove formula (8.5.4). For any point $\tilde{a}_H \in \tilde{\mathscr{U}}(k')$, we can directly calculate both sides of this formula and then compare them. By replacing k with k' and X with $X \otimes_k k'$, we can assume that $\tilde{a}_H$ is a k-point of $\tilde{\mathscr{U}}$.

8.5.6. — Let $a_H \in \mathscr{A}_H(k)$ be the image of $\tilde{a}_H$. Let a be the image of a_H in $\mathscr{A}(k)$. According to 2.5.1, we have a canonical homomorphism

$$J_a \to J_{H,a_H}$$

which is an isomorphism above the open set $U = a^{-1}(\mathfrak{c}_D^{\mathrm{rs}})$. Choose a group scheme J'_a which is smooth, commutative, with connected fibers, and equipped with a homomorphism $J'_a \to J_a$ which is generically an isomorphism. Let $\mathscr{P}'_a$ be the classifying stack of J'_a-torsors over X. By shrinking J'_a if necessary, we can assume that $\mathrm{H}^0(\bar{X}, J'_a)$ is trivial, in which case $\mathscr{P}'_a$ is a finite type algebraic group. We have natural homomorphisms $\mathscr{P}'_a \to \mathscr{P}_a$ and $\mathscr{P}'_a \to \mathscr{P}_{H,a_H}$ which induce an action of $\mathscr{P}'_a$ on $\mathscr{M}_a$ and $\mathscr{M}_{H,a_H}$. We can calculate both sides of formula 8.5.4 by considering the quotients $[\mathscr{M}_a/\mathscr{P}'_a]$ and $[\mathscr{M}_{H,a_H}/\mathscr{P}'_a]$. According to 8.4.4, the left side of 8.5.4 is

$$(\mathscr{P}'_a)^0(k) \prod_{v \in |X-U|} \sharp\, [\mathscr{M}_v(a)/\mathscr{P}_v(J'_a)](k)_\kappa$$

while the right side is

$$q^{r_H^G(D)} (\mathscr{P}'_a)^0(k) \prod_{v \in |X-U|} \sharp\, [\mathscr{M}_{H,v}(a)/\mathscr{P}_v(J'_a)](k).$$

As

$$r_H^G(D) = \sum_v \deg(v) r_{H,v}^G(a_H)$$

where $r_{H,v}^G(a_H)$ is the degree of the divisor $a_H^* \mathfrak{R}_{H,D}^G$ at v, it suffices to prove the following statement which is a particularly simple case of the fundamental lemma of Langlands-Shelstad 1.11.1. Note that this special case of the fundamental lemma is essentially contained in the article [**48**] by Labesse and Langlands on SL(2) and was revisited from a more geometric perspective in [**26**] by Goresky, Kottwitz and MacPherson. □

Lemma 8.5.7. — *Let $\tilde{a}_H \in \tilde{\mathscr{A}}_H(k)$. Let v be a place of X above which $a_H(\bar{X}_v)$ does not intersect or intersects transversally the divisor $\mathfrak{D}_{H,D} + \mathfrak{R}_{H,D}^G$. We have the equality between the numbers*

$$\sharp\, [\mathscr{M}_v(a)/\mathscr{P}_v(J'_a)](k)_\kappa = q^{\deg(v) r_{H,v}^G(a_H)} \sharp\, [\mathscr{M}_{H,v}(a)/\mathscr{P}_v(J'_a)](k)$$

which are non-zero rational numbers.

Proof. — Let $\bar{v}$ be a geometric point above v. As $a_H(\bar{X}_v)$ intersects transversally $\mathfrak{D}_{H,D} + \mathfrak{R}_{H,D}^G$, there are three possibilities for the integers $d_{H,\bar{v}}(a_H)$, $d_{\bar{v}}(a)$ and $r_{H,v}^G(a_H)$:

(1) if $a_H(\bar{v}) \notin \mathfrak{D}_{H,D} \cup \mathfrak{R}_{H,D}^G$, then

$$d_{H,\bar{v}}(a_H) = 0, d_{\bar{v}}(a) = 0 \text{ and } r_{H,v}^G(a_H) = 0,$$

(2) if $a_H(\bar{v}) \in \mathfrak{D}_{H,D}$ then $a_H(\bar{v}) \notin \mathfrak{R}_{H,D}^G$ and we have

$$d_{H,\bar{v}}(a_H) = 1, d_{\bar{v}}(a) = 1 \text{ and } r_{H,v}^G(a_H) = 0,$$

(3) if $a_H(\bar{v}) \in \mathfrak{R}_{H,D}^G$ then $a_H(\bar{v}) \notin \mathfrak{D}_{H,D}$ and we have

$$d_{H,\bar{v}}(a_H) = 0, d_{\bar{v}}(a) = 2 \text{ and } r_{H,v}^G(a_H) = 1.$$

8.5.8. — In the first two cases, Bezrukavnikov's formula 3.7.5 shows that $\delta_{H,\bar{v}}(a_H) = \delta_{\bar{v}}(a) = 0$ so that the affine Springer fibers $\mathscr{M}_{H,v}(a_H)$ and $\mathscr{M}_v(a)$ are both of dimension zero. It follows that $\mathscr{P}_v(J_a)$ acts simply transitively on $\mathscr{M}_v(a)$ and $\mathscr{P}_v(J_{H,a_H})$ acts simply transitively on $\mathscr{M}_{H,v}(a_H)$ *cf.* 3.7.2. Choose a trivialization of D' above X_v as in 8.4.5 to be able to write the orbital integrals nicely. In particular, a_H and a define elements $a_{H,v} \in \mathfrak{c}_H(\mathcal{O}_v)$ and $a_v \in \mathfrak{c}(\mathcal{O}_v)$. In conjunction with 8.2.7, we deduce the formula

$$1 = [\mathcal{M}_{H,v}(a_H)/\mathcal{P}_v(J_{H,a_H})](k) = \mathrm{vol}(J^0_{H,a_H}(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{SO}_{a_{H,v}}(1_{\mathfrak{h}_v}, \mathrm{d}t_v)$$

Comparing with formula 8.2.5

$$\sharp\,[\mathcal{M}_{H,v}(a)/\mathcal{P}_v(J'_a)](k) = \mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)\mathbf{SO}_{a_{H,v}}(1_{\mathfrak{h}_v}, \mathrm{d}t_v)$$

we obtain

$$\sharp\,[\mathcal{M}_{H,v}(a)/\mathcal{P}_v(J'_a)](k) = \frac{\mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)}{\mathrm{vol}(J^0_{H,a_H}(\mathcal{O}_v), \mathrm{d}t_v)}.$$

Noting that $\mathcal{M}_v(a)$ is given with a k-point by the Kostant section, we also have

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)](k)_\kappa = 1.$$

The same reasoning as above then implies

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \frac{\mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)}{\mathrm{vol}(J^0_a(\mathcal{O}_v), \mathrm{d}t_v)}.$$

It remains to note that in the first two cases, the homomorphism $J_a \longrightarrow J_{H,a_H}$ induces an isomorphism on the neutral components $J^0_a \longrightarrow J^0_{H,a_H}$ and we obtain the equality

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \sharp\,[\mathcal{M}_{H,v}(a)/\mathcal{P}_v(J'_a)](k)$$

which was desired.

8.5.9. — Consider now the third case where $a_H(X_v)$ intersects transversally $\mathfrak{R}^G_{H,D}$ and does not intersect $\mathfrak{c}_{H,D}$. In this case, $d_{H,\bar v}(a_H)$ is zero, so that

$$\delta_{H,\bar v}(a_H) = 0 \text{ and } c_{H,\bar v}(a_H) = 0.$$

As $\delta_{H,\bar v}(a_H) = 0$, the affine Springer fiber $\mathcal{M}_{H,v}(a_H)$ is of dimension zero. According to 3.7.2, $\mathcal{P}_v(J_{H,a_H})$ acts simply transitively on $\mathcal{M}_{H,v}(a_H)$ so that we still have

$$\sharp\,[\mathcal{M}_{H,v}(a_H)/\mathcal{P}_v(J_{H,a_H})](k) = 1.$$

The same reasoning as above shows that

$$\sharp\,[\mathcal{M}_{H,v}(a)/\mathcal{P}_v(J'_a)](k) = \frac{\mathrm{vol}(J'_a(\mathcal{O}_v), \mathrm{d}t_v)}{\mathrm{vol}(J^0_{H,a_H}(\mathcal{O}_v), \mathrm{d}t_v)}.$$

Note that since $a_H(X_v)$ does not intersect $\mathfrak{c}_{H,D}$, J_{H,a_H} is a torus and in particular $J_{H,a_H} = J^0_{H,a_H}$.

Since the invariant $c_{\bar v}(a)$ depends only on the generic fiber of $J_a|_{X_v}$, we have

$$c_{\bar v}(a) = c_{H,\bar v}(a_H) = 0.$$

This implies that the affine Springer fiber $\mathcal{M}_{\bar v}(a)$ is of dimension

$$\delta_{\bar v}(a) = \frac{d_{\bar v}(a) - c_{\bar v}(a)}{2} = 1.$$

We are therefore exactly in the situation of 8.3. By applying formula 8.3.8 while noting that the hypothesis $\kappa(\alpha^\vee) \neq 1$ is well verified here, we obtain the formula

$$[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)](k)_\kappa = \sharp\,A_{\pm\alpha}(k_v)q^{\deg(v)}$$

where $A_{\pm\alpha}$ is the one-dimensional torus over k_v defined by

$$A_{\pm\alpha}(\bar k) = J_{H,a_H}(\mathcal{O}_{\bar v})/J_a(\mathcal{O}_{\bar v}).$$

By applying formula 8.2.7, we find

$$\mathbf{O}^\kappa_a(1_{\mathfrak{g}_v}, \mathrm{d}t_v) = \frac{q^{\deg(v)}}{\sharp\,A_{\pm\alpha}(k_v)\mathrm{vol}(J^0_a(\mathcal{O}_v), \mathrm{d}t_v)}.$$

We can also verify the formula

$$\sharp A_{\pm\alpha}(k_v)\mathrm{vol}(J_a^0(\mathcal{O}_v),\mathrm{d}t_v) = \mathrm{vol}(J_{H,a_H}^0(\mathcal{O}_v),\mathrm{d}t_v)$$

which implies

$$[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = \frac{q^{\deg(v)}\mathrm{vol}(J'_a(\mathcal{O}_v,\mathrm{d}t_v))}{\mathrm{vol}(J_{H,a_H}^0(\mathcal{O}_v),\mathrm{d}t_v)}.$$

We therefore obtain the equality

$$\sharp\,[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa = q^{\deg(v)}\sharp\,[\mathcal{M}_{H,v}(a)/\mathcal{P}_v(J'_a)](k)$$

which was desired.

This completes the proof of 8.5.7. □

8.6. The fundamental lemma of Langlands-Shelstad. — We are now able to prove the fundamental lemma of Langlands and Shelstad 1.11.1.

Proof. — Let $X_v = \mathrm{Spec}(\mathcal{O}_v)$ with $\mathcal{O}_v = k[[\epsilon_v]]$ and $X_v^\bullet = \mathrm{Spec}(F_v)$ with $F_v = k((\epsilon_v))$. Let G_v be a reductive group over X_v, a quasi-split form of $\mathbb{G}$ given by an $\mathrm{Out}(\mathbb{G})$-torsor $\rho_{G,v}$ over X_v. Let $(\kappa,\rho_{\kappa,v})$ be an elliptic endoscopic datum *cf.* 1.8. Let H_v be the associated endoscopic group. Let $a_H \in \mathfrak{c}_H(\mathcal{O}_v)$ with image $a \in \mathfrak{c}(\mathcal{O}_v)\cap\mathfrak{c}^{\mathrm{rs}}(F_v)$.

***8.6.1.* —** If the center of G_v contains a split torus C, C is also contained in the center of H_v. By replacing G_v with G_v/C and H_v with H_v/C, the κ-orbital integral and the stable orbital integral considered do not change, so we can assume that the center of G_v does not contain a split torus.

If the center of H_v contains a split torus C, C is naturally included in the torus I_{γ_0} where $\gamma_0 = \epsilon(a)$. The centralizer M_v of C in G_v is a Levi subgroup of G_v. By the descent formula, we can replace the κ-orbital integrals in G_v with a κ-orbital integral in M_v. The center of M_v now contains a split torus, and we are reduced to the situation discussed above. After a finite number of these reductions, we can assume that the centers of G_v and H_v do not contain split tori.

***8.6.2.* —** Let $\mathcal{M}_{H,v}(a_H)$ and $\mathcal{M}_v(a)$ be the associated affine Springer fibers. According to 3.5, there exists a natural integer N such that for any finite extension k' of k, for any $a'_H \in \mathfrak{c}_H(\mathcal{O}_v\otimes_k k')$ such that

$$a_H \equiv a'_H \mod \epsilon_v^N$$

then

– a'_H has an image $a' \in \mathfrak{c}(\mathcal{O}_v\otimes_k k')\cap\mathfrak{c}^{\mathrm{rs}}(F_v\otimes_k k')$;

– the affine Springer fibers $\mathcal{M}_{H,v}(a_H)\otimes_k k'$ and $\mathcal{M}_{H,v}(a'_H)$ equipped with the action of $\mathcal{P}_v(J_{H,a_H})\otimes_k k'$ and $\mathcal{P}_v(J_{H,a'_H})$ respectively, are isomorphic;

– the affine Springer fibers $\mathcal{M}_v(a)\otimes_k k'$ and $\mathcal{M}_v(a')$ equipped with the action of $\mathcal{P}_v(J_a)$ and $\mathcal{P}_v(J_{a'})$ respectively, are isomorphic.

Let $\delta_H(a_H)$ denote the dimension of the affine Springer fiber $\mathcal{M}_{H,v}(a_H)$.

***8.6.3.* —** There exists a geometrically connected smooth projective curve X over k equipped with the following data:

– two distinct k-points denoted v and ∞,

– an isomorphism between the completion of X at v with the scheme X_v above,

– a $\pi_0(\kappa)$-torsor ρ_κ over X equipped with a trivialization α_∞ above ∞ and an isomorphism α_v with $\rho_{\kappa,v}$ above X_v.

Let G and H be the associated X-group schemes as in 1.8. With the data of α_v, the restrictions of G and H to X_v are canonically isomorphic to G_v and H_v. Since the centers of G_v and H_v do not contain split tori, the same applies to G and H. With the data of α_∞, we have a homomorphism

$$\rho_\kappa^\bullet : \pi_1(X,\infty) = \pi_1(\overline{X},\infty)\rtimes\mathrm{Gal}(\bar{k}/k) \to \pi_0(\kappa)$$

trivial on the factor $\mathrm{Gal}(\bar{k}/k)$. Consequently, the subgroup $\pi_1(\overline{X},\infty)$ has the same image in $\pi_0(\kappa)$ as $\pi_1(X,\infty)$. It follows that over $\overline{X}$, the centers of G and H do not contain split tori.

8.6.4. — We now choose an invertible bundle D on X satisfying the following hypotheses:

- there exists an invertible bundle D' on X with $D = D'^{\otimes 2}$,
- a non-zero global section of D' at v,
- $\deg(D) > rN + 2g$ where r is the rank of G and g is the genus of X,
- $\delta_H(a_H)$ is smaller than the integer $\delta_H^{\mathrm{bad}}(D)$ defined in 7.8.2.

According to 5.7.2, the integer δ_H^{bad} tends to ∞ as $\deg(D) \to \infty$, so that for $\deg(D)$ sufficiently large, the last hypothesis is satisfied.

8.6.5. — Consider the Hitchin fibrations associated with curve X, the invertible bundle D, and the groups G and H respectively

$$f : \mathscr{M} \to \mathscr{A} \text{ and } f_H : \mathscr{M}_H \to \mathscr{A}_H.$$

The hypothesis that the centers of G and H do not contain split tori over $\overline{X}$ and the hypothesis $\deg(D) > 2g$ ensure that the anisotropic open sets $\mathscr{A}^{\mathrm{ani}}$ and $\mathscr{A}_H^{\mathrm{ani}}$ are non-empty.

8.6.6. — With the hypothesis $\deg(D) > rN + 2g$, the restriction map of global sections to $\mathrm{Spec}(\mathscr{O}_v/\epsilon_v^N)$

$$\mathrm{H}^0(X, \mathfrak{c}_{H,D}) \to \mathfrak{c}_H(\mathscr{O}_v/\epsilon_v^N)$$

is a surjective linear map. Let Z be the affine subspace of $\mathrm{H}^0(X,\mathfrak{c}_{H,D})$ of elements having the same image in $\mathfrak{c}_H(\mathscr{O}_v/\epsilon_v^N)$ as a_H. This is a closed sub-scheme of codimension rN of $\mathscr{A}_H$. By the same reasoning as in 4.7.1, we can show that the open set Z' of Z consisting of points $a'_H \in Z'(\bar{k})$ such that the curve $a'_H(\bar{X} - \{v\})$ intersects the divisor $\mathfrak{D}_{H,D} + \mathfrak{R}^G_{H,D}$ transversally, is a non-empty open set. Since the complement of $\mathscr{A}_H^{\mathrm{ani}}$ in $\mathscr{A}_H$ is a closed set of codimension greater than or equal to $\deg(D)$ *cf.* 6.3.6, $Z' \cap \mathscr{A}_H^{\mathrm{ani}} \neq \emptyset$. By replacing Z' with $Z' \cap \mathscr{A}_H^{\mathrm{ani}}$, we can assume that $Z' \subset \mathscr{A}_H^{\mathrm{ani}}$. For any $a'_H \in Z'(\bar{k})$, the invariant $\delta_H(a') = \delta_H(a_H)$ so that we have $Z' \subset \mathscr{A}_H^{\mathrm{good}}$ where we have the stabilization theorem *cf.* 8.5.

Let $\tilde{Z}'$ be the inverse image of Z' in $\tilde{\mathscr{A}}_H^{\mathrm{good}}$. As $\tilde{Z}'$ is a scheme of positive dimension, there exists an integer m such that for any finite extension k'/k of degree greater than or equal to m, the set $\tilde{Z}'(k') \neq \emptyset$. Let $\tilde{a}'_H \in Z'(k')$ above $a'_H \in Z(k')$. Let $\tilde{a}'$ be the image of $\tilde{a}'_H$ in $\tilde{\mathscr{A}}$ and a' be the image of a'_H in $\mathscr{A}$.

8.6.7. — The conjunction of the part of the stabilization theorem 6.4.2 proved on the open set $\tilde{\mathscr{A}}_H^{\mathrm{good}}$ with the formula 8.4.4 gives us the equality

$$\begin{aligned} & \sharp\,(\mathscr{P}'_{a'})^0(k') \textstyle\prod_{v'\in|(X-U)\otimes_k k'|} \sharp\,[\mathscr{M}_{H,v'}(a'_H)/\mathscr{P}_{v'}(J'_{a'})](k')_\kappa \\ = \;& q^{r(a'_H)\deg(k'/k)} \sharp\,(\mathscr{P}'_{a'})^0(k') \textstyle\prod_{v'\in|(X-U)\otimes_k k'|} \sharp\,[\mathscr{M}_{v'}(a'_H)/\mathscr{P}_{v'}(J'_{a'})](k') \end{aligned}$$

with a choice of a group scheme $J'_{a'}$ smooth, commutative, with connected fiber, equipped with a homomorphism

$$J'_{a'} \to J_{a'} \to J_{H,a'_H}$$

as in 8.4.1. It will be convenient to choose $J'_{a'} = J^0_{a'}$ above X_v.

8.6.8. — Since $\tilde{a}'_H \in \tilde{\mathscr{U}}(k')$, we can apply 8.5.7 to all places $v' \neq v$. By subtracting the equalities at these places, we find an equality at the initial place v

$$\sharp\,[\mathscr{M}_{H,v}(a'_H)/\mathscr{P}_v(J^0_{a'})](k')_\kappa = q^{r_v(a'_H)\deg(k'/k)} \sharp\,[\mathscr{M}_v(a'_H)/\mathscr{P}_v(J^0_{a'})](k').$$

As we know that the affine Springer fiber $\mathscr{M}_v(a) \otimes_k k'$ equipped with the action of $\mathscr{P}_v(J_a) \otimes_k k'$ is isomorphic to $\mathscr{M}_v(a')$ equipped with the action of $\mathscr{P}_v(J_{a'})$ and the same for group H, we deduce the equality

$$\sharp\,[\mathscr{M}_{H,v}(a_H)/\mathscr{P}_v(J^0_a)](k')_\kappa = q^{r^G_{H,v}(a_H)\deg(k'/k)} \sharp\,[\mathscr{M}_v(a_H)/\mathscr{P}_v(J^0_a)](k').$$

for any extension k' of k of degree greater than m. By applying 8.1.14, we obtain the equality

$$\sharp\,[\mathscr{M}_{H,v}(a_H)/\mathscr{P}_v(J_a^0)](k)_\kappa = q^{r_{H,v}^G(a_H)}\sharp\,[\mathscr{M}_v(a_H)/\mathscr{P}_v(J_a^0)](k).$$

By applying formula 8.2.5, we now obtain the equality

$$\mathbf{O}_a^\kappa(1_{\mathfrak{g}_v}, \mathrm{d}t_v) = q^{r_{H,v}^G(a_H)}\mathbf{SO}_{a_H}(1_{\mathfrak{h}_v}, \mathrm{d}t_v)$$

which was desired.

This completes the proof of the Langlands-Shelstad conjecture 1.11.1. □

8.7. Stabilization over $\tilde{\mathscr{A}}^{\mathrm{ani}}$. — By reversing the local-global process once again, we can now complete the proof of 6.4.2.

Proof. — Let's keep the notations from 8.5. As in 8.5, it suffices to prove the equality of traces 8.5.4

$$\sum_n(-1)^n\mathrm{tr}(\sigma_{k'}, (L_\kappa^n)_{\tilde{a}}) = \sum_n(-1)^n\mathrm{tr}(\sigma_{k'}, (L_{H,\mathrm{st}}^n)_{\tilde{a}_H}). \tag{8.7.1}$$

for any $\tilde{a}_H \in \tilde{\mathscr{A}}_H^{\mathrm{ani}}(k)$ with image $\tilde{a} \in \tilde{\mathscr{A}}^{\mathrm{ani}}(k)$. According to 8.4.4, the left side of 8.5.4 is

$$(\mathscr{P}'_a)^0(k)\prod_{v\in|X-U|}\sharp\,[\mathscr{M}_v(a)/\mathscr{P}_v(J'_a)](k)_\kappa$$

while the right side is

$$q^{r_H^G(D)}(\mathscr{P}'_a)^0(k)\prod_{v\in|X-U|}\sharp\,[\mathscr{M}_{H,v}(a)/\mathscr{P}_v(J'_a)](k).$$

By combining 1.11.1, 8.2.5, we have the equality

$$[\mathscr{M}_v(a)/\mathscr{P}_v(J'_a)](k)_\kappa = q^{r_{H,v}^G(D)}[\mathscr{M}_{H,v}(a)/\mathscr{P}_v(J'_a)](k)$$

whence 8.7.1. □

8.8. Waldspurger's Conjecture. — Now let G_1 and G_2 be two X-group schemes paired in the sense of 1.12.4. Suppose that the centers of G_1 and G_2 do not contain split tori over $\overline{X} = X\otimes_k \bar{k}$.

8.8.1. — Let $f_1 : \mathscr{M}_1 \to \mathscr{A}_1$ and $f_2 : \mathscr{M}_2 \to \mathscr{A}_2$ be the associated Hitchin fibrations. As in 4.18, we have

$$\mathscr{A} = \mathscr{A}_1 = \mathscr{A}_2.$$

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the $\mathscr{A}$-Picard stacks associated with G_1 and G_2. According to 4.18.1, there exists a homomorphism $\mathscr{P}_1 \to \mathscr{P}_2$ which induces an isogeny between their neutral components.

Theorem 8.8.2. — *There exists an isomorphism between the simplifications of the perverse graded sheaves on $\mathscr{A}^{\mathrm{ani}}$*

$$K_1 = \bigoplus_n {}^p\mathrm{H}^n(f_{1,*}^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}} \text{ and } K_2 = \bigoplus_n {}^p\mathrm{H}^n(f_{2,*}^{\mathrm{ani}}\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}.$$

The proof of this theorem essentially follows the same steps as that of 6.4.2. In particular, we will prove along the way the non-standard fundamental lemma conjectured by Waldspurger 1.12.7.

8.8.3. — We first show that such an isomorphism exists above the open set $\mathscr{A}^\diamond$. Above this open set, the morphisms f_1 and f_2 are proper and smooth, so that the restrictions of K_1 and K_2 to $\mathscr{A}^\diamond$ are graded local systems. To show that there exists an isomorphism between their simplifications, it suffices, according to Chebotarev's theorem, to prove the equality of traces

$$\mathrm{tr}(\sigma_{k'}, K_{1,a}) = \mathrm{tr}(\sigma_{k'}, K_{2,a}) \tag{8.8.4}$$

for any finite extension k' of k and for any point $a \in \mathcal{A}^{\diamondsuit}(k')$. By replacing X with $X \otimes_k k'$, we can assume that $k = k'$.

8.8.5. — Let $a \in \mathcal{A}^{\diamondsuit}(k)$. According to 4.7.7, we know:

– $\mathcal{P}_{i,a}$ acts simply transitively on $\mathcal{M}_{i,a}$

– $\mathcal{P}_{i,a}$ is of the form $[P_i/A_i]$ where P_i is an extension of a finite group by an abelian variety and where A_i is a finite group acting trivially on P_i.

Consequently, the quotient $[\mathcal{M}_{i,a}/P_i]$ is isomorphic to the classifying stack of the finite group A_i. According to the fixed-point formula 8.1.6 and 8.1.7, we have

$$\mathrm{tr}(\sigma, K_{i,a}) = \sharp P_i^0(k).$$

Since P_1^0 and P_2^0 are isogenous abelian varieties over k, they have the same number of k-points, hence the equality of traces 8.8.4. Therefore, there exists an isomorphism between the simplifications of the restrictions of K_1 and K_2 to $\mathcal{A}^{\diamondsuit}$.

8.8.6. — According to the support theorem 7.8.3, there exists an isomorphism between the semi-simplifications of the restrictions of K_1 and K_2 to

$$\mathcal{A}^{\mathrm{good}} = \mathcal{A}^{\mathrm{ani}} - \mathcal{A}^{\mathrm{bad}}.$$

8.8.7. — Proceeding as in 8.6, we deduce the non-standard fundamental lemma conjectured by Waldspurger 1.12.7.

8.8.8. — By reversing the local-global process again as in 8.7, we deduce the equality of traces 8.8.4 for any $a \in \mathcal{A}^{\mathrm{ani}}(k')$ for any finite extension k' of k. We deduce Theorem 8.8.2.

8.8.9. — If one grants the conjecture 7.8.1, then Theorem 7.2.1 implies that the stable part of the perverse cohomology sheaves of the Hitchin fibration is the intermediate extension of its restriction to the open subset $\mathcal{A}^{\diamondsuit}$, where it consists of local systems.

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Appendix A. Poincaré Duality and Dimension Counting according to Goresky and MacPherson

Goresky and MacPherson observed that Poincaré duality imposes a constraint on the codimension of support of simple perverse sheaves present in the decomposition theorem. This observation played a crucial role in the proof of the support theorem. It seems to us that it should also have other applications. With their permission, we recall this argument of dimension counting and Poincaré duality in a general context and in the simplest possible form. In full generality, this constraint is relatively weak, but in practice, it provides a valuable starting point with the help of which one can bring into play other more specific arguments.

Let X and Y be finite type schemes over a finite field $k = \mathbb{F}_q$. Let $f : X \to Y$ be a proper morphism. Suppose furthermore that X is a smooth k-scheme. Then, according to Deligne [**18**], the direct image $f_*\overline{\mathbb{Q}}_\ell$ is a pure complex. According to [**7**], it decomposes geometrically as a direct sum of geometrically irreducible perverse sheaves with shifts. There exists an isomorphism over $Y \otimes_k \bar{k}$

$$f_*\overline{\mathbb{Q}}_\ell = \bigoplus_{(K,n)} K[-n]^{m_{K,n}}$$

where the direct sum is extended over the set of equivalence classes of pairs (K, n) consisting of an irreducible perverse sheaf K on $Y \otimes_k \bar{k}$ and an integer n, and where

$$m_{K,n} := \dim \operatorname{Hom}(K, {}^p\mathrm{H}^n(f_*\overline{\mathbb{Q}}_\ell))$$

is a natural integer, zero except for a finite number of pairs (K, n). A geometrically irreducible perverse sheaf K is said to be *present* in $f_*\overline{\mathbb{Q}}_\ell$ if there exists an integer n such that $m_{K,n} \neq 0$.

Poincaré duality implies a symmetry for these integers $m_{K,n}$.

Proposition 1. — *For any geometrically irreducible perverse sheaf K on Y, then we have*

$$m_{K,n+\dim(X)} = m_{DK,-n+\dim(X)}$$

where DK is the Verdier dual of K.

Goresky and MacPherson observed that this symmetry imposes a constraint on the codimension of the supports of the geometrically irreducible perverse sheaves K present in $f_*\overline{\mathbb{Q}}_\ell$.

Theorem 2. — *Let X and Y be finite type schemes over a field k, with X smooth over k. Let $f : X \to Y$ be a proper morphism. Suppose that the morphism f is of relative dimension d. Let K be an irreducible perverse sheaf on $Y \otimes_k \bar{k}$ present in $f_*\overline{\mathbb{Q}}_\ell$. Let Z be the support of K. Then we have the inequality*

$$\operatorname{codim}(Z) \leq d.$$

Proof. — Suppose on the contrary that $\operatorname{codim}(Z) > d$. According to Poincaré duality and by exchanging K and DK which have the same support, we can assume that there exists an integer n

$$n \geq \dim(X)$$

such that $m_{K,n} \neq 0$. According to [**7**], there exists an open set Z' of Z, a local system K' on S such that $K = j_{!*}K'[\dim(Z)]$ where j is the inclusion of the open set Z' in Z. Let y be a geometric point of Z'. The fiber of K at y is then a $\overline{\mathbb{Q}}_\ell$-vector space placed in degree

$$-\dim(Z) = -\dim(Y) + \operatorname{codim}(Z).$$

The fiber of $K[-n]$ at y being a direct factor $\mathrm{R}\Gamma(X_y, \overline{\mathbb{Q}}_\ell)$, this implies that

$$\mathrm{H}^{n-\dim(Y)+\operatorname{codim}(Z)}(X_y) \neq 0.$$

However, we have the inequality

$$n - \dim(Y) + \operatorname{codim}(Z) > 2d$$

because $n \geq \dim(X)$ and $\operatorname{codim}(Z) > d$. This non-vanishing is in contradiction with the hypothesis that the fiber X_y is of dimension less than or equal to d. □

Theorem 3. — *Let us assume the hypotheses of the previous theorem. Suppose in addition that the fibers of f are geometrically irreducible of dimension $d > 0$. Let K be an irreducible perverse sheaf on $Y \otimes_k \bar{k}$ present in $f_* \overline{\mathbb{Q}}_\ell$. Let Z be the support of K. Then we have the strict inequality*

$$\mathrm{codim}(Z) < d.$$

Proof. — Under the hypothesis that the fibers of f are irreducible, the cohomology sheaf of maximal degree $2d$ is the local system

$$\mathrm{H}^{2d}(f_* \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell(-d).$$

Consider the decomposition of $f_* \overline{\mathbb{Q}}_\ell$ on $Y \otimes_k \bar{k}$

$$f_* \overline{\mathbb{Q}}_\ell = \bigoplus_{(L,n)} L[-n]^{m_{L,n}}$$

where L ranges over the set of isomorphism classes of irreducible perverse sheaves on $Y \otimes_k \bar{k}$ and n the set of integers. If $m_{L,n} \neq 0$, then we have $\mathrm{H}^i(L[-n]) = 0$ for all $i > 2d$ and even $\mathrm{H}^{2d}(L[-n]) = 0$ if L is not isomorphic to $\overline{\mathbb{Q}}_\ell[\dim(Y)]$.

The same argument as in the previous theorem shows that for any irreducible perverse sheaf K on $Y \otimes_k \bar{k}$ which is not isomorphic to $\overline{\mathbb{Q}}_\ell[-\dim(Y)]$, then the support Z of K must satisfy the strict inequality $\mathrm{codim}(Z) < d$. Of course, if K is isomorphic to $\overline{\mathbb{Q}}_\ell[-\dim(Y)]$, its support is Y entirely and the inequality is trivially satisfied. □

References

[1] Altman A., Iarrobino A., Kleiman S.: Irreducibility of the compactified Jacobian. in *Real and complex singularities* (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 1–12.

[2] Arthur J., An introduction to the trace formula. in *Harmonic analysis, the trace formula, and Shimura varieties*, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[3] Artin M. Algebraic approximation of structures over complete local rings. *Inst. Hautes études Sci. Publ. Math.* 36 (1969) 23–58.

[4] Beauville A., Laszlo Y.: Un lemme de descente. *C. R. de l'Acad. des Sci. de Paris* 320 (1995) 335-340.

[5] Beauville A., Narasimhan M., Ramanan S. : Spectral curves and generalized theta divisor. *J. Reine Angew. Math.* 398 (1989) 169-179.

[6] Beilinson A., Drinfeld V.: Opers, preprint.

[7] Beilinson A., Bernstein J., Deligne P.: Faisceaux pervers. *Astérisque* 100 (1982).

[8] Bezrukavnikov R.: The dimension of the fixed points set on affine flag manifolds. *Mathematical Research Letters* 3 (1996) 185–189.

[9] Biswas I., Ramanan S. : Infinitesimal study of Hitchin pairs. *J. London Math. Soc.* 49 (1994) 219-231.

[10] Bosch S., Lutkebohmert W., Raynaud M. : *Neron models*. Ergeb. der Math. 21. Springer Verlag 1990.

[11] Bourbaki N. *Groupes et algèbres de Lie*, chapitres 4,5 et 6. Masson, Paris 1981.

[12] Carter R.: *Finite Groups of Lie Type*. Wiley Classics Library.

[13] Chai C.-L., Yu J.-K.: Congruences of Néron models for tori and the Artin conductor. *Ann. of Math. (2)* **154** (2001).

[14] Clozel L., The fundamental lemma for stable base change. Duke Math. J. 61 (1990), no. 1, 255–302.

[15] Cluckers, R., Loeser, F. Fonctions constructibles exponentielles, transformation de Fourier motivique et principe de transfert . *Comptes Rrendus de l'Acad. des Sciences de Paris* 341, 741-746 (2005)

[16] Dat, J.-F., Lemme fondamental et endoscopie, une approche géométrique, *Séminaire Bourbaki* 940 novembre 2004.

[17] Debarre O.: Théorèmes de connexité pour les produits d'espaces projectifs et les grassmanniennes. *Am. J. Math.* **118** (1996), 1347–1367.

[18] Deligne P.: La conjecture de Weil II. *Publ. Math. de l'I.H.E.S.* 52 (1980) 137–252.

[19] Deligne P.: Décomposition dans la catégorie dérivée, in *Motives*, Proc. of Symp. in Pure Math. vol. 55.1(1994) 115–128.

[20] Deligne P. Communication privée, 2007.

[21] Demazure M., Grothendieck A.: *Séminaire de géométrie algébrique du Bois-Marie 3*. LNM 151, 152, 153, Springer Verlag.

[22] S. Diaz, J. Harris, Ideals associated to Deformations of singular plane curves, *Trans. Amer. Math. Soc.* 309, 433-468. (1988).

[23] Donagi R., Gaitsgory D.: The gerb of Higgs bundles. *Transform. Groups* 7 (2002) 109–153.

[24] Faltings G.: Stable G-bundles and projective connections. *J. Alg. Geom.* 2 (1993) 507–568.

[25] Fantechi B., Göttsche L., Van Straten D. : Euler number of the compactified Jacobian and multiplicity of rational curves. *J. Alg. Geom.* 8 (1999), 115-133.

[26] Goresky M., Kottwitz R., MacPherson R.: Homology of affine Springer fiber in the unramified case. *Duke Math. J.* 121 (2004) 509–561.

[27] Goresky M., Kottwitz R., MacPherson R.: Purity of equivalued affine Springer fibers. *Represent. Theory* 10 (2006), 130–146.

[28] Goresky M., Kottwitz R., MacPherson R.: Codimension of root valuation strata. preprint 2006.

[29] Grothendieck A.: Groupes de monodromie en Géométrie algébrique (SGA 7 I), LNM 288, Springer Verlag.

[30] Grothendieck A., Dieudonné J.: éléments de géométrie algébrique IV. étude locale des schémas et de morphismes de schémas, *Pub. Math. de l' I.H.E.S.* 20, 24, 28 et 32.

[31] Hales, T.: The fundamental lemma for Sp(4). *Proc. Amer. Math. Soc.* **125** (1997) no. 1, 301–308.

[32] Hales, T.: A statement of the fundamental lemma. in *Harmonic analysis, the trace formula, and Shimura varieties*, 643–658, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[33] Heinloth J.: Uniformization of $\mathcal{G}$-bundles. *Math. Ann.*

[34] Hitchin N.: Stable bundles and integrable connections. *Duke Math. J.* 54 (1987) 91–114.

[35] Kazhdan D.: On lifting. in *Lie group representations,* II (College Park, Md., 1982/1983), 209–249, Lecture Notes in Math., 1041, Springer, Berlin, 1984.

[36] Kazhdan D., Lusztig G.: Fixed point varieties on affine flag manifolds. *Israel J. Math.* 62 (1988), no. 2, 129–168.

[37] Kleiman S.: Algebraic cycles and Weil conjectures, *in Dix exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968.

[38] Kostant B.: Lie group representations on polynomial rings. *Amer. J. of Math.* 85 (1963) 327–404.

[39] Kottwitz R. Orbital integrals on GL_3. *Amer. J. Math.* 102 (1980), no. 2, 327–384.

[40] Kottwitz R.: Unstable orbital integrals on SL(3). *Duke Math. J.* 48 (1981), no. 3, 649–664

[41] Kottwitz R. Stable trace formula : cuspidal tempered terms. *Duke Math. J.* 51 (1984) 611-650.

[42] Kottwitz R. Isocristal with additionnal structures. *Composotion Math.* 56 (1985) 201–220.

[43] Kottwitz R. Base change for unit elements of Hecke algebras. *Compositio Math.* 60 (1986), no. 2, 237–250.

[44] Kottwitz R.: Stable trace formula : elliptic singular terms. *Math. Ann.* 275 (1986) 365-399.

[45] Kottwitz R.: Shimura varieties and λ-adic representations, in *Automorphic forms, Shimura varieties, and L-functions*, Vol. I 161–209, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[46] Kotttwiz R. Transfert factors for Lie algebra. *Represent. Theory* 3 (1999) 127-138.

[47] Labesse, J.-P. Fonctions élémentaires et lemme fondamental pour le changement de base stable. *Duke Math. J.* 61 (1990), no. 2, 519–530.

[48] Labesse, J.-P. et Langlands, R., L-indistinguishability for SL(2). *Can. J. Math.* **31** (1979) 726–785.

[49] Langlands R. *Base change for* GL(2). Annals of Mathematics Studies, 96. Princeton University Press 1980.

[50] Langlands R.: *Les débuts d'une formule des traces stables*. Publications de l'Université Paris 7, 13 (1983).

[51] Laumon G.: Fibres de Springer et Jacobiennes compactifiées in*Algebraic geometry and number theory*, 515–563, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006..

[52] Laumon, G.: Sur le lemme fondamental pour les groupes unitaires, prépublication `arxiv.org/abs/math/0212245`.

[53] Laumon G., Moret-Bailly L.: *Champs algébriques*. Ergebnisse der Mathematik 39. Springer-Verlag, Berlin, 2000.

[54] Laumon, G. et Ngô B.C.: Le lemme fondamental pour les groupes unitaires, à paraître aux *Annals of Math.*

[55] Levy P.: Involutions of reductive Lie algebras in positive characteristic. *Adv. Math.* **210** (2007), 505–559.

[56] Mumford, D. *Abelian varieties*. Oxford University Press.

[57] Ngô B.C.: Fibration de Hitchin et endoscopie. *Inv. Math.* 164 (2006) 399–453.

[58] Ngô B.C.: Fibration de Hitchin et structure endoscopique de la formule des traces. *International Congress of Mathematicians* Vol. II, 1213–1225, Eur. Math. Soc., Zürich, 2006.

[59] Ngô B.C. : Fibrations de Hitchin pour les groupes classiques. En préparation.

[60] Ngô B.C.: Decomposition theorem and abelian fibration. Article d'exposition pour le projet du livre, disponible à l'adresse `http://fa.institut.math.jussieu.fr/node/44`.

[61] Nitsure : Moduli space of semistable pairs on a curve. *Proc. London Math. Soc.* (3) 62 (1991), no. 2, 275–300.

[62] Ono T., On Tamagawa numbers. In *Algebraic groups and discontinuous subgroups*. Proc. of Symp. in Pure Math. **9** (1966), A.M.S.

[63] Rapoport M.: A guide to the reduction of Shimura varieties in *Automorphic forms*. I. Astérisque No. 298 (2005), 271–318.

[64] Raynaud M. : Communication privée, 2004.

[65] Rogawski, J., *Automorphic representations of unitary groups in three variables.* Annals of Math. Studies 123, 1–259, Princeton University Press, Princeton 1990.

[66] Rosenlicht M.: Some basic theorems on algebraic groups. *Amer. J. Math.* 78 (1956), 401–443.

[67] Saito H.: *Automorphic Forms and Algebraic Extensions of Number Fields*. Lectures in Mathematics **8**, Kinokuniya, 1975.

[68] M. Schöder. Inauguraldissertation, Mannheim 1993.

[69] J.-P. Serre. *Groupes algébriques et corps de classes* Publications de l'Institut de Mathématique de l'Université de Nancago, VII. Hermann, Paris 1959.

[70] J.-P. Serre. *Corps locaux*. Publications de l'Institut de Mathématique de l'Université de Nancago, VIII. Hermann, Paris 1962.

[71] D. Shelstad. Orbital integrals and a family of groups attached to a real reductive group, *Ann. Scient. de l'Ecole Normale Supérieure*, 12 (1979).

[72] Spaltenstein. On the fixed point set of a unipotent element on the variety of Borel subgroups. *Topology* 16 (1977), no. 2, 203–204.

[73] Springer, T. Some arithmetical results on semi-simple Lie algebras. *Pub. Math. de l'I.H.E.S.*, 33 (1966), 115-141.

[74] Springer T.: Reductive groups *in Automorphic forms, representations, and L-functions*. Proc. Symp. in Pure Math. 33-1 AMS 1997.

[75] T. Springer, R. Steinberg.: Conjugacy classes *in Seminar on algebraic groups and related finite groups. Lectures Notes in Math. 131* Springer Verlag 1970.

[76] Steinberg R.: *Lectures on Chevalley groups*.

[77] Teissier B., Résolution simultanée - I. Famille de courbes, in *Séminaire sur les Singularités des Surfaces*, M. Demazure, H. Pinkham, B. Teissier eds, Springer LNM 777 (1980).

[78] Veldkamp F.: The center of the universal enveloping algebra of a Lie algebra in characteristic p. *Ann. Sci. École Norm. Sup.*

[79] Waldspurger, J.-L., Sur les intégrales orbitales tordues pour les groupes linéaires : un lemme fondamental. *Can. J. Math.* **43** (1991) 852–896.

[80] Waldspurger, J.-L. , Le lemme fondamental implique le transfert. *Compositio Math.*, **105** (1997) 153–236.

[81] Waldspurger J.-L. Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés *Astérisque* 269.

[82] Waldspurger J.-L. Endoscopie et changement de caractéristique, prépublication.

[83] Waldspurger J.-L.: L'endoscopie tordue n'est pas si tordue : intégrales orbitales, prépublication 2006.

[84] Weissauer R., A special case of fundamental lemma I-IV, preprint Mannheim 1993.

[85] Whitehouse D.: The twisted weighted fundamental lemma for the transfer of automorphic forms from GSp(4). Formes automorphes. II. Le cas du groupe GSp(4). Astérisque No. 302 (2005), 291–436.

[86] Faltings G.: Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc.* 5 (2003), no. 1, 41–68.

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