On the Harder-Narasimhan Filtration of Principal Bundles

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1

Let X be a nonsingular projective curve over a field k of characteristic $p > 0$ and let $E \to X$ be a principal G-bundle on X with G a reductive group over k, or more generally, let $\mathcal{G} \to X$ be a reductive group scheme over X (we observe that given a principal G-bundle $E \to X$, we can obtain a group scheme $E(G) \to X$ over X, by taking the associated fibre space to E for the conjugacy action of G on itself). If $\mathcal{G} \to X$ is not semistable (see [7] for the notion of semistability of principal bundles, and [4] for that of group schemes), then Behrend (in [4]), Atiyah-Bott (in [1]) and A. Ramanathan (in [7]) have each defined the Harder-Narasimhan filtration (or the canonical filtration) of $\mathcal{G}$. Moreover, if $\mathfrak{p}$ denotes the Lie algebra bundle of the canonical parabolic $\mathcal{P}$ and $\mathfrak{g}$ that of $\mathcal{G}$, then Behrend (see [4, Conjecture 7.6]) conjectures that

$$H^0(X, \mathfrak{g}/\mathfrak{p}) = 0.$$

This conjecture implies that the canonical parabolic descends over purely inseparable field extensions.

We recall (from [5]) that if G is a reductive group over an algebraically closed field k and $\alpha = \sum k_i \alpha_i$ is a root of G where the α_i are simple roots and k_i are integers, then the height of α is defined to be $\sum k_i$. Let $h(G)$ denote the height of the highest root of G. In Section 3 in this article, we show that the canonical parabolic of Behrend coincides with the canonical parabolic of Atiyah-Bott if $\operatorname{char} k = p > \max(2 \dim G, 4h(G))$. As a consequence, we obtain a proof of Behrend's conjecture (see Corollary 3.6) when $\operatorname{char} k = p > \max(2 \dim G, 4h(G))$. In Section 4, we show that if $\operatorname{char} k = p > 3$ and $\mu_{\min}(TX) \geq 0$ (where TX is the tangent bundle of X: in Section 4, X is a nonsingular projective variety of arbitrary dimension), then Behrend's conjecture is true.

For the definition of the Harder-Narasimhan filtration of a vector bundle V, $\mu_{\min}(V)$, $\mu_{\max}(V)$, and the β-subbundle of V, we refer to [1, pages 588–589], where $\mu_{\min}(V)$ is denoted by $\inf V$, $\mu_{\max}(V)$ is denoted by $\sup V$, and the β-subbundle is E_1, the first step in the Harder-Narasimhan filtration.

2

Let X be a nonsingular curve over a field k. Let $E \to X$ be a principal G-bundle on X, where G is a reductive group defined over k. Let $\mathfrak{g}$ be the Lie algebra of G and let $E(\mathfrak{g})$ be the Lie algebra bundle obtained from E by the adjoint representation of G on $\mathfrak{g}$. Let

$$E(\mathfrak{g})_{-r} \subset E(\mathfrak{g})_{-r+1} \subset \cdots \subset E(\mathfrak{g})_{-1} \subset E(\mathfrak{g})_0 \subset E(\mathfrak{g})_1 \subset \cdots \subset E(\mathfrak{g})_s = E(\mathfrak{g}) \qquad (*)$$

be the Harder-Narasimhan filtration of the vector bundle $E(\mathfrak{g})$. Here we have indexed the filtration so that

$$\mu(E_i/E_{i-1}) < 0 \quad \text{for } i \geq 1 \quad \text{and} \quad \mu(E_{-j}/E_{-j-1}) \geq 0 \quad \text{for } j \geq 0,$$

where $\mu(V)$ denotes the slope degree $V/\operatorname{rank} V$ for a vector bundle V on X.

Lemma 2.1. *Let $G = GL(n)$ and $V \to X$ a rank n vector bundle on X so that $E \to X$ is the frame bundle of V. Then the Lie algebra bundle $E(\mathfrak{gl}(n)) = V^* \otimes V$. Let $E(\mathfrak{gl}(n))_0$ be as in $(*)$ above for $E(\mathfrak{gl}(n))$. Then $E(\mathfrak{gl}(n))_0$ is a bundle of parabolic subalgebras of $E(\mathfrak{gl}(n))$ if* char $k = p > 2n$.

Proof. Let

$$V_1 \subset V_2 \subset \cdots \subset V_\ell = V$$

be the Harder-Narasimhan filtration of V. Let

$$0 = E^{-\ell} \subset E^{-\ell+1} \subset \cdots \subset E^{-1} \subset E^0 \subset E^1 \subset \cdots \subset E^\ell = \operatorname{End}(V)$$

be the filtration of $\operatorname{End}(V)$ defined by

$$E^j = \{\varphi \in \operatorname{End}(V) \mid \varphi(V_i) \subset V_{i+j}, \text{ for all } i = 1, \ldots, \ell\}$$

with the convention that $V_i = 0$ for $i \leq 0$ and $V_i = V$ for $i \geq \ell$. Then we can see that

$$E^j/E^{j-1} \cong \bigoplus_i \operatorname{Hom}(V_i/V_{i-1}, V_{i+j}/V_{i+j-1}).$$

From the definition of the E^j, E^0 consists of endomorphisms of V preserving the given flag in V, and hence E^0 is a bundle of parabolic subalgebras of $\operatorname{End}(V)$. Consider the map

$$E^0 \subset \operatorname{End}(V) \to \operatorname{End} V/E_{s-1},$$

where E_{s-1} denotes $E(\mathfrak{gl}(n))_{s-1}$ (and in general, we will let E_i denote $E(\mathfrak{gl}(n))_i$).

Claim. *The map $E^0 \to \operatorname{End} V/E_{s-1}$ is zero.*

Proof of Claim. Consider the composite map

$$E^{-\ell+1} \subset E^0 \to \operatorname{End} V/E_{s-1}.$$

Since $E^{-\ell+1} \cong \operatorname{Hom}(V_\ell/V_{\ell-1}, V_1)$, $E^{-\ell+1}$ is semistable if char $k = p > \operatorname{rank}(V_\ell/V_{\ell-1}) + \operatorname{rank} V_1$, in particular, if $p > 2n$. (Here, and subsequently, we use the fact that if V_1 and V_2 are semistable vector bundles with $\operatorname{rank} V_1 + \operatorname{rank} V_2 < p$, then $V_1 \otimes V_2$ is semistable, from [5]). Also,

$$\mu(E^{-\ell+1}) = -\mu(V_\ell/V_{\ell-1}) + \mu(V_1) > 0.$$

Since $\operatorname{End} V/E_{s-1}$ is semistable with $\mu(\operatorname{End} V/E_{s-1}) < 0$, it follows that the map $E^{-\ell+1} \to \operatorname{End} V/E_{s-1}$ is zero. Hence the map $E^0 \to \operatorname{End} V/E_{s-1}$ factors through a map $E^0/E^{-\ell+1} \to \operatorname{End} V/E_{s-1}$. Now consider the composite map

$$E^{-\ell+2}/E^{-\ell+1} \subset E^0/E^{-\ell+1} \to \operatorname{End} V/E_{s-1}.$$

Since

$$E^{-\ell+2}/E^{-\ell+1} \cong \mathrm{Hom}(V_\ell/V_{\ell-1}, V_2/V_1) \oplus \mathrm{Hom}(V_{\ell-1}/V_{\ell-2}, V_1),$$

and the maps

$$\mathrm{Hom}(V_\ell/V_{\ell-1}, V_2/V_1) \to \mathrm{End}\, V/E_{s-1}$$

and

$$\mathrm{Hom}(V_{\ell-1}/V_{\ell-2}, V_1) \to \mathrm{End}\, V/E_{s-1}$$

are both zero if $p > 2n$ (since μ (left handside) $> 0 > \mu$ (right handside)), it follows that

$$E^{-\ell+2}/E^{-\ell+1} \to \mathrm{End}\, V/E_{s-1}$$

is zero, and so the map

$$E^0/E^{-\ell+1} \to \mathrm{End}\, V/E_{s-1}$$

factors through a map

$$E^0/E^{-\ell+2} \to \mathrm{End}\, V/E_{s-1}.$$

Arguing inductively as above, we obtain that the map

$$E^0 \to \mathrm{End}\, V/E_{s-1}$$

factors through a map

$$E^0/E^{-1} \to \mathrm{End}\, V/E_{s-1}.$$

Since

$$E^0/E^{-1} \cong \bigoplus_i \mathrm{Hom}(V_i/V_{i-1}, V_i/V_{i-1})$$

and the maps

$$\mathrm{Hom}(V_i/V_{i-1}, V_i/V_{i-1}) \to \mathrm{End}\, V/E_{s-1}$$

are zero if $p > 2n$, it follows that the map $E^0 \to \mathrm{End}\, V/E_{s-1}$ is zero if $p > 2n$. This shows the claim. □

We hence obtain that $E^0 \subset E_{s-1} = E(\mathfrak{gl}(n))_{s-1}$. Considering the map

$$E^0 \to E_{s-1}/E_{s-2},$$

and arguing as before, we see that this map is also zero and hence $E^0 \subset E_{s-2}$. Continuing this, we obtain $E^0 \subset E_0$.

Conversely, consider the map

$$E_0 \to \mathrm{End}\, V/E^{\ell-2} = E^{\ell-1}/E^{\ell-2},$$

where $E_0 = E(\mathfrak{gl}(n))_0$. Since $\mathrm{End}\, V/E^{\ell-2} \cong \mathrm{Hom}(V_1, V/V_{\ell-1})$, we have

$$\mu(\mathrm{End}\, V/E^{\ell-2}) = -\mu(V_1) + \mu(V/V_{\ell-1}) < 0.$$

Also, $\mu_{\min}(E_0) \geq 0$. Hence the map

$$E_0 \to \mathrm{End}\, V/E^{\ell-2}$$

is zero if $p > 2n$ (since $p > 2n \implies \mathrm{Hom}(V_1, V/V_{\ell-1})$ is semistable), and $E_0 \subset E^{\ell-2}$. Continuing this argument, we obtain $E_0 \subset E^0$.

It follows that $E_0 = E^0$ and hence $E_0 = E(\mathfrak{gl}(n))_0$ is a bundle of parabolic subalgebras of $\mathrm{End}(V)$. □

Proposition 2.2. *Let G be a reductive group and $E \to X$ a principal G-bundle on X. Let $E_0 = E(\mathfrak{g})_0$ be as in ($*$), a subbundle of $E(\mathfrak{g})$. Then E_0 is a bundle of parabolic subalgebras, if* $\operatorname{char} k = p > \max(2\dim G, 4h(G))$.

Proof. It is easy to see that it is enough to consider the case when G is a group with trivial centre. So we assume that the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is faithful. Let $E(GL(\mathfrak{g}))$ be the principal $GL(\mathfrak{g})$ bundle associated to E by the representation Ad. Let $E(\mathfrak{gl}(\mathfrak{g}))$ be the Lie algebra bundle of $E(GL(\mathfrak{g}))$. Since $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ by the adjoint representation, we obtain $E(\mathfrak{g}) \subset E(\mathfrak{gl}(\mathfrak{g}))$. We now appeal to the Main Theorem in [8] (see also [5]) which states that a G-module is semisimple if $p > 2 \cdot$ height of the module, to obtain that

$$\mathfrak{gl}(\mathfrak{g}) = \mathfrak{g} \oplus W$$

as G-modules, if $\operatorname{char} k = p > 4h(G)$ (we observe that height of $\mathfrak{gl}(\mathfrak{g})$ as a G-module is $2h(G)$). Hence we obtain the decomposition

$$E(\mathfrak{gl}(\mathfrak{g})) = E(\mathfrak{g}) \oplus E(W)$$

of the associated bundles. It is then easy to see that

$$E(\mathfrak{gl}(\mathfrak{g}))_0 = E(\mathfrak{g})_0 \oplus E(W)_0$$

with notation as in ($*$). By Lemma 2.1, $E(\mathfrak{gl}(\mathfrak{g}))_0$ is a bundle of parabolic subalgebras, if $p > 2\dim G$. Consider $E(\mathfrak{g}) \subset E(\mathfrak{gl}(\mathfrak{g}))$ and fix a closed point $x \in X$. There exists $t \in GL(\mathfrak{g})$ such that $tE(\mathfrak{gl}(\mathfrak{g}))_{0,x}t^{-1}$ intersects $E(\mathfrak{g})_x$ in a parabolic subalgebra of $E(\mathfrak{g})_x$, where the subscript x denotes the fibre of the corresponding bundle at x. Consider the subgroup tGt^{-1} of $GL(\mathfrak{g})$ with Lie algebra $t\mathfrak{g}t^{-1}$. The Lie algebra bundle $E_G(\mathfrak{g})$ is isomorphic to the Lie algebra bundle $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})$ where $E_{tGt^{-1}}$ is the principal tGt^{-1} bundle on X obtained by translating the principal G bundle E_G. The isomorphism

$$E_G(\mathfrak{g}) \cong E_{tGt^{-1}}(t\mathfrak{g}t^{-1})$$

takes $E_G(\mathfrak{g})_0$ to $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_0$. But $E(\mathfrak{gl}(\mathfrak{g}))_{0,x}$ intersects $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_x$ in a parabolic subalgebra by choice of t. Thus

$$E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_{0,x} = E(\mathfrak{gl}(\mathfrak{g}))_{0,x} \cap E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_x$$

is a parabolic subalgebra, and hence so is $E_G(\mathfrak{g})_{0,x}$. Since x is an arbitrary closed point of X, we obtain that $E(\mathfrak{g})_0$ is a parabolic subalgebra bundle. □

3

Let $E \to X$ be a principal G-bundle on X, where X is a nonsingular projective curve.

Definition 3.1. *A reduction of structure group $E_P \subset E$ of E to a parabolic subgroup P of G is said to be canonical, or the Harder-Narasimhan filtration of E, if, letting $E(G) \to X$ be the group scheme associated to E by the adjoint action of G on itself, and $E(P) \to X$ the parabolic subgroup scheme of $E(G)$ associated to E_P by the adjoint action of P on itself, the following conditions are satisfied:*

(i) $\deg E(P) > 0$ *(for any group scheme* $\mathcal{G} \to X$ *over* X*,* $\deg \mathcal{G}$ *denotes the degree of the Lie algebra bundle of* $\mathcal{G}$ *on* X*, regarded as a vector bundle on* X*),*

(ii) *for any parabolic subgroup scheme* $Q \subset E(G)$*,* $\deg Q \leq \deg E(P)$*,*

(iii) *for any parabolic subgroup scheme* Q *containing* $E(P)$*,* $\deg Q < \deg E(P)$*,*

(iv) *the associated bundle* $E_P(P/R_u(P))$ *is semistable, where* $R_u(P)$ *is the unipotent radical of* P*.*

With this definition, our notion of the canonical reduction (or Harder-Narasimhan filtration) coincides with that of Behrend (see [4, definition 5.6, and the proof of Proposition 7.2]).

We need the following lemmas:

Lemma 3.2. *Let* $V \to X$ *be a vector bundle on* X *such that* $\mu_{\min}(V) \geq 0$ *(recall that* $\mu_{\min}(V)$ *is the smallest slope that occurs in the Harder-Narasimhan filtration of* V*). Let* $S \subset V$ *be a subbundle of* V*. Then* $\deg S \leq \deg V$.

Proof. We use induction on the rank of V. Let $W \subset V$ be the β-subbundle of V (recall that the β-subbundle is the first step in the Harder-Narasimhan filtration of V). Let $\mathcal{S}$ and $\mathcal{W}$ be the sheaves associated to the bundles S and W respectively. Let $\overline{\mathcal{S} \cap \mathcal{W}}$ denote the saturation of the sheaf intersection $\mathcal{S} \cap \mathcal{W}$. Then $\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of V. We have the exact sequence

$$0 \to \overline{\mathcal{S} \cap \mathcal{W}} \to S \to S/\overline{\mathcal{S} \cap \mathcal{W}} \to 0,$$

and $S/\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of V/W. Since $\operatorname{rank}(V/W) < \operatorname{rank} V$ and $\mu_{\min}(V/W) \geq 0$ (since $\mu_{\min}(V/W) = \mu_{\min}(V)$), we have by induction

$$\deg(S/\overline{\mathcal{S} \cap \mathcal{W}}) \leq \deg(V/W).$$

Also, $\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of W, and hence

$$\deg \overline{\mathcal{S} \cap \mathcal{W}} \leq \deg W.$$

From those two inequalities, we obtain

$$\begin{aligned}\deg S &= \deg(\overline{\mathcal{S} \cap \mathcal{W}}) + \deg(S/\overline{\mathcal{S} \cap \mathcal{W}}) \\ &\leq \deg W + \deg(V/W) = \deg V.\end{aligned}$$

□

Lemma 3.3. *Let* $V \to X$ *be a vector bundle such that* $\mu_{\max}(V) < 0$ *(recall that* $\mu_{\max}(V)$ *is the largest slope that occurs in the Harder-Narasimhan filtration of* V*). Let* $S \subset V$ *be a subsheaf of* V*. Then* $\deg S < 0$.

Proof. We again use induction on the rank of V. Let

$$V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V$$

be the Harder-Narasimhan filtration of V. Consider the composite map

$$\varphi : S \to V/V_{k-1}.$$

We have an exact sequence of sheaves

$$0 \to \ker\varphi \to S \to \operatorname{image}\varphi \to 0.$$

Since image φ is a subsheaf of V/V_{k-1} which is semistable of negative degree, we obtain

$$\deg(\operatorname{image}\varphi) < 0.$$

Also, $\ker\varphi \subset V_{k-1}$, $\mu_{\max}(V_{k-1}) < 0$, $\operatorname{rank} V_{k-1} < \operatorname{rank} V$. Hence by induction

$$\deg(\ker\varphi) < 0.$$

We hence obtain

$$\deg S = \deg(\ker\varphi) + \deg(\operatorname{image}\varphi) < 0.$$

□

Proposition 3.4. *Let $E \to X$ be a principal G bundle and $E_0 = E(\mathfrak{g})_0$ be the subbundle of the adjoint bundle $E(\mathfrak{g})$ as in $(*)$ of Section 2. Then by Proposition 2.2, E_0 is a bundle of parabolic subalgebras if* char $k = p > \max(2\dim\mathfrak{g}, 4h(G))$. *Let P be the parabolic subgroup scheme with Lie algebra E_0 of the group scheme $E(G)$ associated to E. Then P is the Harder-Narasimhan filtration of E (see Definition 3.1) if E is not semistable, and* char $k = p > \max(2\dim G, 4h(G))$.

Remark 3.5. *A parabolic subgroup scheme of $E(G)$ is the same as a reduction of structure group of E to a parabolic subgroup.*

Proof of Proposition 3.4. We assume that E is not semistable. Then $E(\mathfrak{g})$ is not semistable, and $\deg P = \deg E_0 > 0$. Let Q be any parabolic subgroup scheme of $E(G)$, and let $\mathfrak{q}$ be its Lie algebra bundle. The $\mathfrak{q}$ is a subbundle of $E(\mathfrak{g})$. Let

$$\mathfrak{q}_{-\ell} \subset \mathfrak{q}_{-\ell+1} \subset \cdots \subset \mathfrak{q}_{-1} \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m = \mathfrak{q}$$

be the Harder-Narasimhan filtration of $\mathfrak{q}$ with $\mu(\mathfrak{q}_i/\mathfrak{q}_{i-1}) \geq 0$ for $i \leq 0$ and $\mu(\mathfrak{q}_i/\mathfrak{q}_{i-1}) < 0$ for $i \geq 1$. Then $\mu_{\min}(\mathfrak{q}_0) \geq 0$, and the composite map $\mathfrak{q}_0 \to E(\mathfrak{g})/E_0$ is zero, since $\mu_{\max}(E(\mathfrak{g})/(E_0)) < 0$. Hence $\mathfrak{q}_0 \subset E_0$. Since $\mu_{\min}(E_0) = 0$, we have by Lemma 3.2 above that $\deg\mathfrak{q}_0 \leq \deg E_0$. But $\deg\mathfrak{q}_0 \geq \deg\mathfrak{q}$, hence $\deg\mathfrak{q} \leq \deg E_0$, or in other words $\deg Q \leq \deg P$. This verifies condition (ii) of Definition 3.1.

Now suppose Q is a parabolic subgroup scheme of $E(G)$ containing P, and again let $\mathfrak{q}$ denote its Lie algebra bundle. Then $E_0 \subset \mathfrak{q}$ (since $P \subset Q$). We have the exact sequence

$$0 \to E_0 \to \mathfrak{q} \to \mathfrak{q}/E_0 \to 0,$$

and $\mathfrak{q}/E_0 \subset E(\mathfrak{g})/E_0$. Since $\mu_{\max}(E(\mathfrak{g})/E_0) < 0$, we obtain from Lemma 3.3 that $\deg(\mathfrak{q}/E_0) < 0$. Hence

$$\deg\mathfrak{q} = \deg E_0 + \deg(\mathfrak{q}/E_0) < \deg E_0,$$

i.e., $\deg Q < \deg P$. This verifies condition (iii) of Definition 3.1.

Consider the filtration ($*$) of Section 2. If the Killing form of G is non-degenerate (which is the case if $\operatorname{char} k = p > 2\dim G$ as in our hypothesis), then we can see that $E(\mathfrak{g})_i = E(\mathfrak{g})_{-i-1}^{\perp}$ for $i \geq 0$, where $\perp$ is taken with respect to the Killing form. In particular $E(\mathfrak{g})_{-1} = E_0^{\perp}$. Since E_0 is a bundle of parabolic subalgebras, it follows that $E(\mathfrak{g})_{-1}$ is the nilpotent radical of E_0 (if $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then $\mathfrak{p}^{\perp}$ is the nilpotent radical of $\mathfrak{p}$). Therefore, the Lie algebra bundle of $P/R_u(P)$ (where $R_u(P)$ is the unipotent radical of P) is $E(\mathfrak{g})_0/E(\mathfrak{g})_{-1}$. Since $E(\mathfrak{g})_0/E(\mathfrak{g})_{-1}$ is a semistable vector bundle of degree zero, it follows that $P/R_u(P)$ is a semistable group scheme. This verifies condition (iv) of Definition 3.1.

We have thus shown that P is the canonical reduction of E. □

Corollary 3.6. *Let $E \to X$ be a principal G bundle and let $E_P \subset E$ be the canonical reduction (the Harder-Narasimhan filtration) to a parabolic P, and let $E(\mathfrak{g})$ and $E_P(\mathfrak{p})$ be the adjoint bundles of E and E_P. Then*

$$H^0(X, E(\mathfrak{g})/E_P(\mathfrak{p})) = 0$$

if $\operatorname{char} k = p > \max(2\dim G, 4h(G))$.

Proof. We know that $E_P(\mathfrak{p}) = E(\mathfrak{g})_0$, where $E(\mathfrak{g})_0$ is as in the filtration ($*$) of Section 2. Since $\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{g})_0) < 0$, it follows that

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{g})_0) = 0.$$

□

4

In this section, we will consider a nonsingular projective variety X, of arbitrary dimension, over a field k of characteristic $p > 3$, and a polarisation H on X. All notions of semistability of bundles on X will be with respect to this polarisation H. We remark that all principal bundles, reductions of structure group, associated bundles, will be defined over open sets U such that *codimension*$(X - U) \geq 2$ (see section 4 in [6]), so in this section, X will stand for an unspecified open subset U with *codimension*$(X - U) \geq 2$. We show the following:

Theorem 4.1. *Let $E \to X$ be a principal G-bundle on X, where G is a reductive group over k. Suppose that $\mu_{\min}(TX) \geq 0$, where TX is the tangent bundle of X. Then we have:*

(i) If $E \to X$ is semistable, so is $F^*E \to X$, where F is the absolute Frobenius of X.

(ii) If $E \to X$ is not semistable, and $E_P \to X$ the canonical reduction of structure group of E (see Definition 3.1), $E(\mathfrak{g})$ the adjoint bundle of E, and $E(\mathfrak{p})$ the adjoint bundle of E_P, then

$$\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{p})) < 0$$

and, in particular,

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{p})) = 0.$$

Proof. We prove the assertion by induction on the semisimple rank of the group G. So we suppose that (i) and (ii) are true for groups of semisimple rank less than that of G. Let $E \to X$ be a principal G bundle and suppose $F^*E \to X$ is not semistable. Let $\widetilde{E}_P \to X$ be the canonical reduction (the Harder-Narasimhan filtration) of F^*E. Then $\widetilde{E}_P$ is defined by a section $\sigma : X \to (F^*E)/P$ of the G/P bundle $F^*E/P \to X$. Then we have a map

$$TX \to N_\sigma,$$

where N_σ is the pullback by σ of the normal bundle of $\sigma(X)$ in F^*E/P.

This map is defined as follows: Let D be a local section of TX and I the ideal sheaf of $\sigma(X)$ in F^*E/P. Then we have the diagram

$$\begin{array}{ccccc} I & \to & \mathcal{O} & & \\ & & \downarrow D & & \\ & & \mathcal{O} & \to & \mathcal{O}/I \end{array}$$

The composite map $I \to \mathcal{O}/I$ vanishes on I^2 and hence defines a map

$$I/I^2 \to \mathcal{O}/I$$

which is $\mathcal{O}/I$ linear. Hence we get a map of sheaves

$$TX \to N_\sigma = \mathrm{Hom}(I/I^2, \mathcal{O}/I).$$

If this map is zero, then σ descends to a section σ_1 of E/P. It is easy to see that

$$N_\sigma = F^*E(\mathfrak{g})/\widetilde{E}_P(\mathfrak{p}),$$

where $F^*E(\mathfrak{g})$ is the adjoint bundle of F^*E and $\widetilde{E}_P(\mathfrak{p})$ is the adjoint bundle of $\widetilde{E}_P$. If $\mathfrak{g}/\mathfrak{p}$ is regarded as a P-module, then $F^*E(\mathfrak{g})/\widetilde{E}_P(\mathfrak{p})$ is the bundle $\widetilde{E}_P(\mathfrak{g}/\mathfrak{p})$ associated to $\widetilde{E}_P$ by the representation of P on $\mathfrak{g}/\mathfrak{p}$. Let $P = MU$ be the Levi decomposition of P, where U is the unipotent radical of P and M the Levi subgroup of P. Let

$$V_1 \subset V_2 \subset \cdots \subset V_m = \mathfrak{g}/\mathfrak{p}$$

be a Jordan-Holder filtration of $\mathfrak{g}/\mathfrak{p}$ as a P-module. Then each V_i/V_{i-1} is an M-module (since U acts trivially), and the M-modules which occur thus are duals of the M-modules which occur in $\mathfrak{u}$ (where $\mathfrak{u}$ is the Lie algebra of U; see [2, Remark 6, Section 3]). In particular, the vector bundles $\widetilde{E}_M(V_i/V_{i-1})$ are dual to the elementary vector bundles (see [4]) associated to $\widetilde{E}_P$. Since the elementary vector bundles have positive degree (by [4]), it follows that the $\widetilde{E}_M(V_i/V_{i-1})$ have negative degree. Further, by the definition of canonical reduction, $\widetilde{E}_M$ is semistable. Also, since the semisimple rank of M is smaller than that of G, by the inductive hypothesis, $F^{n*}\widetilde{E}_M$ is semistable for all $n \geq 1$. Hence by a theorem of Ramanan and Ramanathan (see [6, Theorem 3.23]), the vector bundles $\widetilde{E}_M(V_i/V_{i-1})$ are semistable too. Thus $\widetilde{E}_P(\mathfrak{g}/\mathfrak{p})$ has a filtration with subquotients which are semistable bundles of negative degree. Hence $\mu_{\max}(\widetilde{E}_P(\mathfrak{g}/\mathfrak{p}))(= \mu_{\max}(N_\sigma)) < 0$. Therefore the map $T_X \to N_\sigma$ is zero. Hence the section $\sigma : X \to F^*E/P$ descends to a section $\sigma_1 : X \to E/P$, and σ_1 defines a reduction of structure group of E contradicting the semistability of E. Hence F^*E is semistable. This proves (i).

As for (ii), we observe that the argument given above to show that

$$\mu_{\max}(\widetilde{E}_P(\mathfrak{g}/\mathfrak{p})) < 0,$$

does the job for us.

It remains to start the induction. So let G be a reductive group of semisimple rank 1 (in fact, we can reduce to the case $G = PSL(2)$) and $E \to X$ a principal G-bundle which is semistable. Suppose $F^*E \to X$ is not semistable and $\sigma : X \to F^*E/P$ is a reduction contradicting semistability. Then we observe that $F^*E/P \to X$ is a $\mathbb{P}^1$ bundle, and hence the normal bundle N_σ of the section $\sigma(X)$ in F^*E/P is a line bundle. Since σ contradicts semistability, $N_\sigma = F^*E(\mathfrak{g}/\mathfrak{p})$ has negative degree, and hence the canonical map $T_X \to N_\sigma$ is zero (since $\mu_{\min}(T_X) \geq 0$). It follows that the section σ descends to a section $\sigma_1 : X \to E/P$ contradicting the semistability of $E \to X$.

□

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