

On the Harder-Narasimhan Filtration of Principal Bundles

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1

Let X be a nonsingular projective curve over a field k of characteristic $p > 0$ and let $E \rightarrow X$ be a principal G -bundle on X with G a reductive group over k , or more generally, let $\mathcal{G} \rightarrow X$ be a reductive group scheme over X (we observe that given a principal G -bundle $E \rightarrow X$, we can obtain a group scheme $E(G) \rightarrow X$ over X , by taking the associated fibre space to E for the conjugacy action of G on itself). If $\mathcal{G} \rightarrow X$ is not semistable (see [7] for the notion of semistability of principal bundles, and [4] for that of group schemes), then Behrend (in [4]), Atiyah-Bott (in [1]) and A. Ramanathan (in [7]) have each defined the Harder-Narasimhan filtration (or the canonical filtration) of $\mathcal{G}$. Moreover, if $\mathfrak{p}$ denotes the Lie algebra bundle of the canonical parabolic $\mathcal{P}$ and $\mathfrak{g}$ that of $\mathcal{G}$, then Behrend (see [4, Conjecture 7.6]) conjectures that

$$H^0(X, \mathfrak{g}/\mathfrak{p}) = 0.$$

This conjecture implies that the canonical parabolic descends over purely inseparable field extensions.

We recall (from [5]) that if G is a reductive group over an algebraically closed field k and $\alpha = \sum k_i \alpha_i$ is a root of G where the α_i are simple roots and k_i are integers, then the height of α is defined to be $\sum k_i$. Let $h(G)$ denote the height of the highest root of G . In Section 3, in this article, we show that the canonical parabolic of Behrend coincides with the canonical parabolic of Atiyah-Bott if $\text{char } k = p > \max(2 \dim G, 4h(G))$. As a consequence, we obtain a proof of Behrend's conjecture (see Corollary (3.6)) when $\text{char } k = p > \max(2 \dim G, 4h(G))$. In Section 4, we show that if $\text{char } k = p > 3$ and $\mu_{\min}(TX) \geq 0$ (where TX is the tangent bundle of X : in Section 4, X is a nonsingular projective variety of arbitrary dimension), then Behrend's conjecture is true.

For the definition of the Harder-Narasimhan filtration of a vector bundle V , $\mu_{\min}(V)$, $\mu_{\max}(V)$, and the β -subbundle of V , we refer to [1, pages 588-589], where $\mu_{\min}(V)$ is denoted by $\inf V$, $\mu_{\max}(V)$ is denoted by $\sup V$, and the β -subbundle is E_1 , the first step in the Harder-Narasimhan filtration.

2

Let X be a nonsingular curve over a field k . Let $E \rightarrow X$ be a principal G -bundle on X , where G is a reductive group defined over k . Let $\mathfrak{g}$ be the Lie algebra of G and let $E(\mathfrak{g})$ be the Lie algebra bundle obtained from E by the adjoint representation of G on $\mathfrak{g}$. Let

$$E(\mathfrak{g})_{-r} \subset E(\mathfrak{g})_{-r+1} \subset \cdots \subset E(\mathfrak{g})_{-1} \subset E(\mathfrak{g})_0 \subset E(\mathfrak{g})_1 \subset \cdots \subset E(\mathfrak{g})_s = E(\mathfrak{g}) \quad (*)$$

be the Harder-Narasimhan filtration of the vector bundle $E(\mathfrak{g})$. Here we have indexed the filtration so that

$$\begin{aligned} \mu(E_i/E_{i-1}) &< 0 \text{ for } i \geq 1 \text{ and} \\ \mu(E_{-j}/E_{-j-1}) &\geq 0 \text{ for } j \geq 0, \end{aligned}$$

where $\mu(V)$ denotes the slope degree $V/\text{rank } V$ for a vector bundle V on X .

Lemma 2.1 *Let $G = GL(n)$ and $V \rightarrow X$ a rank n vector bundle on X so that $E \rightarrow X$ is the frame bundle of V . Then the Lie algebra bundle $E(\mathfrak{gl}(n)) = V^* \otimes V$. Let $E(\mathfrak{gl}(n))_0$ be as in (*) above for $E(\mathfrak{gl}(n))$. Then $E(\mathfrak{gl}(n))_0$ is a bundle of parabolic subalgebras of $E(\mathfrak{gl}(n))$ if $\text{char } k = p > 2n$.*

Proof Let

$$V_1 \subset V_2 \subset \cdots \subset V_\ell = V$$

be the Harder-Narasimhan filtration of V . Let

$$0 = E^{-\ell} \subset E^{-\ell+1} \subset \cdots \subset E^{-1} \subset E^0 \subset E^1 \subset \cdots \subset \text{End}(V)$$

be the filtration of $\text{End}(V)$ defined by

$$E^j = \{\varphi \in \text{End}(V) \mid \varphi(V_i) \subset V_{i+j}, \text{ for all } i = 1, \dots, \ell\}$$

with the convention that $V_i = 0$ for $i \leq 0$ and $V_i = V$ for $i \geq \ell$. Then we can see that

$$E^j/E^{j-1} \simeq \bigoplus_i \text{Hom}(V_i/V_{i-1}, V_{i+j}/V_{i+j-1})$$

From the definition of the E^j , E^0 consists of endomorphisms of V preserving the given flag in V , and hence E^0 is a bundle of parabolic subalgebras of $\text{End}(V)$. Consider the map

$$E^0 \subset \text{End}(V) \rightarrow \text{End } V/E_{s-1}$$

where E_{s-1} denotes $E(\mathfrak{gl}(n))_{s-1}$ (and in general, we will let E_i denote $E(\mathfrak{gl}(n))_i$).

Claim *The map $E^0 \rightarrow \text{End } V/E_{s-1}$ is zero.*

Proof (of Claim) Consider the composite map

$$E^{-\ell+1} \subset E^0 \rightarrow \text{End } V/E_{s-1}.$$

Since $E^{-\ell+1} \simeq \text{Hom}(V_\ell/V_{\ell-1}, V_1)$, $E^{-\ell+1}$ is semistable if $\text{char } k = p > \text{rank}(V_\ell/V_{\ell-1}) + \text{rank } V_1$, in particular, if $p > 2n$. (Here, and subsequently, we use the fact that if V_1 and V_2 are semistable vector bundles with $\text{rank } V_1 + \text{rank } V_2 < p$, then $V_1 \otimes V_2$ is semistable, from [5]). Also $\mu(E^{-\ell+1}) = -\mu(V_\ell/V_{\ell-1}) + \mu(V_1) > 0$. Since $\text{End } V/E_{s-1}$ is semistable with $\mu(\text{End } V/E_{s-1}) < 0$, it follows that the map $E^{-\ell+1} \rightarrow \text{End } V/E_{s-1}$ is zero. Hence the map $E^0 \rightarrow \text{End } V/E_{s-1}$ factors through a map $E^0/E^{-\ell+1} \rightarrow \text{End } V/E_{s-1}$. Now consider the composite map

$$E^{-\ell+2}/E^{-\ell+1} \subset E^0/E^{-\ell+1} \rightarrow \text{End } V/E_{s-1}.$$

Since

$$E^{-\ell+2}/E^{-\ell+1} \simeq \text{Hom}(V_\ell/V_{\ell-1}, V_2/V_1) \oplus \text{Hom}(V_{\ell-1}/V_{\ell-2}, V_1)$$

and the maps

$$\text{Hom}(V_\ell/V_{\ell-1}, V_2/V_1) \rightarrow \text{End } V/E_{s-1}$$

$$\text{Hom}(V_{\ell-1}/V_{\ell-2}, V_1) \rightarrow \text{End } V/E_{s-1}$$

are both zero if $p > 2n$ (since μ (left handside) $> 0 > \mu$ (right handside)) it follows that

$$E^{-\ell+2}/E^{-\ell+1} \rightarrow \text{End } V/E_{s-1}$$

is zero and so the map

$$E^0/E^{-\ell+1} \rightarrow \text{End } V/E_{s-1}$$

factors through a map

$$E^0/E^{-\ell+2} \rightarrow \text{End } V/E_{s-1}.$$

Arguing inductively as above, we obtain that the map

$$E^0 \rightarrow \text{End } V/E_{s-1}$$

factors through a map

$$E^0/E^{-1} \rightarrow \text{End } V/E_{s-1}.$$

Since

$$E^0/E^{-1} \simeq \bigoplus_i \text{Hom}(V_i/V_{i-1}, V_i/V_{i-1})$$

and the maps

$$\text{Hom}(V_i/V_{i-1}, V_i/V_{i-1}) \rightarrow \text{End } V/E_{s-1}$$

are zero if $p > 2n$, it follows that the map $E^0 \rightarrow \text{End } V/E_{s-1}$ is zero if $p > 2n$. This shows the claim.

We hence obtain that $E^0 \subset E_{s-1} = E(\mathfrak{gl}(n))_{s-1}$. Considering the map

$$E^0 \rightarrow E_{s-1}/E_{s-2}$$

and arguing as before, we see that this map is also zero and hence $E^0 \subset E_{s-2}$. Continuing this, we obtain $E^0 \subset E_0$.

Conversely, consider the map

$$E_0 \rightarrow \text{End } V/E^{\ell-2} = E^{\ell-1}/E^{\ell-2}$$

where $E_0 = E(\mathfrak{gl}(n))_0$. Since $\text{End } V/E^{\ell-2} \simeq \text{Hom}(V_1, V/V_{\ell-1})$, we have

$$\mu(\text{End } V/E^{\ell-2}) = -\mu(V_1) + \mu(V/V_{\ell-1}) < 0.$$

Also, $\mu_{\min}(E_0) \geq 0$. Hence the map

$$E_0 \rightarrow \text{End } V/E^{\ell-2}$$

is zero if $p > 2n$ (since $p > 2n \Rightarrow \text{Hom}(V_1, V/V_{\ell-1})$ is semistable), and $E_0 \subset E^{\ell-2}$. Continuing this argument, we obtain $E_0 \subset E^0$.

It follows that $E_0 = E^0$ and hence $E_0 = E(\mathfrak{gl}(n))_0$ is a bundle of parabolic subalgebras of $\text{End}(V)$. □

Proposition 2.2 *Let G be a reductive group and $E \rightarrow X$ a principal G -bundle on X . Let $E_0 = E(\mathfrak{g})_0$ be as in (*), a subbundle of $E(\mathfrak{g})$. Then E_0 is a bundle of parabolic subalgebras, if $\text{char } k = p > \max(2 \dim G, 4h(G))$.*

Proof It is easy to see that it is enough to consider the case when G is a group with trivial centre. So we assume that the adjoint representation $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ is faithful. Let $E(GL(\mathfrak{g}))$ be the principal $GL(\mathfrak{g})$ bundle associated to E by the representation Ad . Let $E(\mathfrak{gl}(\mathfrak{g}))$ be the Lie algebra bundle of $E(GL(\mathfrak{g}))$. Since $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ by the adjoint representation, we obtain $E(\mathfrak{g}) \subset E(\mathfrak{gl}(\mathfrak{g}))$. We now appeal to the Main Theorem in [8] (see also [5]) which states that a G -module is semisimple if $p > 2$ height of the module, to obtain that

$$\mathfrak{gl}(\mathfrak{g}) = \mathfrak{g} \oplus W$$

as G -modules, if $\text{char } k = p > 4h(G)$ (we observe that height of $\mathfrak{gl}(\mathfrak{g})$ as a G -module is $2h(G)$). Hence we obtain the decomposition

$$E(\mathfrak{gl}(\mathfrak{g})) = E(\mathfrak{g}) \oplus E(W)$$

of the associated bundles. It is then easy to see that

$$E(\mathfrak{gl}(\mathfrak{g}))_0 = E(\mathfrak{g})_0 \oplus E(W)_0$$

with notation as in (*). By Lemma 2.1, $E(\mathfrak{gl}(\mathfrak{g}))_0$ is a bundle of parabolic subalgebras, if $p > 2 \dim G$. Consider $E(\mathfrak{g}) \subset E(\mathfrak{gl}(\mathfrak{g}))$ and fix a closed point $x \in X$. There exists $t \in GL(\mathfrak{g})$ such that $tE(\mathfrak{gl}(\mathfrak{g}))_{0,x}t^{-1}$ intersects $E(\mathfrak{g})_x$ in a parabolic subalgebra of $E(\mathfrak{g})_x$, where the subscript x denotes the fibre of the corresponding bundle at x . Consider the subgroup tGt^{-1} of $GL(\mathfrak{g})$ with Lie algebra $t\mathfrak{g}t^{-1}$. The Lie algebra bundle $E_G(\mathfrak{g})$ is isomorphic to the Lie algebra bundle $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})$ where $E_{tGt^{-1}}$ is the principal tGt^{-1} bundle on X obtained by translating the principal G bundle E_G . The isomorphism

$$E_G(\mathfrak{g}) \simeq E_{tGt^{-1}}(t\mathfrak{g}t^{-1})$$

takes $E_G(\mathfrak{g})_0$ to $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_0$. But $E(\mathfrak{gl}(\mathfrak{g}))_{0,x}$ intersects $E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_x$ in a parabolic subalgebra by choice of t . Thus

$$E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_{0,x} = E(\mathfrak{gl}(\mathfrak{g}))_{0,x} \cap E_{tGt^{-1}}(t\mathfrak{g}t^{-1})_x$$

is a parabolic subalgebra, and hence so is $E_G(\mathfrak{g})_{0,x}$. Since x is an arbitrary closed point of X , we obtain that $E(\mathfrak{g})_0$ is a parabolic subalgebra bundle. $\square$

3

Let $E \rightarrow X$ be a principal G -bundle on X , where X is a nonsingular projective curve.

Definition 3.1 A reduction of structure group $E_P \subset E$ of E to a parabolic subgroup P of G is said to be canonical, or the Harder-Narasimhan filtration of E if, letting $E(G) \rightarrow X$ be the group scheme associated to E by the adjoint action of G on itself, and $E(P) \rightarrow X$ the parabolic subgroup scheme of $E(G)$ associated to E_P by the adjoint action of P on itself, the following conditions are satisfied:

- (i) $\deg E(P) > 0$ (for any group scheme $\mathcal{G} \rightarrow X$ over X , $\deg \mathcal{G}$ denotes the degree of the Lie algebra bundle of $\mathcal{G}$ on X , regarded as a vector bundle on X)
- (ii) for any parabolic subgroup scheme $Q \subset E(G)$, $\deg Q \leq \deg E(P)$
- (iii) for any parabolic subgroup scheme Q containing $E(P)$, $\deg Q < \deg E(P)$
- (iv) the associated bundle $E_P(P/R_u(P))$ is semistable, where $R_u(P)$ is the unipotent radical of P .

With this definition, our notion of the canonical reduction (or Harder-Narasimhan filtration) coincides with that of Behrend (see [4], definition 5.6, and the proof of Proposition 7.2).

We need the following lemmas:

Lemma 3.2 Let $V \rightarrow X$ be a vector bundle on X such that $\mu_{\min}(V) \geq 0$ (recall that $\mu_{\min}(V)$ is the smallest slope that occurs in the Harder-Narasimhan filtration of V). Let $S \subset V$ be a subbundle of V . Then $\deg S \leq \deg V$.

Proof We use induction on the rank of V . Let $W \subset V$ be the β -subbundle of V (recall that the β -subbundle is the first step in the Harder-Narasimhan filtration of V). Let $\mathcal{S}$ and $\mathcal{W}$ be the sheaves associated to the bundles S and W respectively. Let $\overline{\mathcal{S} \cap \mathcal{W}}$ denote the saturation of the sheaf intersection $\mathcal{S} \cap \mathcal{W}$. Then $\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of V . We have the exact sequence

$$0 \rightarrow \overline{\mathcal{S} \cap \mathcal{W}} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\overline{\mathcal{S} \cap \mathcal{W}} \rightarrow 0$$

and $\mathcal{S}/\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of V/W . Since $\text{rank}(V/W) < \text{rank } V$ and $\mu_{\min}(V/W) \geq 0$ (since $\mu_{\min}(V/W) = \mu_{\min}(V)$), we have by induction

$$\deg(\mathcal{S}/\overline{\mathcal{S} \cap \mathcal{W}}) \leq \deg(V/W).$$

Also, $\overline{\mathcal{S} \cap \mathcal{W}}$ is a subbundle of W , and hence

$$\deg \overline{\mathcal{S} \cap \mathcal{W}} \leq \deg W.$$

From those two inequalities, we obtain

$$\begin{aligned}\deg S &= \deg(\overline{S \cap W}) + (S/\overline{S \cap W}) \\ &\leq \deg W + \deg(V/W) = \deg V.\end{aligned}$$

□

Lemma 3.3 *Let $V \rightarrow X$ be a vector bundle such that $\mu_{\max}(V) < 0$ (recall that $\mu_{\max}(V)$ is the largest slope that occurs in the Harder-Narasimhan filtration of V). Let $S \subset V$ be a subsheaf of V . Then $\deg S < 0$.*

Proof We again use induction on the rank of V . Let

$$V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V$$

be the Harder-Narasimhan filtration of V . Consider the composite map

$$\varphi : S \rightarrow V/V_{k-1}.$$

We have an exact sequence of sheaves

$$0 \rightarrow \ker \varphi \rightarrow S \rightarrow \text{image } \varphi \rightarrow 0.$$

Since $\text{image } \varphi$ is a subsheaf of V/V_{k-1} which is semistable of negative degree, we obtain

$$\deg(\text{image } \varphi) < 0.$$

Also, $\ker \varphi \subset V_{k-1}$, $\mu_{\max}(V_{k-1}) < 0$, $\text{rank } V_{k-1} < \text{rank } V$. Hence by induction

$$\deg(\ker \varphi) < 0.$$

We hence obtain

$$\deg S = \deg(\ker \varphi) + \deg(\text{image } \varphi) < 0.$$

□

Proposition 3.4 *Let $E \rightarrow X$ be a principal G bundle and $E_0 = E(\mathfrak{g})_0$ be the subbundle of the adjoint bundle $E(\mathfrak{g})$ as in (*) of Section 2. Then by Proposition 2.2, E_0 is a bundle of parabolic subalgebras if $\text{char } k = p > \max(2 \dim \mathfrak{g}, 4h(G))$. Let P be the parabolic subgroup scheme with Lie algebra E_0 of the group scheme $E(G)$ associated to E . Then P is the Harder-Narasimhan filtration of E (see Definition 3.1) if E is not semistable, and $\text{char } k = p > \max(2 \dim G, 4h(G))$.*

Remark 3.5 A parabolic subgroup scheme of $E(G)$ is the same as a reduction of structure group of E to a parabolic subgroup.

Proof (of Proposition 3.4) We assume that E is not semistable. Then $E(\mathfrak{g})$ is not semistable, and $\deg P = \deg E_0 > 0$. Let Q be any parabolic subgroup scheme of $E(G)$, and let $\mathfrak{q}$ be its Lie algebra bundle. The $\mathfrak{q}$ is a subbundle of $E(\mathfrak{g})$. Let

$$\mathfrak{q}_{-\ell} \subset \mathfrak{q}_{-\ell+1} \subset \cdots \subset \mathfrak{q}_{-1} \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m = \mathfrak{q}$$

be the Harder-Narasimhan filtration of $\mathfrak{q}$ with $\mu(\mathfrak{q}_i/\mathfrak{q}_{i-1}) \geq 0$ for $i \leq 0$ and $\mu(\mathfrak{q}_i/\mathfrak{q}_{i-1}) < 0$ for $i \geq 1$. Then $\mu_{\min}(\mathfrak{q}_0) \geq 0$, and the composite map $\mathfrak{q}_0 \rightarrow E(\mathfrak{g})/E_0$ is zero, since $\mu_{\max}(E(\mathfrak{g})/E_0) < 0$. Hence $\mathfrak{q}_0 \subset E_0$. Since $\mu_{\min}(E_0) = 0$, we have by Lemma 3.2 above that $\deg \mathfrak{q}_0 \leq \deg E_0$. But $\deg \mathfrak{q}_0 \geq \deg \mathfrak{q}$. Hence $\deg \mathfrak{q} \leq \deg E_0$, or in other words $\deg Q \leq \deg P$. This verifies condition (ii) of Definition 3.1.

Now suppose Q is a parabolic subgroup scheme of $E(G)$ containing P , and again let $\mathfrak{q}$ denote its Lie algebra bundle. Then $E_0 \subset \mathfrak{q}$ (since $P \subset Q$). We have the exact sequence

$$0 \rightarrow E_0 \rightarrow \mathfrak{q} \rightarrow \mathfrak{q}/E_0 \rightarrow 0$$

and $\mathfrak{q}/E_0 \subset E(\mathfrak{g})/E_0$. Since $\mu_{\max}(E(\mathfrak{g})/E_0) < 0$, we obtain from Lemma 3.3 that $\deg(\mathfrak{q}/E_0) < 0$. Hence

$$\deg \mathfrak{q} = \deg E_0 + \deg(\mathfrak{q}/E_0) < \deg E_0$$

i.e., $\deg Q < \deg P$. This verifies condition (iii) of Definition 3.1.

Consider the filtration (*) of Section 2. If the Killing form of G is non-degenerate (which is the case if $\text{char } k = p > 2 \dim G$ as in our hypothesis), then we can see that $E(\mathfrak{g})_i = E(\mathfrak{g})_{i-1}^\perp$ for $i \geq 0$, where $\perp$ is taken with respect to the Killing form. In particular $E(\mathfrak{g})_{-1} = E_0^\perp$. Since E_0 is a bundle of parabolic subalgebras, it follows that $E(\mathfrak{g})_{-1}$ is the nilpotent radical of E_0 (if $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then $\mathfrak{p}^\perp$ is the nilpotent radical of $\mathfrak{p}$). Therefore, the Lie algebra bundle of $P/R_u(P)$ (where $R_u(P)$ is the unipotent radical of P) is $E(\mathfrak{g})_0/E(\mathfrak{g})_{-1}$. Since $E(\mathfrak{g})_0/E(\mathfrak{g})_{-1}$ is a semistable vector bundle of degree zero, it follows that $P/R_u(P)$ is a semistable group scheme. This verifies condition (iv) of Definition 3.1.

We have thus shown that P is the canonical reduction of E . □

Corollary 3.6 Let $E \rightarrow X$ be a principal G bundle and let $E_P \subset E$ be the canonical reduction (the Harder-Narasimhan filtration) to a parabolic

P , and let $E(\mathfrak{g})$ and $E_P(\mathfrak{p})$ be the adjoint bundles of E and E_P . Then

$$H^0(X, E(\mathfrak{g})/E_P(\mathfrak{p})) = 0$$

if $\text{char } k = p > \max(2 \dim G, 4h(G))$.

Proof We know that $E_P(\mathfrak{p}) = E(\mathfrak{g})_0$, where $E(\mathfrak{g})_0$ is as in the filtration (*) of Section 2. Since $\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{g})_0) < 0$, it follows that

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{g})_0) = 0.$$

□

4

In this section, we will consider a nonsingular projective variety X , of arbitrary dimension, over a field k of characteristic $p > 3$, and a polarisation H on X . All notions of semistability of bundles on X will be with respect to this polarisation H . We remark that all principal bundles, reductions of structure group, associated bundles, will be defined over open sets U such that $\text{codimension}(X - U) \geq 2$ (see section 4 in [6]), so in this section, X will stand for an unspecified open subset U with $\text{codimension}(X - U) \geq 2$. We show the following:

Theorem 4.1 *Let $E \rightarrow X$ be a principal G -bundle on X , where G is a reductive group over k . Suppose that $\mu_{\min}(TX) \geq 0$, where TX is the tangent bundle of X . Then we have*

- (i) *If $E \rightarrow X$ is semistable, so is $F^*E \rightarrow X$, where F is the absolute Frobenius of X .*
- (ii) *If $E \rightarrow X$ is not semistable, and $E_P \rightarrow X$ the canonical reduction of structure group of E (see Definition 3.1), $E(\mathfrak{g})$ the adjoint bundle of E , and $E(\mathfrak{p})$ the adjoint bundle of E_P , then*

$$\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{p})) < 0$$

and, in particular

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{p})) = 0.$$

Proof We prove the assertion by induction on the semisimple rank of the group G . So we suppose that (i) and (ii) are true for groups of semisimple rank less than that of G . Let $E \rightarrow X$ be a principal G bundle and suppose $F^*E \rightarrow X$ is not semistable. Let $\tilde{E}_P \rightarrow X$ be the canonical reduction (the

Harder-Narasimhan filtration) of F^*E . Then $\tilde{E}_P$ is defined by a section $\sigma : X \rightarrow (F^*E)/P$ of the G/P bundle $F^*E/P \rightarrow X$. Then we have a map

$$TX \rightarrow N_\sigma$$

where N_σ = the pullback by σ of the normal bundle of $\sigma(X)$ in F^*E/P . This map is defined as follows: Let D be a local section of TX and I the ideal sheaf of $\sigma(X)$ in F^*E/P . Then we have the diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{O} \\ & & \downarrow D \\ & & \mathcal{O} \longrightarrow \mathcal{O}/I \end{array}$$

The composite map $I \rightarrow \mathcal{O}/I$ vanishes on I^2 and hence defines a map $I/I^2 \rightarrow \mathcal{O}/I$ which is $\mathcal{O}/I$ linear. Hence we get a map of sheaves

$$TX \rightarrow N_\sigma = \text{Hom}(I/I^2, \mathcal{O}/I).$$

If this map is zero, then σ descends to a section σ_1 of E/P . It is easy to see that $N_\sigma = F^*E(\mathfrak{g})/\tilde{E}_P(\mathfrak{p})$, where $F^*E(\mathfrak{g})$ is the adjoint bundle of F^*E and $\tilde{E}_P(\mathfrak{p})$ is the adjoint bundle of $\tilde{E}_P$. If $\mathfrak{g}/\mathfrak{p}$ is regarded as a P -module, then $F^*E(\mathfrak{g})/\tilde{E}_P(\mathfrak{p})$ is the bundle $\tilde{E}_P(\mathfrak{g}/\mathfrak{p})$ associated to $\tilde{E}_P$ by the representation of P on $\mathfrak{g}/\mathfrak{p}$. Let $P = MU$ be the Levi decomposition of P where U is the unipotent radical of P and M the Levi subgroup of P . Let

$$V_1 \subset V_2 \subset \cdots \subset V_m = \mathfrak{g}/\mathfrak{p}$$

be a Jordan-Holder filtration of $\mathfrak{g}/\mathfrak{p}$ as a P -module. Then each V_i/V_{i-1} is an M module (since U acts trivially), and the M -modules which occur thus are duals of the M -modules which occur in $\mathfrak{u}$ (where $\mathfrak{u}$ is the Lie algebra of U ; see [2, Remark 6, Section 3]). In particular, the vector bundles $\tilde{E}_M(V_i/V_{i-1})$ are dual to the elementary vector bundles (see [4]) associated to $\tilde{E}_P$. Since the elementary vector bundles have positive degree (by [4]), it follows that the $\tilde{E}_M(V_i/V_{i-1})$ have negative degree. Further, by the definition of canonical reduction, $\tilde{E}_M$ is semistable. Also, since the semisimple rank of M is smaller than that of G , by the inductive hypothesis, $F^{n*}\tilde{E}_M$ is semistable for all $n \geq 1$. Hence by a theorem of Ramanan and Ramanathan (see [6, Theorem 3.23]), the vector bundles $\tilde{E}_M(V_i/V_{i-1})$ are semistable too. Thus $\tilde{E}_P(\mathfrak{g}/\mathfrak{p})$ has a filtration with subquotients which are semistable bundles of negative degree. Hence $\mu_{\max}(\tilde{E}_P(\mathfrak{g}/\mathfrak{p})) (= \mu_{\max}(N_\sigma)) < 0$. Therefore the map $TX \rightarrow N_\sigma$ is zero. Hence the section $\sigma : X \rightarrow F^*E/P$ descends to a section $\sigma_1 : X \rightarrow E/P$, and σ_1 defines a reduction of structure group of E

contradicting the semistability of E . Hence F^*E is semistable. This proves (i).

As for (ii), we observe that the argument given above to show that $\mu_{\max}(\tilde{E}_P(\mathfrak{g}/\mathfrak{p})) < 0$, does the job for us.

It remains to start the induction. So let G be a reductive group of semisimple rank 1 (in fact, we can reduce to the case $G = PSL(2)$) and $E \rightarrow X$ a principal G -bundle which is semistable. Suppose $F^*E \rightarrow X$ is not semistable and $\sigma : X \rightarrow F^*E/P$ is a reduction contradicting semistability. Then we observe that $F^*E/P \rightarrow X$ is a $\mathbb{P}^1$ bundle, and hence the normal bundle N_σ of the section $\sigma(X)$ in F^*E/P is a line bundle. Since σ contradicts semistability, $N_\sigma = F^*E(\mathfrak{g}/\mathfrak{p})$ has negative degree, and hence the canonical map $T_X \rightarrow N_\sigma$ is zero (since $\mu_{\min}(T_X) \geq 0$). It follows that the section σ descends to a section $\sigma_1 : X \rightarrow E/P$ contradicting the semistability of $E \rightarrow X$.

□

References

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