REDUCTIVE GROUPS OVER A LOCAL FIELD I. VALUED ROOT DATA

by

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Contents

Introduction

The Introduction (pp. 7–12) is discursive expository prose rather than mathematics; what follows is a summary of its content and of the notation it fixes, not a line-by-line rendering.

Programme. The memoir develops the theory of reductive algebraic groups over a local field K along the lines opened by Iwahori and Matsumoto; the main results had been announced in [10]–[16], with conjectures in [9]. The guiding picture: a semisimple group over K — or rather the group G of its K-rational points — is to be studied as an "infinite-dimensional object" over the residue field k. The analogy with the ordinary theory of algebraic groups over k is systematic, and, as there, one separates the *absolute* theory (k algebraically closed) from the *relative* theory (k arbitrary).

Iwahori and parahoric subgroups. The role played in the classical absolute theory by Borel subgroups (maximal connected solvable) and parabolic subgroups is played here, for k algebraically closed, by *Iwahori subgroups* and *parahoric subgroups*. Chapter II shows they carry a natural structure of connected prosolvable proalgebraic group, maximal for that property — so an *a priori* algebro-geometric account (not given here) would presumably define Iwahori subgroups as the maximal connected prosolvable subgroups of G. Here they are introduced otherwise: by the explicit Iwahori–Matsumoto construction [28] from the root structure when G is split; and, in general (Steinberg: G is quasi-split when k is algebraically closed), either by extending that constructive method to the quasi-split case — as Hijikata had done in cases [26], [27] — or by Galois descent to the split case.

Just as a Borel subgroup together with the normalizer of a maximal torus forms a Tits system with finite Coxeter Weyl group, so for G simply connected an Iwahori subgroup B together with the rational points of the normalizer of a suitable maximal torus forms a Tits system, this time with infinite Weyl group; whence again that B is self-normalizing, a Bruhat decomposition, and so on. *Parahoric subgroups* are the subgroups containing an Iwahori B and equal to a finite union of double cosets mod B. The non-simply-connected case behaves like the non-connected case classically; the Introduction restricts itself for simplicity to the simply connected case.

Descent and the relative case. Passage to the relative case goes by descent, using Galois descent techniques — whence the frequent (and probably often superfluous) hypothesis that k be perfect. As classically one replaces Borel subgroups, which need not exist over k, by minimal parabolic k-subgroups, so here one considers the minimal "parahoric k-subgroups", which are the "B" of Tits systems whose Weyl group is the affine Weyl group of a root system — Tits systems *of affine type*. That root system need not be homothetic to the relative root system of G, but has the same ordinary Weyl group.

Why parahoric subgroups matter. (i) For G simply connected, the maximal bounded subgroups are exactly the maximal parahoric subgroups; the maximal bounded subgroups of an arbitrary semisimple group over K are determined completely, generalizing Hijikata's results for PGL_n and groups of nondegenerate sesquilinear forms. (ii) Parahoric subgroups carry natural proalgebraic structures over k, so that questions about G — Galois cohomology in particular — reduce "modulo p" to questions about finite-dimensional semisimple groups over k; this generalizes theorems of Kneser for K locally compact of characteristic zero (anisotropic groups, vanishing of H^1, classification). (iii) If the group splits over an unramified extension of K, the parahoric subgroups appear as groups of "integral points" of group schemes over the ring O of integers of K which give the group back under $O \to K$.

Affine roots and valuations (notation fixed here). To G one associates an *affine root system*, absolute or relative, whose elements are the affine half-spaces

$$\alpha_{a,k} \;=\; \{x \in \mathbf{A} \mid a(x) + k \geqslant 0\}, \qquad a \in \Phi,\ k \in \Gamma_a,$$

of the dual $\mathbf{A}$ of the real vector space spanned by the relative root system Φ of G, where Γ_a is a discrete subgroup of $\mathbf{R}$. Each affine root α has a subgroup $U_\alpha \leqslant G$; for fixed a the $U_{\alpha_{a,k}}$, $k \in \Gamma_a$, form a filtration of the usual root subgroup U_a. These filtrations come from a *valuation* of the root datum formed by the U_a — the guiding remark being that a valuation of a field is hardly more than a filtration of its additive group. Successive quotients are additive groups over k in the absolute case, connected unipotent groups of class $\leqslant 2$ in the relative case.

Plan of Chapter I. The chapter is the "abstract" part of the theory: pure group theory, independent of any algebro-geometric structure, hence applicable to groups unrelated to algebraic groups. It is expounded from two successive points of view — Tits systems of affine type (§§2–5), geometric at the outset and giving pride of place to parahoric subgroups; and valued root data (§§6–8), more analytic, resting on the U_a and their valuations. The two are not strictly equivalent: there are Tits systems of affine type not arising from a valued root datum (automorphism groups of certain trees), while the valuations considered are not generally assumed discrete and only discrete ones yield such Tits systems. Consequently many statements appear twice in slightly different forms; the authors note the redundancy, decline to recast the memoir, and remark that the two accounts at least exhibit two aspects of the objects and call on quite different modes of reasoning.

Section by section:

- §1 — recollections on Tits systems and affine Weyl groups.

– §2 — to each Tits system of affine type (B, N) is attached a *building* $\mathscr{I}$: a geometric polysimplicial complex, related to but not identical with the combinatorial object of [39] or [5] (chap. IV, §2, exercises 10–17), carrying a distance for which it is complete. In rank 1 (Weyl group infinite dihedral) $\mathscr{I}$ is a *tree*. It plays in part the role of the Riemannian symmetric space of real semisimple Lie theory.
– §3 — maximal bounded subgroups of G, for a bornology naturally attached to (B, N) (in the algebraic case, the one defined by the valuation of K), via a *fixed point lemma* valid for these buildings and for Riemannian symmetric spaces alike — so usable also for maximal compact subgroups of real Lie groups.
– §4 — Iwasawa and Cartan decompositions, and relations between double cosets with consequences for representation theory over locally compact local fields. From (B, N) one builds $\mathfrak{B} \leqslant G$, which in the algebraic case is a minimal parabolic K-subgroup; the Iwasawa decomposition is $G = \mathfrak{B}.P$ with P maximal bounded — valid only for the "good" maximal bounded subgroups and under an extra hypothesis on (B, N), satisfied e.g. whenever $G = \mathfrak{B}.N.\mathfrak{B}$. Unlike the real case one cannot generally split off $R \leqslant \mathfrak{B}$ with $G = R.P$, $R \cap P = \{1\}$. Non-simply-connected groups are accommodated by carrying along a second group $\widehat{G}$ with a *B-N-adapted* homomorphism $\varphi : G \to \widehat{G}$ ($\widehat{G}$, G being in applications the rational points of an arbitrary semisimple group and of the simply connected covering of its identity component) — whence the heaviness of the notation.
– §5 — when $G = \mathfrak{B}N\mathfrak{B}$, $(\mathfrak{B}, N)$ is a Tits system with finite Weyl group: a *double Tits system*. This paragraph is an axiomatic study of that situation; it is not used later and may be skipped.
– §6 — valued root data. A root datum of type Φ in G is a family $(T, (U_a)_{a\in\Phi})$ subject to (6.1.1) and (DR1)–(DR6), modelled on G = rational points of a semisimple group over K, T the centralizer of a maximal split torus, Φ the relative roots, U_a the unipotent root subgroups ($U_{(a)}$ of [3]). A valuation is a family $\varphi_a : U_a \to \mathbf{R} \cup \{\infty\}$ subject to (6.2.1), (V0)–(V5). A discrete valuation of a generating root datum yields a double Tits system in a finite-index subgroup (6.5), but the discrete case is not assumed thereafter. Parahoric subgroups become special cases of the groups U_f of no. 6.4; over a local field with residue field k, every parahoric group is an extension of a reductive k-group by a pronilpotent group.
– §§7–8 — §§2, 3, 4 transposed to valued root data: a building $\mathscr{I}$ (that of the associated Tits system in the discrete case; in general with chambers and facets but without the polysimplicial structure and without completeness), the bornology of G, and the maximal bounded subgroups. The order of proof is roughly reversed: subgroups generalizing the parahorics, then Iwasawa and Bruhat decompositions, then the building.
– §9 — technical; the descent theorem carrying valuations and affine Tits systems from split to quasi-split to arbitrary semisimple groups.
– §10 — the classical groups by direct computation, independent of descent; all valuations of their natural root data are determined. Since non-discrete valuations and skew fields of infinite rank over their center are allowed, these results are not entirely recovered later.

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Terminological footnote (p. 8). The authors adhere to Bourbaki's terminology for ease of reference to [5]; hence what some call a (BN)-pair is here called a *Tits system* — one author having insisted, the other resigning himself.

1. Recollections and notation

1.1. Polysimplicial complexes. —

(1.1.1) Let F be a set. By an *affine structure* on F we mean a map $[0,1] \times F \times F \longrightarrow F$, written $(t, x, y) \mapsto tx + (1-t)y$, such that there exists a bijection j of F onto a convex subset possessing an interior point of a finite-dimensional real affine space E, satisfying

$$j\big(tx + (1-t)y\big) = t\, j(x) + (1-t)\, j(y) \tag{1}$$

for all $t \in [0,1]$ and $x, y \in F$. Every convex subset of a finite-dimensional affine space carries a canonical affine structure. A set equipped with an affine structure is called an *affine set*.

If F is an affine set, the affine space E and the bijection j satisfying (1) are clearly determined up to affine isomorphism. One may therefore transport to F the topology of $j(F)$, together with the notions of dimension, of open and closed segment, of parallelism of two segments, of convex subset and convex hull, of *internal point*, of *extreme point*, and so on. For each point x of F, let F_x be the set consisting of x together with those points $y \in F$, $y \neq x$, for which there is an open segment of F containing both x and y. One checks at once that F_x is a convex subset of F and that the sets F_x, $x \in F$, form a partition of F. A subset of F of the form F_x is called a *facet* (*facette*) of F.

Let F and F' be affine sets. A *morphism* of F into F' (also called an *affine map*) is a map $j : F \longrightarrow F'$ satisfying (1). This defines the category of affine sets; every bijective morphism is an isomorphism. The map $\big(t, (x, x'), (y, y')\big) \mapsto \big(tx + (1-t)y,\ tx' + (1-t)y'\big)$ endows $F \times F'$ with an affine structure, the *product* of those of F and F'.

(1.1.2) A *closed* (resp. *open*) *simplex of dimension* r is an affine set isomorphic to a closed (resp. open) simplex of an r-dimensional real vector space E, i.e. to the convex hull (resp. the interior of the convex hull) of $r+1$ affinely independent points of E. A closed simplex F of dimension r has $r+1$ extreme points; a facet of F is an open simplex of dimension $\leqslant r$, namely the set of internal points of the convex hull of a nonempty subset of the set of extreme points of F.

(1.1.3) A *closed polysimplex* is an affine set F isomorphic to the product of finitely many closed simplices $F_1, \dots, F_k$ ($k \geqslant 0$) of dimension > 0. One verifies easily that the simplices F_j and the projections $F \longrightarrow F_j$ are determined up to isomorphism. After identifying F with $F_1 \times \cdots \times F_k$, the facets of F are the subsets $G_1 \times \cdots \times G_k$ with G_j a facet of F_j; they are therefore *open polysimplices*, i.e. affine sets isomorphic to products of open simplices.

(1.1.4) Let A be a set and $\mathscr{F}$ a set of subsets of A, whose elements will be called the *facets* of A. Assume:

(1) *the elements of $\mathscr{F}$ form a partition of A.*

Suppose given moreover an order relation on $\mathscr{F}$, written $F \prec F'$, and for $F \in \mathscr{F}$ let $\bar{F}$ denote the union of the elements of $\mathscr{F}$ bounded above by F (such elements being called the *facets of* F). Finally, for each $F \in \mathscr{F}$ suppose given an affine structure on $\bar{F}$ such that:

(2) *$\bar{F}$ is a closed polysimplex;*

(3) *the set of $F' \in \mathscr{F}$ bounded above by F is the set of facets of $\bar{F}$; the closure of such an F' in $\bar{F}$ coincides with $\bar{F}'$, and the affine structure of $\bar{F}'$ is induced by that of $\bar{F}$.*

It follows at once that each facet F of A is an open polysimplex, namely the set of internal points of $\bar{F}$.

(1.1.5) We assume henceforth that the dimensions of the facets of A are bounded above, and write $\dim A$ for their supremum. For $F \in \mathscr{F}$ put $\operatorname{codim} F = \dim A - \dim F$. The facets of codimension 0 (resp. 1) are called the *chambers* (resp. the *panels*, *cloisons*) of A. Two chambers are *adjacent* (*mitoyennes*) if they are distinct and have a common panel. A *gallery of length n* is a sequence $\Gamma = (C_0, C_1, \ldots, C_n)$ of $n+1$ chambers such that C_{i-1} and C_i are adjacent or equal ($1 \leqslant i \leqslant n$). One says C_0 is the *origin* and C_n the *extremity* of Γ (or that C_0 and C_n are the extremities of Γ). One says Γ is *non-stuttering* (*sans bégaiement*) if $C_{i-1} \neq C_i$ for $1 \leqslant i \leqslant n$.

Let F, F' be facets of A. A gallery $\Gamma = (C_0, \ldots, C_n)$ *joins* F to F' if $F \subset \bar{C}_0$ and $F' \subset \bar{C}_n$. It is *taut* (*tendue*) between F and F' if it joins them and there is no gallery joining F to F' of length strictly less than n. For $x, x' \in A$, a gallery is said to be taut between x and x' if it is taut between the facets F, F' containing x and x' respectively. A gallery is *minimal* if it is taut between its extremities.

***Definition* (1.1.6). —** One says that A (equipped with the set $\mathscr{F}$, the order relation $F \prec F'$, and the affine structures on each $\bar{F}$, satisfying the preceding conditions) is a *polysimplicial complex* if, whatever the facets F and F' of A, there exists a gallery joining F and F'.

If A is a polysimplicial complex, the chambers of A are none other than the maximal facets. If the chambers of a polysimplicial complex are simplices, one says it is a *simplicial complex*; all facets are then (open) simplices.

(1.1.7) Let A, A' be polysimplicial complexes. A *morphism* of A into A' is a map $f : A \to A'$ with the two properties:

(i) the image under f of a facet of A is a facet of A';
(ii) if F is a facet of A, the restriction of f to $\bar{F}$ is an affine map of $\bar{F}$ onto $\overline{f(F)}$.

The morphism f is *chambered* (*chambré*) if for every chamber C of A the restriction of f to C is an isomorphism of C onto a chamber of A'.

(1.1.8) Let A_1, A_2 be polysimplicial complexes. Define the facets of $A_1 \times A_2$ to be the products $F_1 \times F_2$ with F_i a facet of A_i ($i = 1, 2$), and order them by

$$F_1' \times F_2' \prec F_1 \times F_2 \quad \Longleftrightarrow \quad F_1' \prec F_1 \text{ and } F_2' \prec F_2.$$

One then has $\overline{F_1 \times F_2} = \bar{F}_1 \times \bar{F}_2$, and one may endow $\overline{F_1 \times F_2}$ with the product affine structure. One verifies easily that these data make $A_1 \times A_2$ a polysimplicial complex, the *product* of those of A_1 and A_2; the canonical projections are morphisms. This extends in the obvious way to a product of any finite number of polysimplicial complexes.

1.2. Coxeter systems and Tits systems. —

In this number we recall some definitions and results concerning Coxeter systems and Tits systems. For the proofs we refer to ([5], chap. IV), which we designate by *l.c.* throughout 1.2.

(1.2.1) Recall that a *Coxeter system* is a pair (W, S), where W is a group and S a subset of W, satisfying the following axioms (*l.c.*, §1, no. 3):

(i) S generates W and the elements of S are of order 2;
(ii) let $m(s, s')$ be the order of the element ss' of W (for $s, s' \in S$), and let I be the set of pairs (s, s') of elements of S such that $m(s, s')$ is finite; the generating set S together with the relations $(ss')^{m(s,s')} = 1$ for $(s, s') \in I$ form a presentation of W.

When $\operatorname{Card} S = 1$ one has $W = \mathbf{Z}/2\mathbf{Z}$. When $\operatorname{Card} S = 2$, the group W is a dihedral group of order $2m(s, s')$, with $S = \{s, s'\}$ (*l.c.*, §1, no. 2).

(1.2.2) Let (W, S) be a Coxeter system. The *length* of an element $w \in W$, written $\ell(w)$, is the least integer $q \geqslant 0$ such that w is a product of a sequence of q elements of S. A *reduced decomposition* of w is a sequence $(s_1, \ldots, s_q)$ of elements of S with $w = s_1 \ldots s_q$ and $\ell(w) = q$ (*l.c.*, §1, no. 1). The set $\{s_1, \ldots, s_q\}$ depends only on w and not on the chosen reduced decomposition (*l.c.*, §1, no. 8, prop. 7); it is written S_w and called the *support* of w. The elements

$$t_j = (s_1 \ldots s_{j-1})\, s_j\, (s_1 \ldots s_{j-1})^{-1}, \qquad 1 \leqslant j \leqslant q,$$

are pairwise distinct and their set depends only on w (*l.c.*, §1, no. 4, lemma 2).

(1.2.3) Let $(s_1, \dots, s_q)$ be a reduced decomposition of an element $w \in W$ and let $s \in S$. If $\ell(sw) \leqslant \ell(w)$, there is an integer j with $1 \leqslant j \leqslant q$ such that $s\, s_1 \dots s_{j-1} = s_1 \dots s_j$, and $(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q)$ is a reduced decomposition of sw, which is therefore of length $q - 1$ (*l.c.*, §1, no. 4, lemma 3). Let $s_1, \dots, s_q$ be elements of S and put $w = s_1 \dots s_q$; there exists a sequence $1 \leqslant j_1 < j_2 < \dots < j_k \leqslant q$ such that $(s_{j_1}, \dots, s_{j_k})$ is a reduced decomposition of w, and one has $k \equiv q \pmod 2$.

(1.2.4) For every subset X of S, write W_X for the subgroup of W generated by X. It is also the set of elements $w \in W$ whose support is contained in X (*l.c.*, §1, no. 8, cor. 1 of prop. 7). In particular one has $W_X \cap S = X$ and $W_X \cap W_Y = W_{X \cap Y}$ for $X, Y \subset S$. The pair (W_X, X) is a Coxeter system (*l.c.*, §1, no. 8, th. 2).

(1.2.5) The *Coxeter graph* $\operatorname{Cox}(W, S)$ (or $\operatorname{Cox} W$) of the Coxeter system (W, S) is the pair (G, f) obtained as follows: G is the graph whose vertices are the elements of S, two distinct vertices s and s' being joined by an edge if and only if the order $m(s, s')$ of ss' is $\geqslant 3$ (which means that s and s' do not commute); f is the map $\{s, s'\} \mapsto m(s, s')$ from the set of edges into the set consisting of ∞ and the integers $\geqslant 3$ (*l.c.*, §1, no. 9).

(1.2.6) By a *Tits system* we mean a quadruple (G, B, N, S), where G is a group, B and N two subgroups of G, and S a subset of $N/(B \cap N)$, satisfying the following axioms (*l.c.*, §2, no. 1):

(T1) *The set $B \cup N$ generates G, and $B \cap N$ is a normal subgroup of N.*
(T2) *The set S generates the group $W = N/(B \cap N)$ and consists of elements of order* 2.[(1)]
(T3) *For every $s \in S$ and every $w \in W$ one has $sBw \subset BwB \cup BswB$.*
(T4) *For every $s \in S$ one has $sBs \neq B$.*

Note that the elements of W are cosets modulo $B \cap N$, which gives a meaning to the products Bw, BwB, etc.

The set S is the set of elements $w \in W$ such that $B \cup BwB$ is a subgroup of G (*l.c.*, §2, no. 5, cor. of th. 3). It follows that, if G, B and N are given, there exists at most one subset S of $N/(B \cap N)$ such that (G, B, N, S) is a Tits system. This will allow us to say, by abuse of language, that the triple (G, B, N) is a Tits system, or again that the pair (B, N) is a Tits system in G.

The group W is called the *Weyl group* of the Tits system. The pair (W, S) is a Coxeter system (*l.c.*, §2, no. 4, th. 2).

In the remainder of this number, (G, B, N, S) denotes a Tits system, W its Weyl group, and $\nu : N \longrightarrow W$ the canonical homomorphism.

(1.2.7) The map $w \mapsto BwB$ is a bijection of W onto the set $B\backslash G/B$ of double cosets of G modulo B (*l.c.*, §2, no. 3, th. 1).

(1.2.8) Let $s \in S$ and $w \in W$; the three relations "$BsBwB$ contains a single double coset modulo B", $BswB = BsBwB$, and $\ell(sw) = \ell(w) + 1$ are equivalent (*l.c.*, §2, no. 4, th. 2). Consequently, if $(s_1, \dots, s_q)$ is a reduced decomposition of an element $w \in W$, one has $BwB = Bs_1Bs_2B \dots Bs_qB$.

(1.2.9) For every subset X of S, put $B_X = BW_XB$ (the union of the double cosets BwB for w running through the subgroup W_X of W generated by X; this set is denoted G_X in (*l.c.*, §2)). The map $X \mapsto B_X$ is a bijection of the set of subsets of S onto the set of subgroups of G containing B, and one has $B_X \cap B_Y = B_{X \cap Y}$ for $X, Y \subset S$ (*l.c.*, §2, no. 5, th. 3).

(1.2.10) One says that a subgroup P of G is a *parabolic subgroup* if it contains a conjugate of B. There then exists a unique subset X of S such that P is conjugate to B_X (*l.c.*, §2, no. 6,

[(1)]This second hypothesis made on S is in fact a consequence of the remaining axioms ([37]).

prop. 4). One says that X is the *type* of P. For every subset Y of S containing X there is exactly one parabolic subgroup of type Y containing P, and one obtains in this way all the subgroups containing P.

Every parabolic subgroup is its own normalizer; two distinct parabolic subgroups whose intersection is again a parabolic subgroup are not conjugate; two parabolic subgroups contained in one and the same subgroup P and conjugate by an element of G are conjugate by an element of P (*l.c.*, §2, no. 6, th. 4).

(1.2.11) Let (G', B', N', S') be another Tits system, with $G' = G$. One says that the Tits systems (G, B, N, S) and (G, B', N', S') are *equivalent* if the subgroups B and B' are conjugate, or what amounts to the same, if they possess the same parabolic subgroups. The Weyl groups W and W', or more precisely the Coxeter systems (W, S) and (W', S'), are then canonically isomorphic. Indeed, let $g \in G$ be such that $B' = gBg^{-1}$; since B is its own normalizer, such an element is determined up to right multiplication by an element of B. The map $x \mapsto gxg^{-1}$ defines a bijection, independent of the choice of g, of the set $B\backslash G/B$ onto $B'\backslash G/B'$, hence a bijection γ of W onto W' (cf. (1.2.7)). Since the elements s of S (resp. S') are characterized by the fact that $B \cup BsB$ (resp. $B' \cup B'sB'$) is a subgroup (1.2.6), one has $\gamma(S) = S'$. On the other hand, let $(s_1, \dots, s_q)$ be a reduced decomposition of an element $w \in W$. One deduces immediately from (1.2.8) that $(\gamma(s_1), \dots, \gamma(s_q))$ is a reduced decomposition of $\gamma(w)$. It follows at once that, for $s, t \in S$, the order of $\gamma(s)\gamma(t)$ in W' equals that of st in W, and that consequently there exists exactly one homomorphism γ' of W into W' with $\gamma'(s) = \gamma(s)$ for all $s \in S$. Since

$$\gamma(w) = \gamma(s_1) \dots \gamma(s_q) = \gamma'(s_1) \dots \gamma'(s_q) = \gamma'(w),$$

one sees that γ is indeed an isomorphism of (W, S) onto (W', S').

(1.2.12) Put $H = \bigcap_{n \in N} nBn^{-1}$. This is a subgroup of G, normalized by N. Put $\widetilde{\mathrm{N}} = NH$. Then H is a normal subgroup of $\widetilde{\mathrm{N}}$, one has $H = B \cap \widetilde{\mathrm{N}}$, and the injection of N into $\widetilde{\mathrm{N}}$ defines an isomorphism of W onto $\widetilde{\mathrm{N}}/(B \cap \widetilde{\mathrm{N}})$, by means of which we shall identify these two groups. One says that the Tits system (G, B, N, S) is *saturated* if $\widetilde{\mathrm{N}} = N$. In all cases the quadruple $(G, B, \widetilde{\mathrm{N}}, S)$ is a saturated Tits system, equivalent to (G, B, N, S), called the *saturated Tits system associated with* (G, B, N, S) (*l.c.*, §2, exerc. 5).

Two Tits systems will be called *strongly equivalent* if the associated saturated Tits systems are the same, up to inner automorphism. Note that two Tits systems, even saturated, may be equivalent without being strongly equivalent (except however if their Weyl group is finite (cf. *l.c.*, §2, exerc. 15)); in particular the groups N and N' may fail to be conjugate (cf. (2.8.13)).

(1.2.13) A homomorphism φ of the group G into a group $\widehat{G}$ is said to be *B-adapted* (resp. *B-N-adapted*) if the following two conditions are satisfied:

(i) the kernel of φ is contained in B;

(ii) for every $g \in \widehat{G}$ there exists an element $h \in G$ such that $\varphi(hBh^{-1}) = g\varphi(B)g^{-1}$ (resp. $\varphi(hBh^{-1}) = g\varphi(B)g^{-1}$ and $\varphi(hNh^{-1}) = g\varphi(N)g^{-1}$).

In the remainder of this number, $\varphi : G \longrightarrow \widehat{G}$ denotes a B-adapted homomorphism.

(1.2.14) Since G is generated by the union of the conjugates of B (*l.c.*, §2, no. 4, cor. 2 of th. 2), the image $\varphi(G)$ is a normal subgroup of $\widehat{G}$. On the other hand, (i) entails that the kernel of φ is contained in every parabolic subgroup of G, hence in H.

(1.2.15) For every parabolic subgroup P of G and every $g \in \widehat{G}$, the inverse image $\varphi^{-1}\big(g\varphi(P)g^{-1}\big)$ is a parabolic subgroup of G, which we shall denote by ${}^gP$. One thus defines an action of $\widehat{G}$ on the set of parabolic subgroups of G. For every parabolic subgroup P of G we shall write $\mathrm{Stab}_{\widehat{G}} P$, or simply $\mathrm{Stab}\, P$, for the set of $g \in \widehat{G}$ such that ${}^gP = P$. One has $\varphi^{-1}(\mathrm{Stab}\, P) = P$.

(1.2.16) For $g \in \widehat{G}$, the right coset modulo B of the element $h \in G$ satisfying condition (ii) of (1.2.13) is well determined, since B is its own normalizer and the kernel of φ is contained in B. One deduces that there exists exactly one permutation $\xi(g)$ of W such that

$$\varphi\big(B \cdot \xi(g)(w) \cdot B\big) = \varphi(h)^{-1} \cdot g \cdot \varphi(BwB) \cdot g^{-1} \cdot \varphi(h)$$

for every $w \in W$ and every $h \in G$ satisfying (1.2.13)(ii). An argument analogous to that of (1.2.11) shows that $\xi(g)$ is an automorphism of the Coxeter system (W, S) and that ξ *is a homomorphism of* $\widehat{G}$ *into the group of automorphisms of* (W, S)*, or again into the group of automorphisms of the Coxeter graph of* (W, S). We put $\Xi = \xi(\widehat{G})$.

(1.2.17) The kernel $\widehat{G}_0$ of ξ contains $\varphi(G)$. It follows that the restriction of ξ to $\operatorname{Stab} B$ is a surjective map of $\operatorname{Stab} B$ onto Ξ. We put $\widehat{\mathrm{B}} = \widehat{G}_0 \cap \operatorname{Stab} B$. The pair $(\widehat{\mathrm{B}}, \varphi(N))$ is a Tits system in $\widehat{G}_0$, with Weyl group canonically isomorphic to W (*l.c.*, §2, exerc. 8).

More generally, if P is a parabolic subgroup of G, we put $\widehat{\mathrm{P}} = \widehat{G}_0 \cap \operatorname{Stab} P$. If $P \supset B$, one has $\widehat{\mathrm{P}} = \varphi(P) \cdot \widehat{\mathrm{B}}$. The map $P \mapsto \widehat{\mathrm{P}}$ is a bijection of the set of parabolic subgroups of G onto that of $\widehat{G}_0$.

(1.2.18) If P is a parabolic subgroup of type $X \subset S$, the parabolic subgroup ${}^gP$ is of type $\xi(g)(X)$ (for $g \in \widehat{G}$).

For every subset X of S we write Ξ_X for the set of $\xi \in \Xi$ such that $\xi(X) = X$. If $g \in \operatorname{Stab} B_X$, the minimal parabolic subgroups B and ${}^gB$ are contained in B_X, hence are conjugate by an element of B_X (1.2.10). Consequently

$$\operatorname{Stab} B_X = \varphi(B_X) \cdot (\operatorname{Stab} B_X \cap \operatorname{Stab} B) = \varphi(B_X) \cdot \big(\operatorname{Stab} B \cap \xi^{-1}(\Xi_X)\big),$$

since ${}^gB_X = B_{\xi(g)(X)}$ for $g \in \operatorname{Stab} B$.

(1.2.19) Let $X \subset S$ and let H be a subgroup of Ξ_X. Put

$$Q(X, \mathsf{H}) = \varphi(B_X) \cdot \big(\operatorname{Stab} B \cap \xi^{-1}(\mathsf{H})\big).$$

The map $(X, \mathsf{H}) \mapsto Q(X, \mathsf{H})$ is a bijection of the set of pairs $(X, \mathsf{H}) \in \mathfrak{P}(S) \times \mathfrak{P}(\Xi)$ such that H is a subgroup of Ξ_X, onto the set of subgroups of $\widehat{G}$ containing $\widehat{\mathrm{B}}$. One has $Q(X, \Xi_X) = \operatorname{Stab} B_X$; moreover $Q(X, \mathsf{H}) \subset Q(X', \mathsf{H}')$ if and only if $X \subset X'$ and $\mathsf{H} \subset \mathsf{H}'$ (*l.c.*, §2, exerc. 8). The subgroups $Q(X, \mathsf{H})$ and $Q(X', \mathsf{H}')$ are conjugate in $\widehat{G}$ if and only if there exists $t \in \Xi$ such that $X' = t.X$ and $\mathsf{H}' = t.\mathsf{H}.t^{-1}$.

Let P (resp. P') be a parabolic subgroup of G, of type X (resp. X'). In order that $\operatorname{Stab} P \supset \operatorname{Stab} P'$ it is necessary and sufficient that $P \supset P'$ (whence $X \supset X'$) and that $\Xi_X \supset \Xi_{X'}$: this follows from what precedes, by reducing through conjugation to the case $P' = B_{X'}$.

(1.2.20) Suppose φ is B-N-adapted and let E be the intersection of the normalizers of $\varphi(B)$ and of $\varphi(N)$ in $\widehat{G}$. One has $\widehat{G} = E \cdot \varphi(G)$, whence $\widehat{G} = E \cdot \widehat{G}_0$. Let $e \in E$ and $n \in N$, and let ${}^en \in N$ be such that $e\varphi(n)e^{-1} = \varphi({}^en)$. One has ${}^e(nBn^{-1}) = {}^en \,.\, B \,.\, ({}^en)^{-1}$ and $\nu({}^en) = \xi(e)\big(\nu(n)\big)$.

1.3. Affine Weyl groups. —

In this number we recall some definitions and results concerning "Weyl groups of affine type". For further details and for the proofs we refer to [5], designated by *l.c.* throughout 1.3.

(1.3.1) Let $\mathbf{A}$ be a finite-dimensional real affine space. We assume that the space of translations ${}^v\mathbf{A}$ of $\mathbf{A}$ is endowed with a (nondegenerate) scalar product, hence with a norm, and we equip $\mathbf{A}$ with the corresponding (Euclidean) distance: one then says that $\mathbf{A}$ is a *Euclidean affine space*.

Two subsets of $\mathbf{A}$ deduced from one another by a translation are said to be *equipollent*.

We denote moreover by $\mathbf{W}$ a subgroup of the group of displacements of $\mathbf{A}$, generated by orthogonal reflections in (affine) hyperplanes of $\mathbf{A}$ and acting properly on $\mathbf{A}$ when the latter is given the discrete topology; this last condition amounts to saying that $\mathbf{W}$ is a discrete subgroup of the group of displacements of $\mathbf{A}$ (*l.c.*, chap. V, §4, no. 9). For $w \in \mathbf{W}$ we write ${}^v w$ for the corresponding automorphism of ${}^v\mathbf{A}$.

(1.3.2) It is known (*l.c.*, chap. V, §3, no. 8) that $\mathbf{A}$ decomposes canonically into a product of Euclidean affine spaces $\mathbf{A}_0 \times \cdots \times \mathbf{A}_m$, each of them a quotient of $\mathbf{A}$, in such a way that the following conditions are realized: the action of $\mathbf{W}$ passes to the quotient and defines a discrete group $\mathbf{W}_i$ of displacements of $\mathbf{A}_i$, generated by orthogonal reflections in hyperplanes of $\mathbf{A}_i$; the group $\mathbf{W}$ is identified with the direct product of the $\mathbf{W}_i$; one has $\mathbf{W}_0 = \{1\}$, and each $\mathbf{W}_i$ for $1 \leqslant i \leqslant m$ is *irreducible*, in the sense that $\mathbf{W}_i \neq \{1\}$ and that the canonical linear representation of $\mathbf{W}_i$ in the space ${}^v\mathbf{A}_i$ of translations of $\mathbf{A}_i$ is irreducible.

In everything that follows we shall assume that $\mathbf{A}_0$ is reduced to a point, which allows us not to consider it, and that *each* $\mathbf{W}_i$ *for* $1 \leqslant i \leqslant m$ *is infinite*, or, what amounts to the same (*l.c.*, chap. V, §3, no. 9), *has no fixed point in* $\mathbf{A}_i$. Under these conditions we shall say that $\mathbf{W}$ is an *affine Weyl group*; when moreover $m = 1$, we shall say that $\mathbf{W}$ is *irreducible*.

(1.3.3) If L is an affine hyperplane of $\mathbf{A}$, we write s_L for the orthogonal reflection admitting L as its set of fixed points. Conversely, if s is an orthogonal reflection in a hyperplane, we write L_s for that hyperplane, the set of fixed points of s.

By a *wall* of $\mathbf{A}$ (relative to $\mathbf{W}$) we mean any hyperplane L of $\mathbf{A}$ such that the reflection s_L belongs to $\mathbf{W}$. It is known that the set $\mathscr{H}$ of walls of $\mathbf{A}$ is locally finite (*l.c.*, chap. V, §3, no. 1).

By an *affine root* of $\mathbf{A}$ we mean any closed half-space of $\mathbf{A}$ bounded by a wall, this wall being called the *wall of the root*. The set of affine roots of $\mathbf{A}$ will be written $\mathbf{\Sigma}$. If $\alpha \in \mathbf{\Sigma}$, one puts $r_\alpha = s_{\partial\alpha}$, where $\partial\alpha$ is the wall of α; one puts $\alpha^* = \overline{\mathbf{A} - \alpha}$, and one denotes by α_+ the intersection of the affine roots containing a neighborhood of α: it is an affine root equipollent to α.

It is clear that the datum of $\mathbf{\Sigma}$ determines $\mathbf{W}$. A set of closed half-spaces of a Euclidean affine space $\mathbf{A}$ which is the set of affine roots of $\mathbf{A}$ for an affine Weyl group acting in $\mathbf{A}$ will be called an *affine root system*.

By a *facet* of $\mathbf{A}$ (relative to $\mathbf{W}$ or to $\mathbf{\Sigma}$) we mean an equivalence class in $\mathbf{A}$ for the equivalence relation "x and y are contained in the same affine roots".

A facet F is an open convex subset of the affine subspace (called the *support* of F) that it spans. Let us endow the set $\mathscr{F}$ of facets with the order relation "F' is contained in the closure of F"; the set denoted $\bar{F}$ in no. 1.1 coincides with the closure of F, and $\mathbf{A}$, equipped with $\mathscr{F}$, with this order relation and with the canonical affine structures on each of the sets $\bar{F}$, is a polysimplicial complex, whose structure is the product of the polysimplicial complex structures on each of the $\mathbf{A}_i$ (*l.c.*, chap. V, §3, no. 8, prop. 6 and no. 9, prop. 8), which are moreover simplicial complexes.

The *chambers* of $\mathbf{A}$ (relative to $\mathbf{W}$) are the connected components of the complement in $\mathbf{A}$ of the union of the walls, and the *panels* of $\mathbf{A}$ are the facets whose support is a hyperplane (which is then a wall). It is known that $\mathbf{W}$ is simply transitive on the set of chambers (*l.c.*, chap. V, §3, no. 2, th. 1).

(1.3.4) We denote by $\mathbf{C}$ a chamber of $\mathbf{A}$ fixed once and for all. It is known (*l.c.*, chap. V, §3, no. 3, th. 2) that *the closure* $\bar{\mathbf{C}}$ *of* $\mathbf{C}$ *is a fundamental domain for* $\mathbf{W}$ *acting in* $\mathbf{A}$. Moreover, $\mathbf{W}$ is generated by the set $\mathbf{S}$ of reflections in the walls supporting the panels of $\mathbf{C}$ (called the *walls of the chamber* $\mathbf{C}$), and *the pair* $(\mathbf{W}, \mathbf{S})$ *is a Coxeter system* (*l.c.*, chap. V, §3, no. 2, th. 1).

More precisely, for $1 \leqslant i \leqslant m$, the projection $\mathbf{C}_i$ of $\mathbf{C}$ in $\mathbf{A}_i$ is a chamber of $\mathbf{A}_i$ (relative to $\mathbf{W}_i$). If $\mathbf{S}_i$ is the set of reflections in the walls of $\mathbf{C}_i$, the Coxeter system $(\mathbf{W}_i, \mathbf{S}_i)$ is irreducible and $(\mathbf{W}, \mathbf{S})$ is identified with the product of the $(\mathbf{W}_i, \mathbf{S}_i)$; in other words, after identifying the $\mathbf{W}_i$ with subgroups of $\mathbf{W}$, one has $\mathbf{S} = \bigcup_{1 \leqslant i \leqslant m} \mathbf{S}_i$.

The *rank* of $(\mathbf{W}, \mathbf{S})$, that is to say the cardinality of $\mathbf{S}$, is equal to $\sum_{1 \leqslant i \leqslant m} (\dim \mathbf{A}_i + 1)$. On the other hand, one calls *rank* of $\mathbf{W}$ (or of $\mathbf{\Sigma}$) the dimension of $\mathbf{A}$.

The Coxeter system $(\mathbf{W}, \mathbf{S})$ (and also its Coxeter graph) determines the affine Weyl group $\mathbf{W}$ essentially. More precisely, let $\mathbf{A}'$ be a Euclidean affine space, $\mathbf{W}'$ an affine Weyl group in $\mathbf{A}'$, and $\mathbf{S}'$ the set of reflections in the walls of a chamber of $\mathbf{A}'$ relative to $\mathbf{W}'$; then every isomorphism of the Coxeter system $(\mathbf{W}, \mathbf{S})$ onto $(\mathbf{W}', \mathbf{S}')$ is induced by a unique isomorphism $\alpha : \mathbf{A} \to \mathbf{A}'$ of affine spaces; if moreover $\mathbf{W}$ is irreducible, α is a similarity, i.e. transforms the distance of $\mathbf{A}$ into a multiple of that of $\mathbf{A}'$.

(1.3.5) For every facet F there exists exactly one facet of $\mathbf{C}$ which is the transform of F by an element of $\mathbf{W}$. On the other hand, for every subset X of $\mathbf{S}$ such that $X \cap \mathbf{S}_i \neq \mathbf{S}_i$ for $1 \leqslant i \leqslant m$, the set $\mathbf{C}_X$ of points $a \in \bar{\mathbf{C}}$ such that X is exactly the set of $s \in \mathbf{S}$ with $a \in L_s$, is a facet of $\mathbf{C}$. Moreover, the map $X \mapsto \mathbf{C}_X$ is a bijection of the set

$$\mathbf{T} = \{X \subset \mathbf{S} \mid X \cap \mathbf{S}_i \neq \mathbf{S}_i \text{ for } 1 \leqslant i \leqslant m\} = \{X \subset \mathbf{S} \mid \mathbf{W}_X \text{ is finite}\}$$

onto the set of facets of $\mathbf{C}$. One has $\mathbf{C} = \mathbf{C}_\varnothing$. We shall say that $\mathbf{T}$ is the set of *types* of facets, and that a facet F is *of type* $X \in \mathbf{T}$ if F is the transform of $\mathbf{C}_X$ by an element of $\mathbf{W}$.

The stabilizer of $\mathbf{C}_X$, which is also the stabilizer of an arbitrary point of $\mathbf{C}_X$, is the subgroup $\mathbf{W}_X$ of $\mathbf{W}$ generated by X (*l.c.*, chap. V, §3, no. 3, prop. 1).

(1.3.6) In order that a facet F be a panel it is necessary and sufficient that its type reduce to a single element s; one also says that F is *of type* s. If two chambers $\mathbf{C}$ and $\mathbf{C}'$ are adjacent (1.1.5), they have exactly one panel in common. Conversely, every panel of $\mathbf{A}$ is a facet of exactly two chambers.

If $\Gamma = (\mathbf{C}_0, \dots, \mathbf{C}_n)$ is a non-stuttering gallery, one calls *type* of Γ the sequence $\mathbf{s}(\Gamma) = (s_1, \dots, s_n)$ of elements of $\mathbf{S}$ such that the panel common to $\mathbf{C}_{i-1}$ and $\mathbf{C}_i$ is of type s_i (for $1 \leqslant i \leqslant n$). Two galleries of $\mathbf{A}$ with the same origin and the same type are identical.

(1.3.7) One says that a point x of $\mathbf{A}$ is a *special point* if, for every wall L of $\mathbf{A}$, there exists a wall equipollent to L containing x. It is known that every special point is an extreme point of the closure of a chamber, and that for every chamber $\mathbf{C}$ there is at least one special point adherent to $\mathbf{C}$ (*l.c.*, chap. V, §3, no. 10, cor. of prop. 11). If x is a special point, the group $\mathbf{W}$ is the semidirect product of the stabilizer $\mathbf{W}_x$ of x by the normal subgroup $\mathbf{V}$ of translations of $\mathbf{A}$ belonging to $\mathbf{W}$, and the canonical map of $\mathbf{W}_x$ into the group of automorphisms of the vector space ${}^v\mathbf{A}$ is an isomorphism of $\mathbf{W}_x$ onto the image ${}^v\mathbf{W}$ of the whole of $\mathbf{W}$ in $\operatorname{Aut} {}^v\mathbf{A}$ (*l.c.*, chap. V, §3, no. 10, prop. 9). Moreover the group $\mathbf{V}$ is a *lattice* in ${}^v\mathbf{A}$, that is to say a discrete subgroup of maximal rank of ${}^v\mathbf{A}$ (*l.c.*, chap. VI, §2, no. 5, remark). In order that $x \in \mathbf{A}$ be a special point it is necessary and sufficient that the projection of x in $\mathbf{A}_i$ be, for every $i \in \{1, \dots, m\}$, a special point of $\mathbf{A}_i$ for $\mathbf{W}_i$.

Suppose $\mathbf{W}$ irreducible. A vertex s of the Coxeter graph of $(\mathbf{W}, \mathbf{S})$ is said to be *special* if the corresponding extreme point of $\bar{\mathbf{C}}$ (that is to say the point of intersection of the hyperplanes L_t for $t \in \mathbf{S}$, $t \neq s$) is a special point.

(1.3.8) There exists in the dual $({}^v\mathbf{A})^*$ of ${}^v\mathbf{A}$ one and only one reduced root system ${}^v\mathbf{\Sigma}$ possessing the following property: make $\mathbf{A}$ into a vector space and identify it with ${}^v\mathbf{A}$ by choosing as origin a special point of $\mathbf{A}$; for $a \in {}^v\mathbf{\Sigma}$ and $k \in \mathbf{Z}$, denote by $L_{a,k}$ the affine hyperplane consisting of the points $x \in \mathbf{A}$ such that $a(x) = -k$; then the set $\mathscr{H}$ of walls of $\mathbf{A}$ is none other than the set of hyperplanes $L_{a,k}$ for $a \in {}^v\mathbf{\Sigma}$ and $k \in \mathbf{Z}$ (*l.c.*, chap. VI, §2, no. 5, prop. 8). The affine roots of $\mathbf{A}$ are therefore the closed half-spaces

$$\alpha_{a,k} = \{x \in \mathbf{A} \mid a(x) + k \geqslant 0\} \qquad \text{for } a \in {}^v\mathbf{\Sigma},\ k \in \mathbf{Z}.$$

If $\alpha \in \mathbf{\Sigma}$ is of the form $\alpha_{a,k}$, one puts ${}^v\alpha = a$ and $r_{a,k} = r_\alpha$.

In other words, once a special point has been chosen as origin, the group $\mathbf{W}$ is identified with the affine Weyl group of the root system ${}^v\mathbf{\Sigma}$ in the sense of (*l.c.*, chap. VI, §2). In particular, for $a \in {}^v\mathbf{\Sigma}$, let us write $a^\vee$ for the unique element of ${}^v\mathbf{A}$ such that the reflection r_a associated with

a is given by $r_a(v) = v - a(v)a^\vee$ ($v \in {}^v\mathbf{A}$). The set of the $a^\vee$ for $a \in {}^v\boldsymbol{\Sigma}$ forms the *inverse root system* of ${}^v\boldsymbol{\Sigma}$ (*l.c.*, chap. VI, §1, no. 1, prop. 1). The special points are the *weights* of the root system $({}^v\boldsymbol{\Sigma})^\vee$ (*l.c.*, chap. VI, §2, no. 2, prop. 3), the group $\mathbf{V}$ of translations of $\mathbf{W}$ is the group of *radical weights* of $({}^v\boldsymbol{\Sigma})^\vee$ and is generated by $({}^v\boldsymbol{\Sigma})^\vee$ (*l.c.*, chap. VI, §2, no. 1, prop. 1), and the chambers of $\mathbf{A}$ are the *alcoves* of ${}^v\boldsymbol{\Sigma}$ in the sense of (*l.c.*, chap. VI, §2, no. 1). The group ${}^v\mathbf{W}$ (acting in ${}^v\mathbf{A}$ and in its dual) is identified with the (ordinary) Weyl group of ${}^v\boldsymbol{\Sigma}$. It is generated by the reflections r_a for $a \in {}^v\boldsymbol{\Sigma}$.

The elements of ${}^v\boldsymbol{\Sigma}$ will be called the *vector roots* of $\mathbf{A}$ (relative to $\mathbf{W}$).

(1.3.9) To the decomposition $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_m$ of (1.3.2) there corresponds a decomposition of the dual of ${}^v\mathbf{A}$ into a direct sum of the $({}^v\mathbf{A}_i)^*$, and the intersections ${}^v\boldsymbol{\Sigma} \cap ({}^v\mathbf{A}_i)^*$ are none other than the irreducible components of ${}^v\boldsymbol{\Sigma}$ (*l.c.*, chap. VI, §1, no. 2).

(1.3.10) By a *vector chamber* we mean a connected component of the complement in ${}^v\mathbf{A}$ of the union of the (vector) hyperplanes that are kernels of the vector roots, that is to say a chamber of the inverse root system $({}^v\boldsymbol{\Sigma})^\vee$ in the sense of (*l.c.*, chap. VI, §1, no. 5). More generally, by a *vector facet* we mean an equivalence class for the following equivalence relation in ${}^v\mathbf{A}$: "for every vector root a, $a(x)$ and $a(y)$ are simultaneously either zero, or strictly positive, or strictly negative". A vector facet is a simplicial cone, a vector chamber is an open simplicial cone. The group ${}^v\mathbf{W}$ is simply transitive on the set of vector chambers, and the closure of a vector chamber is a fundamental domain for ${}^v\mathbf{W}$ acting on ${}^v\mathbf{A}$ (*l.c.*, chap. VI, §1, no. 5). The opposite of a vector chamber is a vector chamber (*l.c.*, chap. VI, §1, cor. 3 of prop. 17).

(1.3.11) By a *sector* (*quartier*) of $\mathbf{A}$ we mean the set $x + D$ of transforms of a given point $x \in \mathbf{A}$ (called the *apex* of the sector) by the translations belonging to a given vector chamber D (called the *direction* of the sector). Two sectors of opposite direction are said to be *opposite*. Let x be a special point of $\mathbf{A}$; every sector with apex x contains exactly one chamber $\mathbf{C}$ to which x is adherent; the sector Q is then the union of the transforms of $\mathbf{C}$ by the homotheties with center x and ratio > 0, and one obtains in this way a bijection of the set of sectors with apex x (or again of the set of vector chambers) onto the set of chambers of $\mathbf{A}$ whose closure contains x.

(1.3.12) With every vector chamber D is associated a *basis* $B(D)$ of ${}^v\boldsymbol{\Sigma}$ (or system of simple roots), that is to say a linearly independent subset of ${}^v\boldsymbol{\Sigma}$ such that every element of ${}^v\boldsymbol{\Sigma}$ is a linear combination, with integral coefficients all of the same sign, of elements of $B(D)$: the vector chamber D is the set of $x \in {}^v\mathbf{A}$ such that $a(x) > 0$ for all $a \in B(D)$. The map $D \mapsto B(D)$ is a bijection of the set of vector chambers onto the set of bases of ${}^v\boldsymbol{\Sigma}$ (*l.c.*, chap. VI, §1, nos. 5 and 7). For every $a \in B(D)$, the hyperplane $\operatorname{Ker} a$ contains a face of the simplicial cone D, and one obtains in this way a bijection of $B(D)$ onto the set of faces of D.

(1.3.13) The vector roots which are linear combinations with positive coefficients of elements of $B(D)$ are said to be *positive* for the vector chamber D; they are moreover the positive roots for the order relation on the dual of ${}^v\mathbf{A}$ defined by the convex cone D. Their set is written ${}^v\boldsymbol{\Sigma}^+_D$ or ${}^v\Sigma^+(D)$. One has ${}^v\boldsymbol{\Sigma} = {}^v\boldsymbol{\Sigma}^+_D \cup (-{}^v\boldsymbol{\Sigma}^+_D)$. If $\mathbf{W}$ is irreducible, or what amounts to the same, if the root system ${}^v\boldsymbol{\Sigma}$ is irreducible, there exists a greatest root $\tilde{a}$ for this order relation; if one identifies $\mathbf{A}$ and ${}^v\mathbf{A}$ as above by taking a special point as origin, the chamber of $\mathbf{A}$ contained in the sector D and containing the origin in its closure is the set of points $x \in \mathbf{A}$ such that $a(x) > 0$ for all $a \in B(D)$ and $\tilde{a}(x) < 1$ (*l.c.*, chap. VI, §2, no. 2, prop. 5).

(1.3.14) Let D be a vector chamber. It is known that the set $R = R(D)$ of reflections r_a for $a \in B(D)$ generates ${}^v\mathbf{W}$, and that $({}^v\mathbf{W}, R)$ is a Coxeter system (*l.c.*, chap. VI, §1, no. 5, th. 2); one has

$$ {}^v\Sigma^+\big(r_a(D)\big) = \{b \in {}^v\Sigma^+(D) \mid b \neq a\} \cup \{-a\} \quad \text{and} \quad -a \in B\big(r_a(D)\big) \qquad (\textit{ibid.}).$$

(1.3.15) Let M be a subset of ${}^v\boldsymbol{\Sigma}$. We shall say that an element $m \in M$ is *extremal in M* if the half-line $\mathbf{R}_+ m$ is an extremal generator of the convex cone with origin 0 generated by M. An order on M will be called *nibbling* (*grignotant*) if it is total and if every element $m \in M$ is extremal in the set of elements of M greater than m. In order that there exist a nibbling order on a subset M of ${}^v\boldsymbol{\Sigma}$, it is necessary and sufficient that the convex cone with origin 0 generated by M be strictly convex, or again that its polar have nonempty interior, or again that it meet a vector chamber D, which means that $M \subset {}^v\Sigma^+(D)$. In this case there exists a nibbling order on M for which the first element is a given element extremal in M. In particular, there exists a nibbling order on ${}^v\Sigma^+(D)$ for which the first element is an arbitrary simple root $a \in B(D)$.

If $M' \subset M$, the restriction to M' of a nibbling order on M is a nibbling order on M'.

(1.3.16) For every vector chamber D one defines an order relation, said to be *associated with* D and written $\leqslant_D$, on $\mathbf{A}$ (resp. ${}^v\mathbf{A}$) in the following manner. Let D^* be the chamber of the root system ${}^v\boldsymbol{\Sigma}$ associated with the basis $B(D)$ (*l.c.*, chap. VI, §1, no. 5): it is the set of $t \in ({}^v\mathbf{A})^*$ whose scalar product with every element of $B(D)$ is strictly positive. By definition, the relation $x \leqslant_D x'$, for $x, x' \in \mathbf{A}$ (resp. ${}^v\mathbf{A}$), is equivalent to $t(x' - x) \geqslant 0$ for all $t \in D^*$ (or, what amounts to the same, for every dominant weight t of ${}^v\boldsymbol{\Sigma}$ relative to $B(D)$). These order relations are compatible with the vector space structure of ${}^v\mathbf{A}$ and with the action of ${}^v\mathbf{A}$ on $\mathbf{A}$. For $y \in \bar{D}$ and $w \in {}^v\mathbf{W}$ one has $w(y) \leqslant_D y$ (*l.c.*, chap. VI, §1, no. 6, prop. 18).

(1.3.18) Let $\widetilde{\mathbf{W}}$ be the normalizer of $\mathbf{W}$ in the group of automorphisms of $\mathbf{A}$; it is also the group of automorphisms of the polysimplicial complex $\mathbf{A}$. Let $\widetilde{\mathbf{V}}$ be the subgroup of translations belonging to $\widetilde{\mathbf{W}}$, let x be a special point of $\mathbf{A}$ adherent to $\mathbf{C}$, and let D be the vector chamber satisfying $\mathbf{C} \subset x + \mathrm{D}$. For every subset Ω of $\mathbf{A}$ one puts

$$\widetilde{\mathbf{W}}^\dagger_\Omega = \{\, w \in \widetilde{\mathbf{W}} \mid w(\Omega) = \Omega \,\}.$$

Then (*loc. cit.*, ch. VI, §2):

(1) $\widetilde{\mathbf{W}}$ is the semidirect product of $\widetilde{\mathbf{W}}^\dagger_{\mathbf{C}}$ by the normal subgroup $\mathbf{W}$.

(2) $\widetilde{\mathbf{W}}$ is the semidirect product of $\widetilde{\mathbf{W}}^\dagger_x$ by $\widetilde{\mathbf{V}}$, and the map $w \mapsto {}^v w$ is an isomorphism of $\widetilde{\mathbf{W}}^\dagger_x$ onto ${}^v\widetilde{\mathbf{W}}$.

(3) $\widetilde{\mathbf{V}}$ is the weight group of the root system inverse to ${}^v\boldsymbol{\Sigma}$, and it acts simply transitively on the set of special points of $\mathbf{A}$.

(4) $\widetilde{\mathbf{W}}^\dagger_x$ is the semidirect product of $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ by $\mathbf{W}_x$. If one lets $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ act on the dual of ${}^v\mathbf{A}$ by the contragredient of the natural action on ${}^v\mathbf{A}$, then $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ fixes ${}^v\boldsymbol{\Sigma}$ and $\mathrm{B}(\mathrm{D})$, whence a homomorphism of $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ into the automorphism group of the Dynkin graph of ${}^v\boldsymbol{\Sigma}$; this homomorphism is bijective.

(5) The elements of $\widetilde{\mathbf{W}}^\dagger_{\mathbf{C}}$ permute the walls of $\mathbf{C}$; one deduces a homomorphism of $\widetilde{\mathbf{W}}^\dagger_{\mathbf{C}}$ into the automorphism group of $(\mathbf{W}, \mathbf{S})$, equivalently of the Coxeter graph of $\mathbf{W}$, and this homomorphism is bijective. If $\mathbf{W}$ is irreducible, the image of $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ under this isomorphism is the fixer subgroup of the special vertex of Cox $\mathbf{W}$ associated with x.

(6) Put $\widetilde{\mathbf{W}}_{\mathrm{int}} = \mathbf{W}_x \cdot \widetilde{\mathbf{V}}$. This is a normal subgroup of $\widetilde{\mathbf{W}}$, and $\widetilde{\mathbf{W}}$ is the semidirect product of $\widetilde{\mathbf{W}}^\dagger_{(x+\mathrm{D})}$ by $\widetilde{\mathbf{W}}_{\mathrm{int}}$.

1.4. Échelonnages. —

(Note added in proof.) There is an equivalence between the notion of *échelonnage* used here and what I. G. Macdonald calls an *affine root system*.

The notation of no. 1.3 is retained.

(1.4.1) Let Φ be a not necessarily reduced root system in the dual of ${}^v\mathbf{A}$. The following two properties are equivalent:

(i) *Φ and the reduced root system ${}^v\mathbf{\Sigma}$ have the same Weyl group ${}^v\mathbf{W}$;*

(ii) *for every $a \in \Phi$ there is a real number $\lambda(a) > 0$ with $\lambda(a)a \in {}^v\mathbf{\Sigma}$, and the map $a \mapsto \lambda(a)a$ from Φ to ${}^v\mathbf{\Sigma}$ is surjective.*

In what follows Φ is always assumed to satisfy these conditions. One sees readily that if ${}^v\mathbf{\Sigma}$ is irreducible and not of type B_n, C_n, G_2 or F_4, then $\Phi = \lambda\, {}^v\mathbf{\Sigma}$ for some $\lambda \in \mathbf{R}_+^*$. If on the other hand ${}^v\mathbf{\Sigma}$ is of type B_n, C_n, F_4 (resp. G_2), then Φ may be of type C_n, B_n, F_4 (resp. G_2), and one may have $\lambda(a) = 2\lambda(b)$ (resp. $\lambda(a) = 3\lambda(b)$) with a running over the short roots and b over the long roots of Φ; moreover, if ${}^v\mathbf{\Sigma}$ is of type B_n or C_n, then Φ may be of type BC_n.

An **échelonnage of Φ by $\mathbf{\Sigma}$** is a subset $\mathscr{E} \subset \Phi \times \mathbf{\Sigma}$ with the following properties:

(E 1) *if $(a, \alpha) \in \mathscr{E}$, then $a \in \mathbf{R}_+^* \cdot {}^v\alpha$;*

(E 2) *if $(a, \alpha) \in \mathscr{E}$ and $w \in \mathbf{W}$, then $({}^v w(a), w(\alpha)) \in \mathscr{E}$;*

(E 3) *the restrictions to $\mathscr{E}$ of the two projections $\mathrm{pr}_1 \colon \Phi \times \mathbf{\Sigma} \to \Phi$ and $\mathrm{pr}_2 \colon \Phi \times \mathbf{\Sigma} \to \mathbf{\Sigma}$ are surjective.*

Two roots $a \in \Phi$ and $\alpha \in \mathbf{\Sigma}$ with $(a, \alpha) \in \mathscr{E}$ are said to be **associated** by the échelonnage $\mathscr{E}$.

(1.4.2) If $x \in \mathbf{A}$ and F is the facet of $\mathbf{A}$ containing x, then the set $\Phi_x = \Phi_F$ of roots $a \in \Phi$ associated with the affine roots whose wall contains x is a root system whose Weyl group is the canonical image in ${}^v\mathbf{W}$ of the stabilizer of x. It is called the **root system attached to** x (or to F) **by** $\mathscr{E}$.

(1.4.3) Let $\Phi' \subset ({}^v\mathbf{A})^*$ be another root system. Two échelonnages $\mathscr{E} \subset \Phi \times \mathbf{\Sigma}$ and $\mathscr{E}' \subset \Phi' \times \mathbf{\Sigma}$ are said to be **homothetic** if there is a linear transformation $\lambda \colon ({}^v\mathbf{A})^* \to ({}^v\mathbf{A})^*$ such that $(a, \alpha) \in \mathscr{E}$ if and only if $(\lambda(a), \alpha) \in \mathscr{E}'$. In that case λ induces on each $({}^v\mathbf{A}_i)^*$ a homothety of positive ratio. An échelonnage is said to be **similar** to $\mathscr{E}$ if it is isomorphic (in the evident sense) to an échelonnage homothetic to $\mathscr{E}$.

(1.4.4) In the remainder of this number one considers quadruples (G, f, M, F) consisting of a graph G, a function f from the set of edges of G to $\{3, 4, 6, \infty\}$, a set M of vertices of G, and a set F of pairs (r, s) of vertices such that $\{r, s\}$ is an edge. The vertices belonging to M are called **multipliable**. The pair (G, f) is the **underlying Coxeter graph** of (G, f, M, F). Such a quadruple is drawn according to the following conventions: an edge $\{r, s\}$ is drawn as a single, double, triple or thick stroke according as $f(\{r, s\}) = 3,\ 4,\ 6$ or ∞; it carries an inequality sign $>$ running from r towards s when $(r, s) \in F$; and multipliable vertices are marked with a cross.

Let Π be a finite subset of $({}^v\mathbf{A})^* \setminus \{0\}$ such that for every pair (a, b) of elements of Π one has

$$\frac{-2(a \mid b)}{(a \mid a)} \in \mathbf{N}, \qquad \text{or} \quad a = 2b, \qquad \text{or else} \quad b = 2a,$$

where $(\ \mid\ )$ denotes a scalar product on $({}^v\mathbf{A})^*$ invariant under ${}^v\mathbf{W}$. The **Dynkin graph of** Π is the quadruple (G, f, M, F) defined as follows: the vertices of G are the elements a of Π with $a/2 \notin \Pi$; a pair $\{a, b\}$ of vertices is an edge if $(a \mid b) \neq 0$; the function f is defined by

$$\cos^2 \frac{\pi}{f(\{a, b\})} = \frac{(a \mid b)^2}{(a \mid a)(b \mid b)}\,;$$

a vertex a belongs to M if $2a \in \Pi$; and a pair (a, b) of vertices belongs to F if $(a \mid a) > (b \mid b)$.

Let Φ be a root system in a subspace of $({}^v\mathbf{A})^*$ whose Weyl group leaves invariant the restriction of $(\ \mid\ )$ to that subspace. The **Dynkin graph of** Φ, written $\operatorname{Dyn}\Phi$, is by definition the Dynkin graph of the set of elements of Φ that are positive multiples of the elements of a base. Up to canonical isomorphism this graph does not depend on the chosen base.

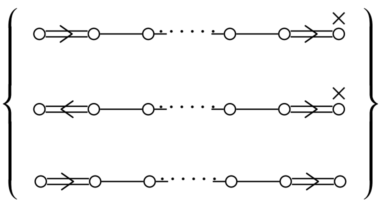
(1.4.5) The **Dynkin graph of an échelonnage** $\mathscr{E} \subset \Phi \times \boldsymbol{\Sigma}$, written $\operatorname{Dyn}\mathscr{E}$, is the Dynkin graph of the set of elements of Φ associated with the affine roots that contain $\mathbf{C}$ and whose wall contains a panel of $\mathbf{C}$. Up to canonical isomorphism this graph depends only on $\mathscr{E}$, and not on the choice of fundamental chamber $\mathbf{C}$; its vertex set is in canonical bijection with the set of panels of $\mathbf{C}$, hence also with $\mathbf{S}$.

Knowing $\operatorname{Dyn}\mathscr{E}$, one recovers the Coxeter graph of $\mathbf{W}$ and the graphs $\operatorname{Dyn}\Phi$ and $\operatorname{Dyn}\Phi_x$ for every $x \in \mathbf{A}$ (hence also the systems Φ, $\boldsymbol{\Sigma}$ and Φ_x up to isomorphism) by the following rules, whose proofs are immediate.

(1) *The Coxeter graph of* $\mathbf{W}$ *is the Coxeter graph underlying* $\operatorname{Dyn}\mathscr{E}$.

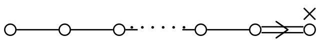
Suppose $\operatorname{Dyn}\mathscr{E}$ connected. If its rank (that is, its number of vertices minus one) equals 1, then $\operatorname{Dyn}\Phi$ has a single vertex, always multipliable except when $\operatorname{Dyn}\mathscr{E}$ is the graph . If $\operatorname{Dyn}\mathscr{E}$ has rank ≥ 2 and is not one of the following graphs:

(2)



($n + 1$ vertices)

then $\operatorname{Dyn}\Phi$ is obtained from $\operatorname{Dyn}\mathscr{E}$ by deleting a special vertex, and the result does not depend on which special vertex is chosen. Finally, if $\operatorname{Dyn}\mathscr{E}$ is one of the graphs (2), then $\operatorname{Dyn}\Phi$ is the graph



(n vertices).

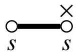
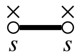
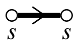
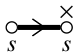
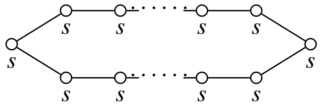
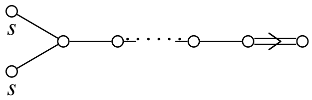
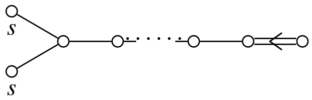
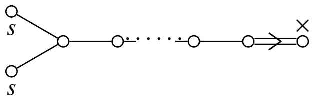
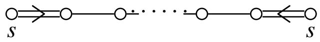
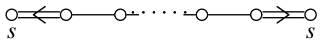
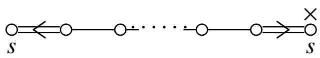
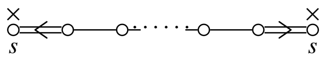
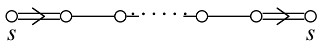
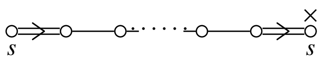
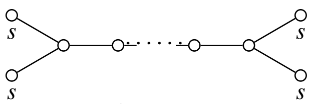
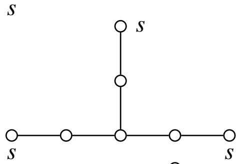
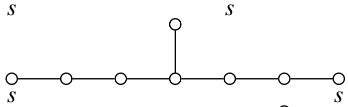
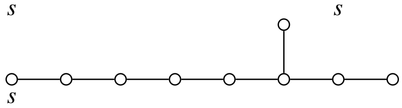
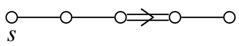
When $\operatorname{Dyn}\mathscr{E}$ is not connected, $\operatorname{Dyn}\Phi$ is obtained by applying the preceding rules to the connected components of $\operatorname{Dyn}\mathscr{E}$.

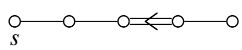
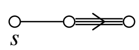
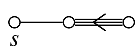
(3) *If* $x \in \mathbf{A}$ *lies in a facet of type* $X \subset \mathbf{S}$*, the graph* $\operatorname{Dyn}\Phi_x$ *is the subgraph of* $\operatorname{Dyn}\mathscr{E}$ *whose vertices correspond to the elements of* X.

Classification (1.4.6). — *An échelonnage is determined up to similarity by its Dynkin graph. The table below is the complete list of the Dynkin graphs of the irreducible échelonnages.*

The proof proceeds by classifying, for each irreducible affine Weyl group, the corresponding échelonnages. For $\mathbf{W}$ of type C_n, for instance, inspection of the classification of root systems shows that up to homothety Φ is one of: the system $\Phi_0 = {}^v\boldsymbol{\Sigma}$ of type C_n; the system Φ_1 of type B_n obtained by keeping the long roots of ${}^v\boldsymbol{\Sigma}$ and replacing the short roots by their doubles; or the system $\Phi_2 = \Phi_0 \cup \Phi_1$ of type BC_n — these being the only systems admitting ${}^v\mathbf{W}$ as Weyl group. For $\Phi = \Phi_0$ and for $\Phi = \Phi_1$ there is exactly one échelonnage of Φ by $\boldsymbol{\Sigma}$, since for each $\alpha \in \boldsymbol{\Sigma}$ there is exactly one root $a \in \Phi$ with ${}^v\alpha \in \mathbf{R}_+ \cdot a$; for $\Phi = \Phi_2$ there are four pairwise non-similar ones. The remaining types are handled in the same way, more easily. (See the original text, pp. 28–29, for the details.)

Table of the irreducible échelonnages. —

Name	**Rank**	**Type of ${}^\vee\Sigma$**	**Type of Φ**	**Dynkin graph and special vertices**
A_1	1	A_1	A_1	
$C\text{-}BC_1^{\mathrm{I}}$	1	A_1	BC_1	
$C\text{-}BC_1^{\mathrm{II}}$	1	A_1	BC_1	
$C\text{-}BC_1^{\mathrm{III}}$	1	A_1	BC_1	
$C\text{-}BC_1^{\mathrm{IV}}$	1	A_1	BC_1	
A_n	$n \geq 2$	A_n	A_n	
B_n	$n \geq 3$	B_n	B_n	
$B\text{-}C_n$	$n \geq 3$	B_n	C_n	
$B\text{-}BC_n$	$n \geq 3$	B_n	BC_n	
C_n	$n \geq 2$	C_n	C_n	
$C\text{-}B_n$	$n \geq 2$	C_n	B_n	
$C\text{-}BC_n^{\mathrm{I}}$	$n \geq 2$	C_n	BC_n	
$C\text{-}BC_n^{\mathrm{II}}$	$n \geq 2$	C_n	BC_n	
$C\text{-}BC_n^{\mathrm{III}}$	$n \geq 2$	C_n	BC_n	
$C\text{-}BC_n^{\mathrm{IV}}$	$n \geq 2$	C_n	BC_n	
D_n	$n \geq 4$	D_n	D_n	
E_6	6	E_6	E_6	
E_7	7	E_7	E_7	
E_8	8	E_8	E_8	
F_4	4	F_4	F_4	

Name	Rank	Type of $^v\Sigma$	Type of Φ	Dynkin graph and special vertices
F_4^{I}	4	F_4	F_4	 s
G_2	2	G_2	G_2	 s
G_2^{I}	2	G_2	G_2	 s

1.5. Conventions and notation. —

(1.5.1) A Tits system (G, B, N, S) with Weyl group W is said to be **of affine type** if there exist an affine Weyl group $\mathbf{W}$ acting on a euclidean affine space $\mathbf{A}$, a chamber $\mathbf{C}$ of $\mathbf{A}$, and an isomorphism of the Coxeter system (W, S) onto the Coxeter system $(\mathbf{W}, \mathbf{S})$, where $\mathbf{S}$ is the set of reflections in the walls of $\mathbf{C}$ (1.3.4). The **rank** of such a Tits system is the rank of $\mathbf{W}$, that is, the dimension of $\mathbf{A}$ (1.3.4).

A **parahoric subgroup** of G is then a parabolic subgroup of G whose type X is, after the identification of W with $\mathbf{W}$, an element of the set $\mathbf{T}$ of (1.3.5) — that is, is such that $\mathbf{W}_X$ is *finite*. A subgroup P containing B is therefore parahoric if and only if the set of double cosets $B\backslash P/B$ is finite. When $\mathbf{W}$ is irreducible the parahoric subgroups are exactly the proper parabolic subgroups, i.e. those different from G. Every parabolic subgroup contained in a parahoric subgroup is parahoric; the minimal parabolic subgroups, i.e. the conjugates of B, are parahoric.

(1.5.2) *Throughout the rest of the chapter the hypotheses and notation of* 1.3 *are retained.* In particular $\mathbf{W}$ denotes an affine Weyl group acting on a euclidean affine space $\mathbf{A}$, and the symbols $^v\mathbf{A}$, $\mathbf{C}$, $\mathbf{\Sigma}$, etc. have the same meaning as in 1.3.

In §§2–4, (G, B, N, S) *denotes a saturated Tits system of affine type.* The Weyl group W of G is identified with $\mathbf{W}$ as in (1.5.1), and the symbols B_X, H, etc. have the same meaning as in 1.2.

2. The building of a Tits system of affine type

With the Tits system (G, B, N, S) of (1.5.2) one associates a set $\mathscr{I}$, equipped with: a structure of polysimplicial complex (2.1); a set of chambered morphisms of polysimplicial complexes from $\mathbf{A}$ into $\mathscr{I}$, called the *structural maps* of $\mathscr{I}$ (2.2); a distance making it a complete metric space, for which the structural maps are isometries (2.5); and an action of G respecting all of these structures (2.1.4). The object $\mathscr{I}$ so defined is called the **building** of the Tits system (G, B, N, S), or, by abuse of language, of G. It depends only on the conjugacy class of the pair (B, N) (2.7).

2.1. The polysimplicial complex $\mathscr{I}$. —

(2.1.1) Let $\mathscr{P}$ be the set of parahoric subgroups of G. For $P \in \mathscr{P}$ let $\tau(P) \in \mathbf{T}$ denote the *type* of P (1.2.10). Let $\mathscr{I}$ be the set of pairs (P, x) with $P \in \mathscr{P}$ and $x \in \mathbf{C}_{\tau(P)}$ (1.3.5). For $P \in \mathscr{P}$ the set

$$F = F(P) = \{(P, x) \mid x \in \mathbf{C}_{\tau(P)}\}$$

is called a **facet** of $\mathscr{I}$, of *type* $\tau(F) = \tau(P)$ and of *codimension* $\operatorname{Card} \tau(P)$. If $P' \in \mathscr{P}$ and $P \subset P'$, one says that $F(P')$ is a facet of $F(P)$; one then has $\tau(P') \supset \tau(P)$. Conversely $F(P)$ possesses, for each type $Y \supset \tau(P)$, one and only one facet of type Y (1.2.10). We write $\overline{F}$ for the union of the facets of a given facet F.

(2.1.2) The map $(P, x) \mapsto x$ is called the **canonical map** of $\mathscr{I}$ into $\overline{\mathbf{C}}$. If F is a facet of $\mathscr{I}$ of type X, the restriction of this map to F (resp. $\overline{F}$) is a bijection of F (resp. $\overline{F}$) onto $\mathbf{C}_X$ (resp. $\overline{\mathbf{C}}_X$), whose inverse is called the canonical map of $\mathbf{C}_X$ (resp. $\overline{\mathbf{C}}_X$) onto F (resp. $\overline{F}$). *We endow F (resp. $\overline{F}$) with the structure of open (resp. closed) affine polysimplex obtained by transport of structure from that of $\mathbf{C}_X$ (resp. $\overline{\mathbf{C}}_X$) via this canonical map.* Writing $\mathscr{F}$ for the set of facets of $\mathscr{I}$ and $F' \prec F$ for the incidence relation $F' \subset \overline{F}$, the conditions of (1.1.4) are satisfied. The **chambers** of $\mathscr{I}$ are the facets of type $\emptyset$, associated with the minimal parahoric subgroups. At the other extreme, if P is a maximal parahoric subgroup, the facet $F(P)$ contains a single point, written a_P and called the **vertex** of $\mathscr{I}$ associated with P.

(2.1.3) If two chambers of $\mathscr{I}$ are *adjacent*, that is, distinct with a panel in common (1.1.5), this panel is well determined: indeed, if $F(P_1)$ and $F(P_2)$ are two distinct panels of one chamber $F(P)$, then $P = P_1 \cap P_2$.

This permits the following definition. If $\Gamma = (C_0, \ldots, C_n)$ is a non-stuttering gallery of $\mathscr{I}$ (1.1.5), the **type** of Γ is the sequence $\mathbf{s}(\Gamma) = (s_1, \ldots, s_n)$ of elements of $\mathbf{S}$ such that the panel common to the adjacent chambers C_{i-1} and C_i is of type s_i for $1 \le i \le n$.

(2.1.4) The group G acts on $\mathscr{I}$ by

$$g \cdot (P, x) = (gPg^{-1}, x) \qquad (P \in \mathscr{P},\ x \in \mathbf{C}_{\tau(P)}). \tag{2}$$

This is indeed an action, since gPg^{-1} is a parahoric subgroup of the same type as P.

Proposition. —
- *(i) For every type $X \in \mathbf{T}$, the group G permutes the facets of type X transitively.*
- *(ii) For every parahoric subgroup P, the restriction to $F(P)$ (resp. $\overline{F(P)}$) of the action of $g \in G$ is an affine bijection of $F(P)$ (resp. $\overline{F(P)}$) onto $F(gPg^{-1})$ (resp. $\overline{F(gPg^{-1})}$), whose composite with the canonical map of $\mathbf{C}_{\tau(P)}$ onto $F(P)$ is the canonical map of $\mathbf{C}_{\tau(P)}$ onto $F(gPg^{-1})$.*

Proof: see the original text, (i) follows from the conjugacy of parahoric subgroups of a given type (1.2.10); (ii) is immediate.

Proposition (2.1.5). — *Let Q be a parahoric subgroup of G.*
- *(i) Q is the stabilizer in G of the facet $F(Q)$ and of each of its points;*
- *(ii) $\overline{F(Q)}$ is the set of points of $\mathscr{I}$ fixed by Q.*

Proof: see the original text, both follow at once from the fact that a parahoric subgroup is its own normalizer.

Corollary (2.1.6). — *Let C be a chamber of $\mathscr{I}$. Then $\overline{C}$ is a fundamental domain for G in $\mathscr{I}$.*

Proof: see the original text, every facet of $\mathscr{I}$ is a transform of exactly one facet of C; (2.1.5) then shows each orbit meets $\overline{C}$ in exactly one point.

Proposition (2.1.7). — *Let C and C' be two chambers of $\mathscr{I}$. There exists one and only one element $w = w(C, C') \in \mathbf{W}$ such that $g^{-1}g' \in BwB$ for all $g, g' \in G$ with $C = g \cdot F(B)$ and $C' = g' \cdot F(B)$. One has $w(h \cdot C, h \cdot C') = w(C, C')$ for every $h \in G$, and $w(C', C) = w(C, C')^{-1}$. Setting $w(C) = w(F(B), C)$: for every $n \in \nu^{-1}(w(C))$ there exists $b \in B$ with $C = bn \cdot F(B)$.*

Proof: see the original text, p. 33.

Lemma (2.1.8). — *In order that a chamber $C = bn \cdot F(B)$ be adjacent to $F(B)$ it is necessary and sufficient that $\nu(n) \in \mathbf{S}$. The panel common to C and $F(B)$ is then of type $\nu(n)$.*

Proof: see the original text, pp. 33–34.

Proposition (2.1.9). — *Let $\Gamma = (C_0, \ldots, C_m)$ be a sequence of chambers with $C_0 = F(B)$. The following conditions are equivalent:*

(i) Γ is a non-stuttering gallery;

(ii) there exist $b_j \in B$, $s_j \in \mathbf{S}$ and $\bar{s}_j \in \nu^{-1}(s_j)$ $(j = 1, \ldots, m)$ such that

$$C_j = b_1\bar{s}_1 \cdots b_j\bar{s}_j \cdot F(B) \qquad \text{for } j = 1, \ldots, m.$$

Under these conditions Γ is of type $(s_1, \ldots, s_m)$.

Proof: see the original text, p. 34; one direction uses (2.1.8), the other is an induction on j.

Corollary (2.1.10). — *Let C be a chamber of $\mathscr{I}$ and put $w = w(F(B), C)$. For every reduced decomposition $\mathbf{s} = (s_1, \ldots, s_m)$ of w there exists a minimal gallery with origin $F(B)$, extremity C and type $\mathbf{s}$.*

Proof: see the original text, p. 34. The result is made more precise in (2.3.9).

Corollary (2.1.11). — *A non-stuttering gallery is minimal if and only if its type $\mathbf{s} = (s_1, \ldots, s_m)$ is a reduced decomposition of $s_1 \cdots s_m$.*

Proposition (2.1.12). — *The set $\mathscr{I}$, equipped with the family of facets, the incidence relation between facets, and the affine structures on each of the sets $\overline{F(P)}$ for $P \in \mathscr{P}$, is a polysimplicial complex, and G acts on $\mathscr{I}$ by automorphisms of polysimplicial complex.*

Proof: see the original text, pp. 34–35.

Example (2.1.13). — *Suppose $\mathbf{A}$ of dimension 1. Then $\operatorname{Card}\mathbf{S} = 2$ and $\mathbf{W}$ is an infinite dihedral group. Every finite sequence (s_j) of elements of $\mathbf{S}$ with $s_j \neq s_{j+1}$ for all j is a reduced decomposition, and every non-stuttering gallery $(C_0, \ldots, C_m)$ in which the panels $\overline{C}_{j-1} \cap \overline{C}_j$ and $\overline{C}_j \cap \overline{C}_{j+1}$ are distinct for $1 \leq j \leq m-1$ is minimal. The building $\mathscr{I}$ is a simplicial complex of dimension 1, i.e. a connected graph, and the above shows that this graph contains no circuit (which would furnish a minimal gallery of length > 0 with origin and extremity equal).* Hence $\mathscr{I}$ is a tree.

2.2. Structural maps and apartments. —

Proposition (2.2.1). — *There exists one and only one map $j\colon \mathbf{A} \to \mathscr{I}$ with the following two properties:*

(i) the restriction of j to $\overline{\mathbf{C}}$ is the canonical bijection (2.1.2) of $\overline{\mathbf{C}}$ onto $\overline{F(B)}$;

(ii) $j(\nu(n)\cdot x) = n \cdot j(x)$ for all $n \in N$ and $x \in \mathbf{A}$.

The map j is injective. If F is a facet of $\mathbf{A}$, then $j(F)$ is a facet of $\mathscr{I}$ of the same type, one has $j(\overline{F}) = \overline{j(F)}$, and the restriction of j to $\overline{F}$ is an affine bijection of $\overline{F}$ onto $\overline{j(F)}$.

Proof: see the original text, p. 35.

Definition (2.2.2). — *The map j of (2.2.1) is called the **canonical map** of $\mathbf{A}$ into $\mathscr{I}$. A map φ of $\mathbf{A}$ into $\mathscr{I}$ is called a **structural map** of $\mathscr{I}$ if there is $g \in G$ with $\varphi(x) = g \cdot j(x)$ for all $x \in \mathbf{A}$. An **apartment** (resp. **half-apartment**, **sector**, **wall**) of $\mathscr{I}$ is a subset of $\mathscr{I}$ which is the image of $\mathbf{A}$ (resp. of an affine root, of a sector, of a wall of $\mathbf{A}$) under a structural map.*

(2.2.3) It follows at once from (2.2.1) that a structural map φ is a chambered morphism (1.1.7) of polysimplicial complexes from $\mathbf{A}$ into $\mathscr{I}$, and is even an isomorphism of polysimplicial complexes of $\mathbf{A}$ onto the apartment $\varphi(\mathbf{A})$, which is a "polysimplicial subcomplex" of $\mathscr{I}$ in the sense that an apartment containing a facet F contains $\overline{F}$, and that every apartment is the union of the facets of the chambers it contains.

(2.2.4) Let M be a half-apartment, the image of an affine root α of $\mathbf{A}$ under a structural map φ. The wall $\varphi(\partial\alpha)$ is the union of the sets $\overline{F}$ for F running over the panels of $\mathscr{I}$ belonging to a single chamber contained in M. This wall therefore depends only on M and not on the choice of φ: it is called **the wall of** M and written ∂M.

Proposition (2.2.5). — *With the preceding notation:*

(i) $H = \bigcap_{n\in N} nBn^{-1} = \{g \in G \mid g \cdot j(x) = j(x)$ for all $x \in \mathbf{A}\}$;

(ii) $N = \{g \in G \mid g \cdot j(\mathbf{A}) = j(\mathbf{A})\}$.

Proof: see the original text, p. 36; (ii) uses that the Tits system (G, B, N) is saturated.

Corollary (2.2.6). — *The group G permutes transitively the pairs (F, A) consisting of a facet F of a given type X and an apartment A containing F.*

Corollary (2.2.7). — *Let φ be a structural map and ψ a map of $\mathbf{A}$ into the apartment $\varphi(\mathbf{A})$. In order that ψ be a structural map it is necessary and sufficient that there exist $w \in \mathbf{W}$ with $\psi = \varphi \circ w$.*

(2.2.8) By (2.2.7) there exists on every apartment A one and only one structure of euclidean affine space for which every structural map with image A is an isomorphism of euclidean affine spaces from $\mathbf{A}$. *We henceforth regard A as endowed with this euclidean structure, and in particular with the corresponding distance, written d_A.* For every $g \in G$ the map $x \mapsto g \cdot x$ is then an isomorphism of A onto the apartment $g \cdot A$.

2.3. Retractions. —

Proposition (2.3.1). — *Two facets of $\mathscr{I}$ are contained in a common apartment.*

Proof: see the original text, p. 37; reduce to two chambers via (2.2.3), then use (2.1.7).

Proposition (2.3.2). — *Let A and A' be two apartments containing a common chamber C. There exists one and only one map $\rho = \rho_{A',A;C}$ of A' onto A which is the restriction to A' of an action of G and satisfies $\rho(C) = C$. The restriction of ρ to $A \cap A'$ is the identity, and ρ is an isometry of A' onto A.*

Proof: see the original text, p. 37.

Corollary (2.3.3). — *Let F_1, F_2 be two facets and A_1, A_2 two apartments each containing both F_1 and F_2. There exists $g \in G$ with $g \cdot A_1 = A_2$ and $g \cdot F_i = F_i$ for $i = 1, 2$.*

Theorem (2.3.4). — *Let A be an apartment and C a chamber contained in A. There exists one and only one map ρ of $\mathscr{I}$ into A with the following properties:*

(i) $\rho(C) = C$;

(ii) for every apartment A' containing C there exists $g \in G$ with $\rho(x) = g \cdot x$ for all $x \in A'$.

The restriction of ρ to A is the identity. If $x \in \overline{C}$ then $\rho^{-1}(x) = \{x\}$. If F is a facet of type X, so is $\rho(F)$; one has $\rho(\overline{F}) = \overline{\rho(F)}$, and the restriction of ρ to $\overline{F}$ is an affine bijection of $\overline{F}$ onto $\overline{\rho(F)}$.

Proof: see the original text, pp. 37–38; the key point is that $\rho_{A',A;C}(x)$ depends only on x.

Definition (2.3.5). — *The map ρ of (2.3.4) is called the **retraction of $\mathscr{I}$ onto A with center C**, and is written $\rho_{A;C}$.*

Proposition (2.3.6). — *Let C be a chamber, F a facet and Γ a taut gallery between C and F. Every apartment containing C and F contains each term of Γ.*

Proof: see the original text, p. 38.

Corollary (2.3.7). —

(i) Let Γ and Γ' be two minimal galleries of the same type. There exists $g \in G$ with $\Gamma = g \cdot \Gamma'$.

(ii) Let C, C', C'' be three pairwise distinct chambers possessing a common panel. There exists $g \in G$ with $g \cdot C = C$ and $g \cdot C' = C''$.

Lemma (2.3.8). — *Let A be an apartment and C a chamber of A.*

(i) For every $w \in \mathbf{W}$ there exists one and only one chamber C' contained in A with $w(C, C') = w$ (notation of (2.1.7)).

(ii) Two non-stuttering galleries of the same type and the same origin, contained in A, coincide.

Corollary (2.3.9). — *Let C and C' be two chambers of $\mathscr{I}$ and put $w = w(C', C)$ (2.1.7). Let E be the set of minimal galleries with origin C' and extremity C. The map assigning to $\Gamma \in E$ its type $\mathbf{s}(\Gamma)$ (2.1.3) is a bijection of E onto the set of reduced decompositions of w.*

Proof: see the original text, p. 39.

(2.3.10) Let $w \in \mathbf{W}$. Applying (2.3.9) to $j(\mathbf{C})$ and $j(w \cdot \mathbf{C})$ shows that the set of minimal galleries with origin $\mathbf{C}$ and extremity $w(\mathbf{C})$ is in bijection with the reduced decompositions of

w. If $\Gamma = (C_0 = \mathbf{C}, \dots, C_m = w(\mathbf{C}))$ is such a gallery, of type $\mathbf{s} = (s_1, \dots, s_m)$, the reflection in the wall L_i supporting the panel common to C_{i-1} and C_i is

$$t_i = (s_1 \cdots s_{i-1})\, s_i\, (s_1 \cdots s_{i-1})^{-1},$$

and $w = s_1 \cdots s_m = t_m \cdots t_1$. It was recalled in (1.2.2) that the t_i are pairwise distinct and that their set — written T_w in what follows — is independent of the reduced decomposition chosen. The t_i are called the **reflections associated with** w.

In order that the reflection r in a wall L of $\mathbf{A}$ *belong to T_w, it is necessary and sufficient that* $\mathbf{C}$ *and $w(\mathbf{C})$ lie on opposite sides of L.* In particular, *the length $\ell(w)$ equals the number of walls of* $\mathbf{A}$ *separating* $\mathbf{C}$ *and $w \cdot$* $\mathbf{C}$, and a minimal gallery of $\mathbf{A}$ is entirely contained in every affine root containing its extremities. More generally:

Proposition (2.3.11). — *Every minimal gallery of $\mathscr{I}$ is contained in every half-apartment of $\mathscr{I}$ containing its extremities.*

Proposition (2.3.12). — *Let A be an apartment of $\mathscr{I}$, C a chamber of A, and $\rho = \rho_{A;C}$. Let F be a panel of $\mathscr{I}$; let C_1 be the chamber of A admitting $\rho(F)$ as a panel and lying on the same side as C of the wall of A containing $\rho(F)$, and let C_2 be the other chamber of A admitting $\rho(F)$ as a panel. Then there exists one and only one chamber C' of $\mathscr{I}$ admitting F as a panel with $\rho(C') = C_1$; and for every chamber C'' of $\mathscr{I}$ admitting F as a panel and distinct from C', one has $\rho(C'') = C_2$.*

Proof: see the original text, pp. 39–40.

2.4. Closed subsets. —

Definition (2.4.1). — *Let M be a subset of $\mathscr{I}$ (resp.* $\mathbf{A}$*) meeting at least one chamber. M is said to be* ***closed*** *if, for every gallery $(C_0, \dots, C_m)$ taut between a chamber C_0 meeting M and a point of M, one has $\overline{C}_k \subset M$ for $0 \le k \le m$.*

Every closed subset meeting a chamber is the union of the sets $\overline{C}$ for the chambers C meeting M. An apartment, a half-apartment and an affine root are closed ((2.3.6) and (2.3.11)). The intersection of a family of closed subsets containing a common chamber is again closed. Hence:

(2.4.2) If M is a subset of $\mathscr{I}$ (resp. $\mathbf{A}$) meeting at least one chamber, there is a smallest closed subset of $\mathscr{I}$ (resp. $\mathbf{A}$) containing M. It is called the **enclosure** of M and written $\mathrm{cl}(M)$.

(2.4.3) The notions of closed subset and of enclosure depend only on the polysimplicial structure of $\mathscr{I}$ (resp. $\mathbf{A}$). If M meets at least one chamber, then

$$j(\mathrm{cl}(M)) = \mathrm{cl}(j(M)) \quad \text{for } M \subset \mathbf{A},$$
$$g \cdot \mathrm{cl}(M) = \mathrm{cl}(g \cdot M) \quad \text{for } M \subset \mathscr{I},\ g \in G.$$

Lemma (2.4.4). — *Let C and C' be two chambers of* $\mathbf{A}$*. The enclosure of $C \cup C'$ coincides with the intersection of the affine roots of* $\mathbf{A}$ *containing $C \cup C'$, and with the union of the closures of the chambers belonging to at least one minimal gallery with extremities C and C'.*

Proof: see the original text, pp. 40–41; the argument compares $\mathrm{Card}\,\mathscr{S}(C, C')$ with $\mathrm{Card}\,\mathscr{S}(C, C'') + \mathrm{Card}\,\mathscr{S}(C', C'')$, where $\mathscr{S}(X, Y)$ denotes the set of walls of $\mathbf{A}$ separating X and Y.

Proposition (2.4.5). — *Let M be a subset of* $\mathbf{A}$ *meeting at least one chamber. The enclosure of M coincides with the intersection of the affine roots containing M, and is the convex hull of the union of finitely many points and half-lines.*

Proof: see the original text, p. 41.

(2.4.6) Proposition (2.4.5) allows definitions (2.4.1) and (2.4.2) to be extended to arbitrary subsets of $\mathbf{A}$:

Definition. — *For every subset M of $\mathbf{A}$, the **enclosure** of M, written $\mathrm{cl}(M)$, is the intersection of the affine roots of $\mathbf{A}$ containing M. M is said to be **closed** if $\mathrm{cl}(M) = M$, equivalently if M is an intersection of affine roots.*

A subset M of $\mathbf{A}$ is closed if and only if it is convex and a union of closures of facets.

Proposition (2.4.7). — *Let M be a subset of $\mathscr{I}$ contained in the union of finitely many facets. In order that M be contained in an apartment it is necessary and sufficient that there exist two chambers C and C' with M contained in the enclosure of $C \cup C'$.*

Proposition (2.4.8). — *Let D be a vector chamber.*

(i) For every bounded subset M of $\mathbf{A}$ there exist $x, x' \in \mathbf{A}$ with $M \subset (x + \mathrm{D}) \cap (x' - \mathrm{D})$.

(ii) For all $x, x', y, y' \in \mathbf{A}$ with $y \in x - \overline{\mathrm{D}}$ and $y' \in x' + \overline{\mathrm{D}}$, one has

$$(x + \overline{\mathrm{D}}) \cap (x' - \overline{\mathrm{D}}) \subset \mathrm{cl}(\{y, y'\}).$$

Proof: see the original text, pp. 41–42.

Corollary (2.4.9). — *Let D be a vector chamber and x a point of $\mathbf{A}$. For any points $y \in x + \mathrm{D}$ and $y' \in x - \mathrm{D}$, the enclosure of $\{y, y'\}$ is a neighborhood of x.*

Proposition (2.4.10). — *Let F be a facet of $\mathbf{A}$, L the affine subspace generated by F, and Δ a connected component of the complement in $\mathbf{A}$ of the union of the walls containing F. For every bounded subset M of $\overline{\Delta}$ there exist a point $y \in L$ and a sector $\mathfrak{C} \subset \Delta$ such that $M \subset \mathrm{cl}(\{y, z\})$ for every point $z \in \mathfrak{C}$.*

Proof: see the original text, p. 42.

Proposition (2.4.11). — *With the notation of (2.4.10): for every bounded subset M of $\overline{\Delta}$ and every chamber C of $\mathscr{I}$ admitting $j(F)$ as a facet, there exists an apartment of $\mathscr{I}$ containing C and $j(M)$.*

Proof: see the original text, p. 42.

Corollary (2.4.12). — *Let α be a half-apartment, M a bounded subset of α, and C a chamber of $\mathscr{I}$ possessing a panel contained in the wall $\partial\alpha$ of α. Then there exists an apartment containing C and M.*

Proposition (2.4.13). — *Let M be a subset of $\mathscr{I}$ meeting at least one chamber, and let $g \in G$ satisfy $g \cdot x = x$ for all $x \in M$. Then $g \cdot x = x$ for all $x \in \mathrm{cl}(M)$. In other words, **the fixers of** M **and of** $\mathrm{cl}(M)$ **coincide**.*

Proof: see the original text, p. 43; it suffices to show that the fixed-point set $\{x \in \mathscr{I} \mid g \cdot x = x\}$ is closed.

2.5. The metric of the building. —

We saw in (2.2.8) that each apartment A of $\mathscr{I}$ carries a structure of euclidean affine space, and in particular a distance d_A. We show that there exists on $\mathscr{I}$ one and only one distance whose restriction to each apartment A is d_A. This distance is G-invariant and makes $\mathscr{I}$ a complete, contractible metric space in which any two points are joined by a unique geodesic.

Lemma (2.5.1). — *There exists one and only one function $d\colon \mathscr{I}\times\mathscr{I}\to\mathbf{R}_+$ such that for every apartment A of $\mathscr{I}$ the restriction of d to $A\times A$ is the function d_A.*

Proof: see the original text, p. 43; uniqueness from (2.3.1), existence from (2.3.3) and (2.2.8).

(2.5.2) The function d is clearly invariant under G.

Proposition (2.5.3). — *Let C be a chamber of $\mathscr{I}$ and A an apartment containing C. Let $\rho=\rho_{A;C}$ be the retraction onto A with center C (2.3.5), and let $x,y\in\mathscr{I}$. Then:*

(i) $d(\rho(x),\rho(y))\le d(x,y)$;

(ii) if $x\in\overline{C}$, then $d(\rho(x),\rho(y))=d(x,y)$;

(iii) if there is a facet F of $\mathscr{I}$ with $x,y\in\overline{F}$, then $d(\rho(x),\rho(y))=d(x,y)$.

Proof: see the original text, p. 43.

Proposition (2.5.4). —

(i) The function d is a distance on $\mathscr{I}$.

(ii) Let $x,y\in\mathscr{I}$ and $D=\{z\in\mathscr{I}\mid d(x,y)=d(x,z)+d(z,y)\}$. Then D is contained in every apartment A containing x and y, and coincides with the segment $[xy]$ of the affine space A.

(iii) If an isometry of $\mathscr{I}$ — in particular an action of G — fixes the two points x and y, it fixes every point of D.

Proof: see the original text, p. 44.

Definition (2.5.5). — *The **building** of the Tits system (G,B,N,S) is the set $\mathscr{I}$ equipped with its structure of polysimplicial complex defined in* 2.1*, with the family of structural maps defined in* 2.2*, and with the distance d.*

Definition (2.5.6). — *With the notation of (2.5.4), D is called the **closed segment with extremities** x **and** y, written $[xy]$. A subset M of $\mathscr{I}$ is **convex** if $x,y\in M$ implies $[xy]\subset M$.*

The intersection of a family of convex subsets of $\mathscr{I}$ is convex, so one may speak of the **convex hull** of a subset of $\mathscr{I}$.

(2.5.7) For a subset of $\mathscr{I}$ contained in an apartment A to be convex in $\mathscr{I}$ in the sense of (2.5.6), it is necessary and sufficient that it be convex for the affine structure of A. In particular every apartment is convex in $\mathscr{I}$. The intersection $A\cap A'$ of two apartments is convex in each of them, and the affine structures induced on $A\cap A'$ by that of A and by that of A' are identical; since $A\cap A'$ is a union of closures of facets, $A\cap A'$ is closed in A in the sense of (2.4.6).

If M is a subset of $\mathscr{I}$ meeting at least one chamber, then M is closed if and only if it is convex and a union of closures of chambers.

More generally, a subset M of $\mathscr{I}$ is called **closed** if it is convex and a union of closures of facets (cf. (2.4.6)). For $M\subset\mathscr{I}$ one calls *enclosure* . . .

. . . of M, and we shall denote by $\mathrm{cl}(M)$ the intersection of the closed subsets of $\mathscr{I}$ containing M; it is the smallest closed subset containing M. p. 45

Proposition **(2.5.8)**. — *Let A and A' be two apartments. There exists $g \in G$ such that $g.A = A'$ and $g.x = x$ for every $x \in A \cap A'$.*

Let F be a facet open in $A \cap A'$ and let $g \in G$ be such that $g.A = A'$ and $g.F = F$ (2.2.6). Let $y \in F$ and let $x \in A \cap A'$; let D be the segment $[xy]$, which is contained in $A \cap A'$. The image of D under g is a segment of A' with endpoint y and of length $d(x, y)$. Since F is open in $A \cap A'$, $D \cap F$ is a subsegment not reduced to $\{y\}$ and fixed pointwise by g. One deduces that $g.D = D$ and that g fixes all the points of D, and in particular x.

Corollary **(2.5.9)**. — *Let $g \in G$; set $A = j(\mathbf{A})$. There exists $n \in N$ such that $g.x = n.x$ for every $x \in A \cap g^{-1}.A$. If moreover $g.x = x$ for x belonging to a nonempty open subset of A, then $g.x = x$ for every $x \in A \cap g^{-1}.A$.*

Indeed, there exists $h \in G$ such that $hg.A = A$ and $h.y = y$ for $y \in A \cap g.A$. One then has $hg = n \in N$ and $g.x = n.x$ for $x \in A \cap g^{-1}.A$. The second assertion follows, since an element of N fixing the points of a nonempty open subset of A belongs to H.

Proposition **(2.5.10)**. — *If F is a facet of $\mathscr{I}$, the closure of F in the metric space $\mathscr{I}$ is none other than the union $\overline{F}$ of the facets of F.*

This follows at once from the compactness of $\overline{F}$ in any apartment containing F.

Lemma **(2.5.11)**. — *Let F be a facet of $\mathscr{I}$ and $x \in F$. There exists an open ball D with center x such that, if F' is a facet of $\mathscr{I}$ meeting D, then $F \subset \overline{F'}$. If D is such a ball and if $g \in G$ is such that $g.x \in D$, then $g.x = x$.*

Let x_0 and F_0 be the images of x and F under the canonical map of $\overline{F}$ into $\overline{\mathbf{C}}$ (2.1.2). There exists $\varepsilon > 0$ such that every wall of $\mathbf{W}$ meeting the open ball D_0 with center x_0 and radius ε contains x_0. Let F_1 be a facet of $\mathbf{A}$ meeting D_0 and let $y \in F_1 \cap D_0$. Every wall meeting the open segment $]yx_0[$ contains x_0, hence also $[yx_0]$. This entails that $]yx_0[\ \subset F_1$ and $x_0 \in \overline{F_1}$. We therefore have $F_0 \subset \overline{F_1}$.

Now let D be the open ball of $\mathscr{I}$ with center x and radius ε, and let F' be a facet of $\mathscr{I}$ meeting D. Let α be a structural map such that $\alpha(\overline{\mathbf{C}}) \supset F$ and $\alpha(\mathbf{A}) \supset F'$. Since α is an isometry, and since $\alpha^{-1}|F$ is the canonical map of F into $\overline{\mathbf{C}}$, one has $\alpha^{-1}(x) = x_0$, $\alpha^{-1}(F) = F_0$ and $\alpha^{-1}(D) = D_0$. Set $F'_0 = \alpha^{-1}(F')$. Since $F'_0 \cap D_0 \neq \emptyset$, one has $F_0 \subset \overline{F'_0}$ by the first part of the proof, whence $F \subset \overline{F'}$.

Finally, let $g \in G$ be such that $g.x \in D$ and let F' be the facet containing $g.x$. One then has $x,\ g.x \in \overline{F'}$, whence $g.x = x$ by (2.1.6).

Theorem **(2.5.12)**. — *(i) The metric space $\mathscr{I}$ is complete.*
(ii) Let $\mathscr{F}$ be a set of facets. Then $\bigcup_{F \in \mathscr{F}} \overline{F}$ is closed in $\mathscr{I}$.

It suffices to show that, with the notation of (ii), $M = \bigcup_{F \in \mathscr{F}} \overline{F}$ is *complete*, that is to say that a Cauchy sequence (y_q) of M possesses an accumulation point in M. Let C be a chamber of $\mathscr{I}$ and let $g_q \in G$ be such that $x_q = g_q^{-1}.y_q \in \overline{C}$ (for $q \in \mathbf{N}$). Up to replacing the sequence (y_q) by a subsequence, one may suppose that the sequence (x_q) converges to $x \in \overline{C}$. Let D be a ball with center x possessing the property of Lemma (2.5.11). Since p. 46

$$\begin{aligned} d(g_p.x, g_q.x) &\le d(g_p.x, g_p.x_p) + d(g_p.x_p, g_q.x_q) + d(g_q.x_q, g_q.x) \\ &\le d(x, x_p) + d(y_p, y_q) + d(x_q, x) \end{aligned}$$

there exists an integer m such that $g_q^{-1}g_p.x \in D$ for $p, q \ge m$, and one has $g_p.x = g_q.x$ for $p, q \ge m$. For $p \ge m$ one has $d(y_p, g_m.x) = d(x_p, x)$, and y_p tends to $y = g_m.x$. Moreover, Lemma (2.5.11) shows that y is adherent to the facet containing y_p for p large enough, and hence belongs to M.

(2.5.13) *Whatever the points $x, y \in \mathscr{I}$, the segment $[xy]$ is the unique geodesic joining x to y.*

This follows at once from (2.5.4)(ii).

For every real number t of the interval $[0,1]$ we shall denote by $tx + (1-t)y$ the unique point $z \in [xy]$ satisfying $d(x,z) = (1-t)d(x,y)$.

Lemma **(2.5.14)**. — *Let x, y, z, z' be four points of $\mathscr{I}$ such that $z \in [xy]$, and let $\varepsilon \in \mathbf{R}_+$ be such that $d(x,z') \leq d(x,z) + \varepsilon\, d(x,y)$ and $d(y,z') \leq d(y,z) + \varepsilon\, d(x,y)$. One then has $d(z,z')^2 \leq 4\varepsilon\, d(x,z)\, d(y,z) + \varepsilon^2 d(x,y)^2$.*

Let A be an apartment containing $[xy]$ and C a chamber of A such that $z \in \overline{C}$. Set $z'' = \rho_{A;C}(z)$. One has $d(z,z') = d(z,z'')$ and the four points x, y, z, z'' satisfy the same hypotheses as x, y, z, z'. One sees that it suffices to prove the lemma when the four points under consideration belong to one and the same apartment, and it then results from an elementary computation of plane Euclidean geometry.

Lemma **(2.5.15)**. — *The map $(t,x,y) \mapsto tx + (1-t)y$ from $[0,1] \times \mathscr{I} \times \mathscr{I}$ into $\mathscr{I}$ is continuous.*

This follows easily from Lemma (2.5.14).

Proposition **(2.5.16)**. — *The metric space $\mathscr{I}$ is contractible.*

This follows from Lemma (2.5.15).

(2.5.17) Let $x, y \in \mathscr{I}$ and let $\mathscr{L}(x,y)$ be the set of finite sequences $s = (x_0, \ldots, x_m)$ of points of $\mathscr{I}$ such that $x_0 = x$, $x_m = y$, and such that x_{j-1} and x_j belong to the closure of one and the same chamber for $1 \leq j \leq m$. One has

$$d(x,y) = \inf_{s \in \mathscr{L}(x,y)} \sum_{j=1}^{m} d(x_{j-1}, x_j).$$

It follows at once that *the metric of $\mathscr{I}$ is completely determined by the data of the polysimplicial complex structure of $\mathscr{I}$ and of the metric space structures on the subsets of $\mathscr{I}$ of the form $\overline{C}$, where C is a chamber of $\mathscr{I}$.*

2.6. Irreducible components. —

Lemma **(2.6.1)**. — *Let $(\mathbf{S}', \mathbf{S}'')$ be a partition of $\mathbf{S}$ such that every element of $\mathbf{S}'$ commutes with every element of $\mathbf{S}''$. Then $\mathbf{W}_{\mathbf{S}'}$ is a normal subgroup of $\mathbf{W}$ and $(G, B_{\mathbf{S}'}, N)$ is a Tits system whose Weyl group is canonically isomorphic to $\mathbf{W}/\mathbf{W}_{\mathbf{S}'}$ and to $\mathbf{W}_{\mathbf{S}''}$.* p. 47

Since $\mathbf{W}_{\mathbf{S}'} \cap \mathbf{W}_{\mathbf{S}''} = \mathbf{W}_{\mathbf{S}' \cap \mathbf{S}''} = \{e\}$, the group $\mathbf{W}$ is the direct product of $\mathbf{W}_{\mathbf{S}'}$ and $\mathbf{W}_{\mathbf{S}''}$. Set $H' = B_{\mathbf{S}'} \cap N$: the image of H' in $\mathbf{W}$ is $\mathbf{W}_{\mathbf{S}'}$, and N/H' is identified with $\mathbf{W}/\mathbf{W}_{\mathbf{S}'}$, or again with $\mathbf{W}_{\mathbf{S}''}$. Let $s \in \mathbf{S}''$ and $w \in \mathbf{W}_{\mathbf{S}''}$. One has:

$$\begin{aligned} sB_{\mathbf{S}'}w = sB\mathbf{W}_{\mathbf{S}'}Bw \subset sB\mathbf{W}_{\mathbf{S}'}wB = sBw\mathbf{W}_{\mathbf{S}'}B \subset (Bsw\mathbf{W}_{\mathbf{S}'}B) \cup (Bw\mathbf{W}_{\mathbf{S}'}B) \\ \subset (B_{\mathbf{S}'}wB_{\mathbf{S}'}) \cup (B_{\mathbf{S}'}swB_{\mathbf{S}'}). \end{aligned}$$

Finally $sB_{\mathbf{S}'}s \neq B_{\mathbf{S}'}$, since $B_{\mathbf{S}'}$ is its own normalizer. We have thus verified the axioms of a Tits system.

Let us remark that this lemma is valid for an arbitrary Tits system, not necessarily of affine type.

(2.6.2) Let $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_m$ be the canonical decomposition of $\mathbf{A}$ into a product of its irreducible components under $\mathbf{W}$, and let $\mathbf{W} = \mathbf{W}_1 \times \cdots \times \mathbf{W}_m$, $\mathbf{C} = \mathbf{C}_1 \times \cdots \times \mathbf{C}_m$ and $\mathbf{S} = \mathbf{S}_1 \cup \cdots \cup \mathbf{S}_m$ be the corresponding decompositions of $\mathbf{W}$, $\mathbf{C}$ and $\mathbf{S}$ ((1.3.2) and (1.3.4)).

For every subset J of $\{1, \ldots, m\}$ let p_J be the canonical projection of $\mathbf{A}$ onto $\mathbf{A}_J = \prod_{i \in J} \mathbf{A}_i$, let $\mathbf{C}_J = \prod_{i \in J} \mathbf{C}_i$, and let $\mathbf{W}_J = \prod_{i \in J} \mathbf{W}_i$, regarded as a group of affine transformations of $\mathbf{A}_J$. Set $\mathbf{S}_J = \bigcup_{i \in J} \mathbf{S}_i$, $B_J = B_{\mathbf{S}_J} = B\mathbf{W}_J B$ and $B^J = B_{\{1,\ldots,m\}-J}$. It follows from Lemma (2.6.1) that

(G, B^J, N) is a Tits system with Weyl group $\mathbf{W}_J$, hence of affine type. Let $\mathscr{I}_J$ be the building of (G, B^J, N) (constructed from $\mathbf{A}_J$ and $\mathbf{C}_J$), d_J its distance, $j_{0,J}$ the canonical map of $\overline{\mathbf{C}_J}$ onto $\overline{F(B^J)}$, etc. The conjugacy class of (B^J, N), hence also the building $\mathscr{I}_J$, depends only on the conjugacy class of (B, N).

Proposition **(2.6.3)**. — *(i) Let $\mathscr{P}$ (resp. $\mathscr{P}_J$) be the set of parahoric subgroups of (G, B, N) (resp. of (G, B^J, N)). For every $P \in \mathscr{P}$ there exists a smallest element, denoted $\pi_J(P)$, in the set of the elements of $\mathscr{P}_J$ which contain P.*
(ii) There exists one and only one map ϖ_J from $\mathscr{I}$ into $\mathscr{I}_J$ such that

$$\varpi_J(F(P)) = F(\pi_J(P)) \quad \textit{for every } P \in \mathscr{P}; \qquad p_J \circ \lambda = \lambda_J \circ \varpi_J$$

where λ (resp. λ_J) denotes the canonical map of $\mathscr{I}$ (resp. $\mathscr{I}_J$) into $\overline{\mathbf{C}}$ (resp. $\overline{\mathbf{C}_J}$).
(iii) For every $g \in G$ and $a \in \mathscr{I}$ one has $g.\varpi_J(a) = \varpi_J(g.a)$, and ϖ_J is surjective.
(iv) For every structural map $\varphi : \mathbf{A} \longrightarrow \mathscr{I}$ there exists one and only one structural map $\varphi_J : \mathbf{A}_J \longrightarrow \mathscr{I}_J$ such that $\varpi_J \circ \varphi = \varphi_J \circ p_J$.

To prove (i) one may suppose $P = B_X$ with $X \in \mathbf{T}$. The subgroups of G containing P are then the B_Y with $X \subset Y \subset \mathbf{S}$, and if such a subgroup belongs to $\mathscr{P}_J$, it must contain a conjugate p. 48
of B^J and Y must contain $\mathbf{S} - \mathbf{S}_J$. One thus sees that $\pi_J(P) = B_{X \cup (\mathbf{S} - \mathbf{S}_J)}$ possesses the desired property.

Let us prove (ii). If ϖ_J exists, it sends the point $(P, a) \in \mathscr{I}$ (with $a \in \lambda(F(P))$) to the point $(\pi_J(P), p_J(a)) \in \mathscr{I}_J$. It therefore suffices to show that

$$(3) \qquad p_J(\lambda(F(P))) \subset \lambda_J(F(\pi_J(P))).$$

By an operation of G one reduces to the case $P = B_X$. One then has $\lambda(F(P)) = \mathbf{C}_X$ and $\lambda_J(F(\pi_J(P))) = \lambda_J((B^J)_{X \cap \mathbf{S}_J}) = (\mathbf{C}_J)_{X \cap \mathbf{S}_J}$ (the facet of type $X \cap \mathbf{S}_J$ of $\mathbf{C}_J$), whence (1).

The first assertion of (iii) follows from the uniqueness of ϖ_J. It entails $\varpi_J(\mathscr{I}) = G.\varpi_J(\overline{F(B)}) = G.\overline{F(B^J)} = \mathscr{I}_J$.

Let us prove (iv). Taking (iii) into account, (iv) will follow from the relation

$$(4) \qquad \varpi_J(j(x)) = j_J(p_J(x)) \qquad \text{for } x \in \mathbf{A}$$

(where $j : \mathbf{A} \longrightarrow \mathscr{I}$ and $j_J : \mathbf{A}_J \longrightarrow \mathscr{I}_J$ are defined as in (2.2.1)). It suffices, again by virtue of (iii) and of (2.2.1), to establish (2) for $x \in \mathbf{C}_X$. Now in this case one has:

$$\varpi_J(j(x)) = \varpi_J(B_X, x) = \big((B^J)_{X \cap \mathbf{S}_J},\ p_J(x)\big) = j_J(p_J(x)).$$

Proposition **(2.6.4)**. — *Let $\{J \mid J \in \mathscr{J}\}$ be a partition of the set $\{1, \ldots, m\}$. Let $\mathscr{I}'$ be the product of the buildings $\mathscr{I}_J$ for $J \in \mathscr{J}$, endowed with the product polysimplicial complex structure and with the distance $\big(\sum_J d_J^2\big)^{1/2}$; let $\mathscr{P}'$ be the product of the sets $\mathscr{P}_J$ endowed with the product order structure, let $\varpi = (\varpi_J) : \mathscr{I} \longrightarrow \mathscr{I}'$ be the product map of the ϖ_J and $\pi = (\pi_J) : \mathscr{P} \longrightarrow \mathscr{P}'$ the product map of the π_J.*
(i) ϖ is an isometry and an isomorphism of polysimplicial complexes from $\mathscr{I}$ onto $\mathscr{I}'$.
(ii) For every apartment A of $\mathscr{I}$, the restriction of ϖ to A is an isomorphism of Euclidean affine spaces and of polysimplicial complexes from A onto the product of the apartments $\varpi_J(A)$.
(iii) π is an isomorphism of ordered sets from $\mathscr{P}$ onto $\mathscr{P}'$. For every $P \in \mathscr{P}$ one has $P = \bigcap_{J \in \mathscr{J}} \pi_J(P)$.

Assertion (ii) follows from (2.6.2) and from (2.6.3)(iii). It entails that ϖ is isometric and is a morphism of polysimplicial complexes. In order to establish (i) it therefore only remains to show that ϖ is surjective. By induction on Card $\mathscr{J}$ one reduces to the case where $\mathscr{J} = (J_1, J_2)$. Let then $x_k \in \mathscr{I}_{J_k}$ and $y_k \in \mathscr{I}$ with image x_k (for $k = 1, 2$) ((2.6.3)(iii)). Let A be an apartment of $\mathscr{I}$ containing y_1 and y_2. One then has $x_k \in \varpi_{J_k}(A)$ for $k = 1, 2$, whence $(x_1, x_2) \in \varpi(A)$ by (ii).

Finally, if $P \in \mathscr{P}$, one has $\varpi(F(P)) = \prod_J F(\pi_J(P))$. Since ϖ commutes with the action of G, one deduces that $P = \bigcap_{J\in\mathscr{J}} \pi_J(P)$ (2.1.5). It follows that π is injective. Let us show that π is surjective: indeed, if $P_J \in \mathscr{P}_J$ for $J \in \mathscr{J}$, then $\prod_J F(P_J)$ is a facet F' of $\mathscr{I}'$ and there exists $P \in \mathscr{P}$ such that $F' = \varpi(F(P))$, or again $P_J = \pi_J(P)$ for every J. Finally, it is clear that $P \subset Q$ is equivalent to $\pi_J(P) \subset \pi_J(Q)$ for every $J \in \mathscr{J}$ $(P, Q \in \mathscr{P})$.

(2.6.5) When J is a subset with a single element i, we agree to write i instead of $\{i\}$ in the preceding notation. The Tits systems (G, B^i, N) are irreducible: they are called the *irreducible components* of (G, B, N). Proposition (2.6.4) thus furnishes a *canonical decomposition of the building $\mathscr{I}$ into a product of simplicial complexes*. p. 49

(2.6.6) Let $J \subset \{1, \ldots, m\}$. It is clear that $(B_J, B, N \cap B_J)$ is also a Tits system with Weyl group $\mathbf{W}_J$. Its building is canonically isomorphic to $\mathscr{I}_J$, the correspondence $P \mapsto P'$ between parahoric subgroups of (G, B^J, N) and of $(B_J, B, N \cap B_J)$ being describable as follows: $P' = P \cap B_J$, and P is the unique parahoric subgroup of (G, B, N) of type $\tau(P) \cup (\mathbf{S} - \mathbf{S}_J)$ containing P'.

2.7. B-adapted homomorphisms. —

Let $\varphi : G \to \widehat{G}$ be a B-adapted homomorphism ((1.2.13) sqq., whose notation we take up again). We saw in 1.2 that $\widehat{G}$ operates on the set $\mathscr{P}$ of parahoric subgroups of G, and we defined a homomorphism ξ of $\widehat{G}$ into the group of automorphisms of the Coxeter system $(\mathbf{W}, \mathbf{S})$, a group which is identified with $\widetilde{\mathbf{W}}^{\dagger}_{\mathbf{C}}$, with the notation of (1.3.18), that is to say with the group of automorphisms of the Euclidean space $\mathbf{A}$ which preserve the chamber $\mathbf{C}$.

(2.7.1) Let then $y = (P, x) \in \mathscr{I}$ (with $P \in \mathscr{P}$ and $x \in \mathbf{C}_{\tau(P)}$). Since ${}^gP$ is a parahoric subgroup of G of type $\xi(g)(\tau(P))$ (1.2.18), the pair $g.y = ({}^gP, \xi(g).x)$ is again a point of $\mathscr{I}$, and one thus defines a *law of operation* of $\widehat{G}$ on the set $\mathscr{I}$. For this law the stabilizer of a facet $F(P)$ is none other than the subgroup $\operatorname{Stab} P$ (1.2.15), and $\widehat{B} = \operatorname{Stab} B \cap \xi^{-1}(1)$ is the fixer of the chamber $F(B)$.

Proposition **(2.7.2)**. — *(i) If $g \in G$ and $y \in \mathscr{I}$, one has $\varphi(g).y = g.y$. In other words, the laws of operation of G and of $\widehat{G}$ on $\mathscr{I}$ are compatible with the homomorphism φ. One has*

$$\operatorname{Ker}\varphi \subset \bigcap_{g\in G} gHg^{-1} = \bigcap_{g\in G} gBg^{-1}.$$

(ii) For every $g \in \widehat{G}$ the map $y \mapsto g.y$ is an isometric automorphism of polysimplicial complex of $\mathscr{I}$.
(iii) If, moreover, the homomorphism φ is B-N-adapted (1.2.13), this map permutes among themselves the apartments (resp. the half-apartments, the sectors, the walls) of $\mathscr{I}$.

The first assertion of (i) is immediate, taking into account that for $g \in G$ and $P \in \mathscr{P}$ one has ${}^{\varphi(g)}P = gPg^{-1}$ and $\xi(\varphi(g)) = 1$. The second follows, since if $g \in \operatorname{Ker}\varphi$ then $\varphi(g)$, hence also g, operates trivially on $\mathscr{I}$.

Let us prove (ii). Let $g \in \widehat{G}$ and let γ be the map $y \mapsto g.y$ of $\mathscr{I}$ into itself. Since the action of $\widehat{G}$ on $\mathscr{P}$ respects the inclusion relations in $\mathscr{P}$, one sees that, for $P \in \mathscr{P}$, the image under γ of the facet $F(P)$ is the facet $F({}^gP)$ and that $\gamma(\overline{F(P)}) = \overline{F({}^gP)}$. Moreover, the restriction of γ to $\overline{F(P)}$ is obtained by composing the canonical map of $\overline{F(P)}$ onto $\overline{\mathbf{C}}_{\tau(P)}$, the map $\xi(g)$, and the canonical map of $\overline{\mathbf{C}}_{\xi(g)(\tau(P))}$ onto $\overline{F({}^gP)}$. Since $\xi(g)$ is an automorphism of the Euclidean space $\mathbf{A}$, one deduces that γ is an automorphism of polysimplicial complex and that the restriction of γ to $\overline{F(P)}$ is isometric. Consequently γ itself is isometric by (2.5.17). p. 50

Let us prove (iii). Up to multiplying g on the left by an element of $\varphi(G)$, one may suppose that g normalizes $\varphi(B)$ and $\varphi(N)$. One then has, for every $w \in \mathbf{W}$ (1.2.20)

$${}^g(wBw^{-1}) = w'Bw'^{-1} \quad \text{with } w' = \xi(g)(w) = \xi(g)w\xi(g)^{-1}$$

and more generally

$$^{g}(wB_Xw^{-1}) = w'B_{\xi(g)(X)}w'^{-1} \qquad \text{for } X \in \mathbf{T}.$$

Let then $a \in \mathbf{A}$ and let $X \in \mathbf{T}$ and $w \in \mathbf{W}$ be such that $a \in w.\mathbf{C}_X$. One has $j(a) = (wB_Xw^{-1}, w^{-1}(a))$ and $g.j(a) = (w'B_{\xi(g)(X)}w'^{-1}, \xi(g).w^{-1}.a)$, whence

$$g.j(a) = (w'B_{\xi(g)(X)}w'^{-1},\ w'^{-1}.\xi(g).a) = j(\xi(g).a).$$

Consequently γ preserves the apartment $j(\mathbf{A})$ and reduces on $j(\mathbf{A})$ to the map $j \circ \xi(g) \circ j^{-1}$, and hence permutes among themselves the walls (resp. the sectors, the half-apartments) of $j(\mathbf{A})$. One deduces (iii) by using (i) and the transitivity of G on the apartments.

(2.7.3) On the other hand, $\widehat{G}$ *does not permute among themselves the structural maps*, unless $\xi(\widehat{G}) = \{1\}$.

(2.7.4) What precedes applies in particular if one takes for $\widehat{G}$ the group $\mathrm{Aut}_B\, G$ (resp. $\mathrm{Aut}_{(B,N)}\, G$) of the automorphisms of G which preserve the conjugacy class of B (resp. of the pair (B, N)), and for φ the homomorphism which associates to $g \in G$ the inner automorphism defined by g. One then has $\gamma.g.x = \gamma(g).\gamma.x$ for $\gamma \in \widehat{G}$, $g \in G$ and $x \in \mathscr{I}$. Proposition (2.7.2) then shows that *the building* $\mathscr{I}$ *with all its structures depends* (up to an automorphism of $\mathbf{A}$ preserving $\mathbf{C}$, as far as the structural maps are concerned) *only on the conjugacy class of the pair* (B, N). If one regards $\mathscr{I}$ as endowed only with its metric and its polysimplicial structure, it depends only on the conjugacy class of B.

2.8. Characterization of apartments. —

We saw (2.4.7) that, in order for a subset M of $\mathscr{I}$ to be contained in an apartment and bounded therein, it is necessary and sufficient that there exist two chambers C and C' such that M is contained in the enclosure of $C \cup C'$. In particular, the family of the subsets of $\mathscr{I}$ which are *bounded subsets of an apartment* is completely determined by the data of the polysimplicial complex structure of $\mathscr{I}$.

The same is not true in general of the family of the apartments: there exist saturated equivalent Tits systems of affine type $(G, B, N, \mathbf{S})$ and $(G, B, N', \mathbf{S}')$ for which N and N' are not conjugate. The associated polysimplicial complex $\mathscr{I}$ is the same for the two systems, but the families of apartments are distinct: otherwise there would exist an element $g \in G$ such that $j'(\mathbf{A}) = g.j(\mathbf{A})$, denoting by j and j' the canonical maps of $\mathbf{A}$ into $\mathscr{I}$ corresponding to these two Tits systems p. 51
with Weyl group $\mathbf{W}$, and one would have $N' = gNg^{-1}$ by (2.2.5).

In this number we shall first give another characterization of the bounded subsets of apartments, then, in the case where the group G is *complete* for a suitable topology, of the apartments themselves. Finally, we shall show that the apartment $j(\mathbf{A})$ is entirely determined by the polysimplicial structure of $\mathscr{I}$ and by the group N regarded as a group of permutations of $\mathscr{I}$.

Henceforth we shall identify $\mathbf{A}$ *and* $j(\mathbf{A})$ *by means of* j*, and shall therefore regard* $\mathbf{A}$ *as an apartment of* $\mathscr{I}$.

Proposition **(2.8.1)**. — *Let M be a bounded subset of $\mathscr{I}$ possessing one of the two following properties:*
a) the interior of M is not empty;
b) M is convex.
In order that there exist an apartment containing M, it is necessary and sufficient that there exist an isometry of M onto a subset of $\mathbf{A}$.

That the condition is necessary is evident. Let us suppose it satisfied, and suppose first that the interior of M is not empty. There then exists a chamber C_0 such that $C_0 \cap M$ is of nonempty interior. On the other hand, every isometry $\varphi : X \to X'$ of one subset of $\mathbf{A}$ onto another extends to an automorphism of $\mathbf{A}$, which is unique as soon as X contains a nonempty open subset of $\mathbf{A}$. One deduces easily that, if A is an apartment containing C_0, there exists one and only one isometry τ of M onto a subset of A whose restriction to $C_0 \cap M$ is the identity. Moreover, τ is the restriction to M of the retraction ρ of center C_0 of $\mathscr{I}$ onto A, since one has $d(\rho(x), y) = d(x, y) = d(\tau(x), y)$ for every $x \in M$ and for every y belonging to $C_0 \cap M$, which is of nonempty interior.

Let E be the enclosure of $\rho(M)$, and let $\mathscr{C}$ be the set of the chambers $C \subset E$ such that the restriction of ρ to $M \cup \overline{C}$ is again isometric. The restriction of ρ to the set M', the union of M and of the closures $\overline{C}$ of the chambers belonging to $\mathscr{C}$, is also isometric. We shall then argue by induction on the number k of chambers contained in E and not belonging to $\mathscr{C}$; what precedes shows that if $k = 0$ the restriction of ρ to $M \cup E$ is isometric, which entails $M \subset A$.

So suppose $k > 0$. Considering a minimal gallery between C_0 (which obviously belongs to $\mathscr{C}$) and a chamber contained in E and not belonging to $\mathscr{C}$, one sees that there exist two adjacent chambers C and C' contained in E such that $C \in \mathscr{C}$ and $C' \notin \mathscr{C}$. The first part of the proof shows that one may suppose $C = C_0$.

Let then F be the panel common to C and C' and let $a \in F$. Let us distinguish two cases:

1) *For every $x \in M'$, $x \neq a$, the intersection $[ax] \cap (\overline{C} \cup \overline{C'})$ is a subsegment of $[ax]$ not reduced to $\{a\}$.* Let then $x \in M'$, $x \neq a$, $z \in \overline{C'}$, and let A' be an apartment containing C' and x. The triangle T, the union of the segments $[az]$, $[zx]$ and $[ax]$, is contained in A', hence is "Euclidean". The triangle $T' = \rho(T)$ is then isometric to it, since on the one hand ρ is isometric on $[ax]$ and on the other hand the angles at the vertex a of T and T' are equal, ρ being the identity on $\overline{C} \cup \overline{C'}$. One therefore has $d(z, x) = d(z, \rho(x))$, and the restriction of ρ to $M \cup \overline{C'}$ is isometric, contrary to the hypothesis according to which $C' \notin \mathscr{C}$. p. 52

2) *There exists $x \in M'$, $x \neq a$, such that* $[ax] \cap (\overline{C} \cup \overline{C'}) = \{a\}$. There then exists a chamber C'' admitting F as panel, distinct from C and from C', such that $[ax] \cap C'' \neq \emptyset$. Let then A' be an apartment containing C and x, hence C'', and let ρ' (resp. ρ'') be the retraction of $\mathscr{I}$ onto A' of center C (resp. C''). Let $y \in M'$; since $\rho = \rho \circ \rho'$, one has

$$\begin{aligned} d(x, y) &= d(\rho(x), \rho(y)) = d(\rho(\rho'(x)), \rho(\rho'(y))) \\ &\leq d(\rho'(x), \rho'(y)) \leq d(x, y). \end{aligned}$$

Since $x = \rho'(x) = \rho''(x)$, one deduces $d(x, \rho'(y)) = d(x, y) \geq d(x, \rho''(y))$. Let R' (resp. R'') be the half-apartment of A' whose wall contains F and which contains C (resp. C''); one has $x \in R''$. Let us show that $[ay] \cap (\overline{C} \cup \overline{C''}) \neq \{a\}$ as soon as $y \neq a$. Otherwise there would exist a chamber C''' admitting F as panel, distinct from C and from C'', with $[ay] \cap C''' \neq \emptyset$. Since $\rho'(C''') = C''$ (resp. $\rho''(C''') = C$) and since ρ' (resp. ρ'') restricted to $[ay]$ is an isometry, one would have $\rho'(y) \in \mathring{R}''$ and $\rho''(y) \in \mathring{R}'$, with moreover $\rho''(y) = r.\rho'(y)$, where r is the reflection in A' with respect to the hyperplane which is the common wall of R' and R''. But this would entail $d(x, \rho''(y)) > d(x, \rho'(y))$, contrary to what we saw above.

In particular, if $y \in M' \cap A$ with $y \neq a$, one has $[ay] \cap C \neq \{a\}$ since $C'' \cap A = \emptyset$ and y belongs to the half-apartment R of A containing C and whose wall contains F. By (2.4.12) there therefore exists an element $g \in G$ such that $g.y = y$ for every $y \in M' \cap A$ and $g.C'' = C'$. One then has $g.C = C$, and this entails $\rho(g.z) = \rho(z)$ for every $z \in \mathscr{I}$. The restriction of ρ to $M'' = g.M'$ is therefore again an isometry, and the enclosure of $\rho(M'') = \rho(M')$ is still equal to E. Moreover, for every $y \in M''$ with $y \neq a$, one has $[ay] \cap (\overline{C} \cup \overline{C'}) \neq \{a\}$ and the study made above in 1) shows that the restriction of ρ to $M'' \cup \overline{C'}$ is isometric. We thus find ourselves in the same situation, with M'' instead of M, but the family $\mathscr{C}$ is increased by at least one chamber, namely C', and the number k decreased by at least one unit.

The induction hypothesis therefore entails that there exists an apartment A'' containing M'', and the apartment $g^{-1}.A''$ contains M, which completes the proof in case a).

Let us now suppose that M is convex. Let F be a facet of $\mathscr{I}$ meeting M, of the greatest possible dimension, and let $x \in F \cap M$. For every $y \in M$, $y \neq x$, the intersection $F \cap [xy]$ is not reduced to $\{x\}$. Let A be an apartment containing F and let C be a chamber of A whose closure contains F. It suffices for us, in order to reduce to the case studied previously, to show that the restriction to $C \cup M$ of the retraction $\rho = \rho_{A;C}$ is isometric, or again that $d(\rho(y), \rho(z)) = d(y, z)$ for $y, z \in M$. Now, the two triangles which are the convex hulls of $\{x, y, z\}$ and of $\{x, \rho(y), \rho(z)\}$ are both isometric to Euclidean triangles; their angles at the vertex x are equal since $[xy] \cap [x\rho(y)] \neq \{x\} \neq [xz] \cap [x\rho(z)]$, and the sides $[xy]$ and $[x\rho(y)]$ (resp. $[xz]$ and $[x\rho(z)]$) are of equal lengths. Consequently one has $d(y, z) = d(\rho(y), \rho(z))$ p. 53
([22], book I, prop. IV), which terminates the proof.

(2.8.2) Let us suppose that G is endowed with a structure of *complete* topological group for which B is a closed subgroup. Let us suppose moreover that there exists a decreasing sequence of subgroups $(V_n)_{n\geq 1}$ forming a fundamental system of neighborhoods of the neutral element, and an increasing sequence of bounded subsets Ω_n of $\mathbf{A}$ such that, denoting by P_n the fixer of $j(\Omega_n)$ in G, one has

(5) $$P_n = (P_n \cap V_n).H \qquad \text{for every } n \geq 1.$$

It is immediate that (1) remains valid if one replaces Ω_n by a larger subset of $\mathbf{A}$. One may therefore suppose moreover that $\mathbf{C} \subset \Omega_1$, that the Ω_n are of the form $\mathrm{cl}(C_n \cup C'_n)$, where C_n and C'_n are chambers of $\mathbf{A}$ (2.4.8), and have as union the whole of $\mathbf{A}$. Let us remark that P_n, being then an intersection of conjugates of B, is a closed subgroup of G, and hence $P_n \cap V_n$ is also closed.

Proposition **(2.8.3)**. — *Under the hypotheses of (2.8.2), every subset M of $\mathscr{I}$ which is the union of an increasing sequence of subsets M_n $(n \geq 0)$, each contained in an apartment A_n, is itself contained in an apartment.*

One may obviously suppose that the M_n are bounded and are convex unions of closures of facets, and that M_0 is open in M_n. Let us choose $u_n \in G$ such that $A_{n+1} = u_n.A_n$ and $u_n.x = x$ for every $x \in M_n \subset A_n \cap A_{n+1}$ (2.5.8). Set $g_n = u_{n-1} \cdots u_0$ and $M'_n = g_n^{-1}.M_n \subset A_0$. One thus obtains an increasing sequence of bounded subsets of $A = A_0$, whose union M' is a convex union of closures of facets.

Let us show that one may reduce to the case where $M' = A$. Suppose $M' \neq A$ and let $\Gamma = (C_0, \dots)$ be an "infinite gallery" having as terms all the chambers of A (each of them being able to figure there several times) and such that C_0 contains a facet of maximal dimension of M'_0. Set $D_{-1} = \emptyset$ and define D_i and the chamber C'_i by induction on $i \geq 0$ in the following way: C'_i is the first term of Γ not contained in $\mathrm{cl}(M' \cup D_{i-1})$, and $D_i = D_{i-1} \cup C'_i$. Up to replacing the sequence (M_n) by an extracted subsequence, one may suppose that C'_i is also the first term of Γ not contained in $\mathrm{cl}(M'_i \cup D_{i-1})$. On the other hand, let us denote by C''_i the term preceding C'_i in Γ; when $C'_0 = C_0$, the chamber C'_0 has no predecessor in Γ and one then designates by C''_0 a chamber adjacent to C'_0 such that the panel common to C'_0 and C''_0 contains a facet of maximal dimension of M'. We shall show by induction on the integer n that there exist, for $k \geq n \geq -1$, elements $h_{n,k} \in G$ such that, setting $h_n = h_{n,n}$:

(1) $h_{n,k}$ coincides with g_k on M'_k;

(2) $h_{n,k}$ coincides with h_n on $\mathrm{cl}(M'_n \cup D_n)$ for $n \geq 0$;

(3) h_n coincides with h_{n-1} on $\mathrm{cl}(M'_{n-1} \cup D_{n-1})$ for $n \geq 1$.

One takes $h_{-1} = \mathrm{id}$ and $h_{-1,k} = g_k$ for $k \geq 0$. Let $n \geq 0$ and suppose the $h_{i,k}$ constructed for $i < n$. Let $k \geq n$. By definition of C'_n and C''_n, the enclosure of $M'_k \cup D_{n-1}$ is contained in the p. 54
half-apartment of A_0 containing C''_n and whose wall contains the panel F common to C'_n and C''_n. In view of the induction hypothesis, the facet $F' = h_{n-1,k}.F$ does not depend on k and one has

$h_{n-1,k}.(M'_k \cup D_{n-1}) = M_k \cup h_{n-1}.D_{n-1}$. By (2.4.12) there exists an apartment A'_k containing $h_{n-1,k}.\operatorname{cl}(M'_k \cup D_{n-1})$ and $h_{n-1}.C'_n$. By (2.5.8) there exists $h_{n,k} \in G$ such that $h_{n,k}.A_0 = A'_k$ and $h_{n,k}.x = h_{n-1,k}.x$ for every $x \in \operatorname{cl}(M'_k \cup D_{n-1})$. It is then immediate that (1) and (3) are satisfied. Condition (2) is so as well, for the restrictions of $h_{n,k}$ and of h_n to $\operatorname{cl}(M'_n \cup D_n)$ have the same image $\operatorname{cl}(M_n \cup h_{n-1}.D_n)$ and coincide on the chamber C_0.

Replacing M_n by $\operatorname{cl}(M_n \cup h_n.C'_n)$ and g_n by h_n, one is indeed reduced to the case where $M' = A$.

So let us suppose henceforth that $M' = A$. Since each M'_n is a convex union of closures of facets, one sees that A is also the union of the interiors of the M'_n and, up to extracting a subsequence, one may suppose that $\Omega_n \subset M'_n$. Since $g_{n+1}.x = g_n.x$ for $x \in M'_n$, one has $g_n^{-1}g_{n+1} \in P_n$, and up to replacing g_{n+1} by an element of the form $g'_{n+1} = g_{n+1}h_{n+1}$ with $h_{n+1} \in H$ (or again u_n by $u'_n = g_n h_{n+1} g_n^{-1}$), one may suppose that $g_n^{-1}g_{n+1} \in P_n \cap V_n$.

The sequence g_n then converges to an element $g \in G$ and one has $g.x = g_n.x$ for every $x \in M'_n$, whence $M_n \subset g.A$, which completes the proof.

Corollary **(2.8.4)**. — *Let us suppose the hypotheses of (2.8.2) satisfied and let M be a subset of $\mathscr{I}$ of nonempty interior.*

(i) In order that M be contained in an apartment, it is necessary and sufficient that it be isometric to a subset of **A**.

(ii) In order that M be an apartment, it is necessary and sufficient that it be isometric to **A**.

This follows from (2.8.1) and (2.8.3).

Proposition **(2.8.5)**. — *Under the hypotheses of (2.8.2), in order that a subset M of $\mathscr{I}$ be an apartment, it is necessary and sufficient that M be a closed subset containing at least one chamber of $\mathscr{I}$ and that, for every panel F of $\mathscr{I}$ contained in M, there exist exactly two chambers of $\mathscr{I}$ contained in M and admitting F as panel.*

That the condition is necessary is quite clear (and independent of (2.8.2)!). Let us suppose it realized. For every chamber $C \subset M$ and every $s \in \mathbf{S}$ there then exists one and only one chamber $C' \subset M$, adjacent to C and such that the common panel is of type s. One concludes immediately, using (2.3.9) and the fact that M is closed, that for every $w \in \mathbf{W}$ and every chamber $C \subset M$ there exists one and only one chamber $C' \subset M$ such that $w(C, C') = w$ (2.1.7).

Let C_n and C'_n be as in (2.8.2). Let us choose a chamber $C^* \subset M$, set $w_n = w(\mathbf{C}, C_n)$ and $w'_n = w(\mathbf{C}, C'_n)$, and let C^*_n (resp. C'^*_n) be the chamber of M such that $w(C^*, C^*_n) = w_n$ (resp. $w(C^*, C'^*_n) = w'_n$). Since $\mathbf{C} \subset \operatorname{cl}(C_n \cup C'_n)$, one has

$$\ell(w_n w'^{-1}_n) = \ell(w_n) + \ell(w'_n) \qquad \text{and} \qquad w(C'_n, C_n) = w_n w'^{-1}_n.$$

One deduces at once that $C^* \subset M_n = \operatorname{cl}(C^*_n \cup C'^*_n)$, that $w(C'^*_n, C^*_n) = w_n w'^{-1}_n$, and hence that there exists $g_n \in G$ such that $g_n.C_n = C^*_n$, $g_n.C'_n = C'^*_n$. For every $k < n$ one has $g_n.C_k \subset \operatorname{cl}(C^*_n \cup C'^*_n)$ and $w(C^*, g_n.C_k) = w_k$, whence $C^*_k = g_n.C_k$. Similarly $C'^*_k = g_n.C'_k$, which entails $M_k \subset M_n$. In view of (2.8.3) there therefore exists $g \in G$ such that $g.\mathbf{C} = C^*$ and $M_n \subset g.\mathbf{A}$ for every n, whence $g.C_n = C^*_n$ and $g.C'_n = C'^*_n$, and consequently $g.\mathbf{A} = \bigcup_n M_n \subset M$. If moreover one had $M \neq \mathbf{A}$, there would be at least one panel of M belonging to three distinct chambers contained in M, contrary to the hypothesis. Consequently $M = g.\mathbf{A}$ is indeed an apartment of $\mathscr{I}$. p. 55

Corollary **(2.8.6)**. — *Let φ be an injection of* **A** *into $\mathscr{I}$. Suppose that φ is a chambered morphism (1.1.7) of polysimplicial complexes and sends each panel of* **C** *onto a panel of the same type. Then $\varphi(\mathbf{A})$ is an apartment and φ is a structural map.*

One shows immediately by induction on $\ell(w(\mathbf{C}, C))$ that φ sends a panel of an arbitrary chamber C of **A** onto a panel of the same type. One deduces that the image of a minimal gallery is a minimal gallery of the same type, which entails that $\varphi(\mathbf{A})$ is closed, and hence is an apartment by (2.8.5). There therefore exists a structural map ψ such that $\psi(\mathbf{A}) = \varphi(\mathbf{A})$ and $\psi(\mathbf{C}) = \varphi(\mathbf{C})$. But $\psi^{-1} \circ \varphi$ is then an automorphism of polysimplicial complex of $\mathbf{A}$ whose restriction to $\mathbf{C}$ is an affine automorphism of $\mathbf{C}$ preserving each panel of $\mathbf{C}$, which entails that $\psi^{-1} \circ \varphi$ is the identity, whence the corollary.

(2.8.7) One thus sees that, under the hypotheses of (2.8.2), *the family of the apartments of $\mathscr{I}$ depends only on the polysimplicial structure of $\mathscr{I}$*. One deduces in particular that *every B-adapted homomorphism $\varphi : G \to \widehat{G}$ is automatically B-N-adapted.*

Moreover, one shows easily that *the metric of $\mathscr{I}$ determines its polysimplicial structure*: the points of $\mathscr{I}$ interior to a chamber are those which possess a neighborhood isometric to an open subset of $\mathbf{A}$. It then follows from (2.8.4) that, under the hypotheses of (2.8.2), every isometry of $\mathscr{I}$ onto itself is an automorphism of the building $\mathscr{I}$, up to an automorphism of $(\mathbf{A}, \mathbf{W})$.

Lemma **(2.8.8)**. — *Let A be an apartment, C a chamber of A, and y, y' two points of $\mathscr{I}$. If $C \subset \mathrm{cl}(\{\rho_{A;C}(y), \rho_{A;C}(y')\})$, then $C \subset \mathrm{cl}(\{y, y'\})$.*

Set $\rho = \rho_{A;C}$, $z = \rho(y)$ and $z' = \rho(y')$. Let (Γ, C) (resp. Γ') be a taut gallery between y and C (resp. between C and y'), C' the origin of Γ and C'' the extremity of Γ'. In view of (2.4.4) the gallery $(\rho(\Gamma), \rho(\Gamma'))$ is minimal. The gallery (Γ, Γ'), which is of the same type, is therefore minimal as well (2.1.11), and C is contained in $\mathrm{cl}(C' \cup C'')$. Consequently there exists $g \in G$ such that $g.C = C$, $g.C' = \rho(C')$ and $g.C'' = \rho(C'')$. One then has

$$\mathrm{cl}(\{y, y'\}) = g^{-1}.\,\mathrm{cl}(\{g.y, g.y'\}) = g^{-1}.\,\mathrm{cl}(\{z, z'\}) \supset g^{-1}.C = C.$$

Proposition **(2.8.9)**. — *Let D be a vector chamber, $x, x' \in \mathbf{A}$ and $y, y' \in \mathscr{I}$. Suppose that there exist $y_1 \in x + D$ and $y'_1 \in x' - D$ such that $d(y, y_1)$ (resp. $d(y', y'_1)$) is less than the distance from y_1 (resp. y'_1) to the complement of $x + D$ (resp. of $x' - D$) in $\mathbf{A}$. Then $\mathrm{cl}(\{y, y'\}) \supset (x' + D) \cap (x - D)$.*

Let C be a chamber meeting $(x - D) \cap (x' + D)$ and let ρ be the retraction of $\mathscr{I}$ onto $\mathbf{A}$ of center C. One has $\rho(y) \in x + \overline{D}$ and $\rho(y') \in x' - \overline{D}$. In view of (2.4.8) and (2.8.8) one therefore has $C \subset \mathrm{cl}(\{y, y'\})$, whence our assertion. p. 56

Corollary **(2.8.10)**. — *Let D be a vector chamber, v an element of D and λ a real number such that $0 \le \lambda < 1$. There exists a real number $\varepsilon > 0$ such that, for $x, x' \in \mathbf{A}$ and $y, y' \in \mathscr{I}$, the relations $x' - x \in \mathbf{R}v$, $d(x, y) < \varepsilon\, d(x, x')$ and $d(x', y') < \varepsilon\, d(x, x')$ entail the existence of $z, z' \in [yy'] \cap \mathbf{A}$ with $d(z, z') \ge \lambda\, d(x, x')$.*

Let x_i $(i = 1, \ldots, 6)$ be six distinct points of a line L of $\mathbf{A}$, such that $x_i \in [x_{i-1}x_{i+1}]$ for $i = 2, \ldots, 5$, that $x_6 - x_1 \in \mathbf{R}_+ v$ and that $d(x_3, x_4) > \lambda\, d(x_1, x_6)$. Let $r \in \mathbf{R}_+^*$ and let X_i be the ball of $\mathbf{A}$ with center x_i and radius r $(1 \le i \le 6)$. Suppose r small enough that $X_1 \subset x_2 - D$, $X_3 \cup X_4 \subset (x_2 + D) \cap (x_5 - D)$ and $X_6 \subset x_5 + D$, and set $\varepsilon = r\, d(x_1, x_6)^{-1}$.

Let $x, x' \in \mathbf{A}$ and $y, y' \in \mathscr{I}$ satisfying the conditions of the statement. Up to transforming the system of the x_i by a homothety or a translation and multiplying r, v and D by the ratio of the homothety, which does not change ε, one may suppose that $x = x_1$ and $x' = x_6$. Let A be an apartment containing $\{y, y'\}$. One has $d(x, y) < \varepsilon\, d(x, x') = r$ and, similarly, $d(x', y') < r$. From (2.8.9) one then deduces that $A \supset (x_2 + D) \cap (x_5 - D)$. Let C be a chamber of $\mathbf{A}$ meeting this intersection and ρ the restriction to A of the retraction $\rho_{\mathbf{A};C}$. One has $C \subset A \cap \mathbf{A}$, and ρ is an isometry of A onto $\mathbf{A}$ leaving fixed all the points of $A \cap \mathbf{A}$ (2.3.4). Since $\rho_{\mathbf{A};C}$ decreases distances (2.5.3), one also has $\rho(y) \in X_1$ and $\rho(y') \in X_6$. One deduces at once that there exist $z \in [\rho(y)\rho(y')] \cap X_3$ and $z' \in [\rho(y)\rho(y')] \cap X_4$ such that $d(z, z') \ge \lambda\, d(x, x')$. Since $X_3 \cup X_4 \subset (x_2 + D) \cap (x_5 - D) \subset A \cap \mathbf{A}$, one has $z = \rho^{-1}(z)$ and $z' = \rho^{-1}(z')$, whence $z, z' \in [yy'] \cap \mathbf{A}$.

Proposition **(2.8.11)**. — *The apartment* $\mathbf{A}$ *is the smallest nonempty convex subset of* $\mathscr{I}$ *invariant under* $\nu^{-1}(V)$. *It is also the smallest nonempty closed subset of* $\mathscr{I}$ *invariant under* $\nu^{-1}(V)$.

The second assertion is an immediate consequence of the first.

Let M be a nonempty convex subset of $\mathscr{I}$ invariant under $\nu^{-1}(V)$, $y \in M$, $x \in \mathbf{A}$, and $g \in \nu^{-1}(V)$ such that $\nu(g)$ belongs to the kernel of no vector root. Set $v = \nu(g)$ and let λ, ε be real numbers possessing the properties of Corollary (2.8.10). For $n \in \mathbf{N}$ sufficiently large one has $d(x, y) = d(g^n.x, g^n.y) < \varepsilon\, d(x, g^n.x)$, and hence there exists $z \in [y(g^n.y)] \cap \mathbf{A} = M \cap \mathbf{A}$. Since the convex hull of $\nu^{-1}(V)(z)$ is manifestly $\mathbf{A}$, one has $\mathbf{A} \subset M$.

(2.8.12) We shall show by an example that assertions (2.8.3) to (2.8.7) do not remain valid if one suppresses the hypotheses of (2.8.2).

Suppose $\operatorname{card} \mathbf{S} = 2$, set $\mathbf{S} = \{s_1, s_2\}$, and let X be a subgroup of B and $n'_i \in Bs_iB$ $(i = 1, 2)$ such that $n_i'^2 \in X$ and $n'_i X n_i'^{-1} = X$; let N' be the group generated by n'_1, n'_2 and X. It is clear that X is normal in N' and that there exists a surjective homomorphism $\eta : \mathbf{W} \to N'/X$, and only one, such that $\eta(s_i) = n'_i X$. From (1.2.8) it follows that if $w \in \mathbf{W}$ one has $\eta(w) \subset BwB$. Consequently η is bijective. One deduces at once that (G, B, N') is a Tits system with Weyl group $\mathbf{W}$, equivalent to (G, B, N) (1.2.11). Consequently the building $\mathscr{I}'$ of (G, B, N') is identified, as a metric space and as a polysimplicial complex, with the building $\mathscr{I}$ of (G, B, N) (2.7.4), and the apartments of $\mathscr{I}'$ are those of $\mathscr{I}$ if and only if there exists $b \in B$ with $N' \subset bNb^{-1}$ (2.2.6), that is to say $\eta(w) \subset b.wH.b^{-1}$ for every $w \in \mathbf{W}$. p. 57

Example **(2.8.13)**. — Let p be a prime number, and let ω_p be the p-adic valuation of $\mathbf{Q}$. Take $G = \mathrm{SL}_2(\mathbf{Q})$,

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;\middle|\; \omega_p(a) = \omega_p(d) = 0,\ \omega_p(b) \geq 0,\ \omega_p(c) > 0 \right\},$$

$$N = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in G \right\}.$$

By [28], (G, B, N) is a Tits system of affine type of rank 1 (see also (6.2.3), a)). One has

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbf{Q},\ \omega_p(a) = 0 \right\}, \qquad s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.H, \qquad s_2 = \begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}.H.$$

Set

$$n'_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad n'_2 = \begin{pmatrix} -1 & 2p \\ -p^{-1} & 1 \end{pmatrix}, \qquad X = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

One verifies easily that the hypotheses of (2.8.12) are satisfied and that the eigenvalues of $n'_1 n'_2$ are not rational. It follows that $n'_1 n'_2$ is not conjugate to an element of $s_1 s_2 H$, and hence that $\mathscr{I}$ and $\mathscr{I}'$ do not have the same apartments.

2.9. Sectors. —

Proposition **(2.9.1)**. — *Let A be an apartment of $\mathscr{I}$ and $\mathfrak{C}$ a sector of A. There exists one and only one map $\rho_{A;\mathfrak{C}}$ from $\mathscr{I}$ into A possessing the following property: for every bounded subset M of $\mathscr{I}$ there exists a sector $\mathfrak{C}' \subset \mathfrak{C}$ such that, for every chamber C meeting $\mathfrak{C}'$, one has $\rho_{A;\mathfrak{C}}(x) = \rho_{A;C}(x)$ for every $x \in M$.*

We must show that, given a bounded subset M of $\mathscr{I}$, there exists a sector $\mathfrak{C}' \subset \mathfrak{C}$ such that, if C denotes a chamber meeting $\mathfrak{C}'$, the restriction of $\rho_{A;C}$ to M does not depend on the choice of C. One may obviously suppose that M is a union of closures of chambers. Let us take a point a of A and set $d = \sup\{d(a, x) \mid x \in M\}$. Let M' be the ball with center a and radius d in A and let $\mathfrak{C}_1$ be a sector opposite to $\mathfrak{C}$ and containing M' ((2.4.8)(i)). Let b be the vertex of $\mathfrak{C}_1$ p. 58
and let $\mathfrak{C}''$ be the sector of A opposite to $\mathfrak{C}_1$ (hence parallel to $\mathfrak{C}$) with vertex b. We shall show that the sector $\mathfrak{C}' = \mathfrak{C} \cap \mathfrak{C}''$ answers the question.

Let C_0 be a chamber of A to which b is adherent; set $\rho = \rho_{A;C_0}$. Since a retraction decreases distances, one has $\rho(C') \subset M' \subset \mathfrak{C}_1$ for every chamber C' contained in M.

Let C be a chamber meeting $\mathfrak{C}''$. By (2.4.9), the enclosure of $C \cup \rho(C')$ contains C_0. There therefore exists a minimal gallery Γ of the form

$$(C, C_1, \dots, C_m, C_0, C_{m+1}, \dots, C_n, \rho(C'))$$

and there exists a minimal gallery of the form $(C_0, C'_{m+1}, \dots, C'_n, C')$ with $\rho(C'_j) = C_j$ for $m+1 \leq j \leq n$. The gallery $(C, C_1, \dots, C_m, C_0, C'_{m+1}, \dots, C'_n, C')$ is non-stuttering and has the same type as Γ. It is therefore minimal, and its image under the retraction $\rho_{A;C}$ is Γ, the unique minimal gallery of type $\mathbf{s}(\Gamma)$ and origin C. Consequently one has

$$\rho_{A;C}(C') = \rho_{A;C_0}(C')$$

for every chamber C' contained in M and every chamber C meeting $\mathfrak{C}''$, whence the proposition.

Definition **(2.9.2)**. — *The map $\rho_{A;\mathfrak{C}}$ defined by (2.9.1) is called the retraction of $\mathscr{I}$ onto the apartment A relative to the sector $\mathfrak{C}$.*

It is clear that $\rho = \rho_{A;\mathfrak{C}}$ does not change if one replaces $\mathfrak{C}$ by a sector of A parallel to $\mathfrak{C}$, and therefore depends only on the "sector germ" of A defined by $\mathfrak{C}$ (cf. (7.2.3), (7.4.12)). It is also clear that ρ decreases distances and that its restriction to the closure of a chamber is isometric (2.5.3).

Corollary **(2.9.3)**. — *Let A be an apartment of $\mathscr{I}$, $\mathfrak{C}$ a sector of A and C a chamber of $\mathscr{I}$. There exists a sector $\mathfrak{C}' \subset \mathfrak{C}$ such that the restriction of $\rho_{A;\mathfrak{C}}$ to $\mathfrak{C}' \cup C$ is isometric.*

In view of (2.5.3)(ii), it suffices to take a sector $\mathfrak{C}' \subset \mathfrak{C}$ such that $\rho_{A;\mathfrak{C}}(C) = \rho_{A;C'}(C)$ for every chamber C' meeting $\mathfrak{C}'$.

Corollary **(2.9.4)**. — *Suppose G endowed with a structure of complete topological group satisfying the hypotheses of (2.8.2). Let $\mathfrak{C}$ be a sector and C a chamber of $\mathscr{I}$. There exist a sector $\mathfrak{C}' \subset \mathfrak{C}$ and an apartment containing C and $\mathfrak{C}'$.*

This follows from (2.9.3) and (2.8.4).

Proposition **(2.9.5)**. — *Let A and A' be two apartments, $\mathfrak{C}$ (resp. $\mathfrak{C}'$) a sector of A (resp. of A'). There exist sectors $\mathfrak{C}_1 \subset \mathfrak{C}$ and $\mathfrak{C}'_1 \subset \mathfrak{C}'$ such that the restriction to $\mathfrak{C}_1 \cup \mathfrak{C}'_1$ of the retraction of $\mathscr{I}$ onto A relative to $\mathfrak{C}$ is isometric.*

One may suppose that $A = \mathbf{A}$ (identified with $j(\mathbf{A})$). Let us denote by ρ the retraction of $\mathscr{I}$ onto $\mathbf{A}$ relative to the sector $\mathfrak{C}$. For every chamber C of A', let $m(C)$ be the center of gravity of C and let $Q(C)$ be the intersection of C with the sector of A' with vertex $m(C)$ parallel to p. 59
$\mathfrak{C}'$. One verifies easily that there exists one and only one element $w(C)$ of the Weyl group ${}^v\mathbf{W}$ of the root system ${}^v\mathbf{\Sigma}$ such that $\rho(Q(C)) \subset \rho(m(C)) + w.D$, where D is the vector chamber which is the direction of $\mathfrak{C}$. Let us then choose a chamber C_0 of A' in such a way that the length of $w(C_0)$ in ${}^v\mathbf{W}$ (relative to the base $B(D)$ of ${}^v\mathbf{\Sigma}$ associated with D) is the largest possible. One may suppose that $\mathfrak{C}'$ has vertex $m(C_0)$, that $\mathfrak{C}$ is contained in $\rho(m(C_0)) + D$ and that one has $\rho(C_0) = \rho_{\mathbf{A};C}(C_0)$ for every chamber C meeting $\mathfrak{C}$.

Let us make $\mathbf{A}$ and A' into vector spaces by choosing as origin the points $m(C_0)$ and $\rho(m(C_0))$ respectively. One then has $\mathfrak{C}' = \mathbf{R}_+.Q(C_0)$.

We shall show that, for every $v \in Q(C_0)$, every $t \in \mathbf{R}_+$ and every chamber C of $\mathbf{A}$ meeting $\mathfrak{C}$, one has

$$\rho_{\mathbf{A};C}(tv) = t\rho(v) = \rho(tv). \tag{6}$$

By continuity it suffices to prove (1) when the line $\mathbf{R}v$ meets no facet of codimension 2, which we suppose henceforth. There then exists a strictly increasing sequence $(t_0, \dots, t_{n+1})$ of points of the interval $[0, t]$, with $t_0 = 0$ and $t_{n+1} = t$, and a minimal gallery $(C_0, \dots, C_n)$ contained in

A' such that $C_i \cap [0, tv] \supset]t_i v,\ t_{i+1}v[$ for $0 \le i \le n$. Let us argue by induction on the integer n, relation (1) being true when $n = 0$. Suppose $n > 0$; in view of the induction hypothesis, relation (1), after substitution of s for t, is true for $0 \le s \le t_n$. One deduces that $\rho(C_{n-1}) = \rho_{\mathbf{A};C}(C_{n-1})$.

It suffices to show that $\rho_{\mathbf{A};C}(C_n)$ is the chamber C'_n adjacent to $C'_{n-1} = \rho_{\mathbf{A};C}(C_{n-1})$ along the panel F, the image under ρ of the panel common to C_{n-1} and C_n. Indeed, this implies that the restriction of $\rho_{\mathbf{A};C}$ to $\overline{C}_{n-1} \cup \overline{C}_n$ is isometric, and hence that the map $t \mapsto \rho_{\mathbf{A};C}(tv)$ from $[t_{n-1}, t_{n+1}]$ into $\mathbf{A}$ is affine. The first equality (1) follows from it. The second reduces to the first by choosing the chamber C "sufficiently far" in $\mathfrak{C}''$ so that ρ and $\rho_{\mathbf{A};C}$ coincide on $C_0 \cup \cdots \cup C_n$.

Let then L be the wall of $\mathbf{A}$ containing F and let α be the affine root with wall L containing C'_{n-1}. One then has $\rho([t_{n-1}v, t_n v]) \subset \alpha$, whence, in view of the induction hypothesis, $\rho([0, t_n v]) \subset \alpha$ and $C'_0 = \rho(C_0) \subset \alpha$. Moreover $\alpha^* + w(C_0).D = \alpha^*$. If $\alpha \supset C$, the sought relation $\rho_{\mathbf{A};C}(C_n) = C'_n$ follows immediately from (2.3.12). If α contains a sector parallel to $\mathfrak{C}$, one has $\alpha \supset \rho(C_0) + D$, and α contains $\mathfrak{C}$, hence C. There finally remains to examine the case where α^* contains C and a sector of direction D. If $\rho(C_n) \ne C'_n$, one has $\rho(C_n) = C'_{n-1}$ and the segment $[\rho(t_n v), \rho(t_{n+1}v)]$ is contained in the symmetric, with respect to L, of the half-line $[t_n, \infty[\ .v$. One deduces that $w(C_n) = s.w(C_0)$, where s is the image in ${}^v\mathbf{W}$ of the reflection with respect to L. But D and $w(C_0).D$ are on the same side of the hyperplane of the fixed points of s, which entails that the length of $sw(C_0)$ is strictly greater than that of $w(C_0)$, whence a contradiction in view of the choice of C_0. Consequently one has $\rho(C_n) = C'_n$. Let us now show that $\rho_{\mathbf{A};C}(C_n) = C'_n$. Suppose on the contrary that $\rho_{\mathbf{A};C}(C_n) = C'_{n-1}$; there then exists a chamber C'' adjacent to C_n and to C_{n-1} such that $\rho_{\mathbf{A};C}(C'') = C'_n$ (2.3.12). The length h of a taut gallery between C and C'' is then equal to that of a taut gallery between C and C'_n and, since C and C'_{n-1} are on either side of L, the length of a taut gallery between C and C'_{n-1} is equal to $h + 1$. One deduces that $\rho(C'') = C'_n$; indeed, $\rho(C'')$ admits F as panel, hence is equal to C'_n p. 60
or to C'_{n-1}, and cannot be equal to C'_{n-1} since the image under ρ of a gallery stretched between C and C'' is a gallery of length h joining C to $\rho(C'')$. But this contradicts (2.3.12): taking a chamber C''' contained in a sufficiently small sector contained in $\mathfrak{C}$, one deduces indeed that the two adjacent chambers C_n and C'' are sent by the retraction $\rho_{\mathbf{A};C'''}$ onto the same chamber C'_n, which is situated on the same side of L as C'''. Consequently we have indeed shown that $\rho_{\mathbf{A};C}(C_n) = C'_n$, which completes the proof of (1).

Let us then show that the restriction of ρ to $\mathfrak{C} \cup \mathfrak{C}'$ is isometric. Relation (1) entails that, for $x \in \mathfrak{C}$ and $y \in \mathfrak{C}'$, one has

$$d(\rho(x), \rho(y)) = d(x, \rho_{\mathbf{A};C}(y)) = d(x, y)$$

on taking for C a chamber containing x in its closure and applying (2.5.3)(ii). If $x, y \in \mathfrak{C}'$, there exists $t \in \mathbf{R}_+^*$ such that tx and ty belong to $Q(C_0)$, and one then has, in view of (1) and of the fact that the restriction of ρ to $Q(C_0)$ is isometric,

$$\begin{aligned} d(\rho(x), \rho(y)) &= d(t^{-1}\rho(tx),\ t^{-1}\rho(ty)) = t^{-1}d(\rho(tx), \rho(ty)) \\ &= t^{-1}d(tx, ty) = d(x, y). \end{aligned}$$

Since ρ is the identity on $\mathfrak{C}$, this completes the proof.

Corollary **(2.9.6)**. — *Let us take up again the hypotheses of (2.8.2), and let $\mathfrak{C}$ and $\mathfrak{C}'$ be two sectors of $\mathscr{I}$. There exist sectors $\mathfrak{C}_1 \subset \mathfrak{C}$ and $\mathfrak{C}'_1 \subset \mathfrak{C}'$ contained in one and the same apartment.*

This follows from (2.9.5) and (2.8.4).

3. Bounded subgroups

We keep the notation of the preceding paragraphs. p. 61

3.1. Bornology defined by a Tits system. —

Definition **(3.1.1)**. — *A* ***bornology*** *on a set X is a set $\mathscr{B}$ of subsets of X, called* ***bounded subsets****, stable under finite union, containing the finite subsets of X, and such that $M \in \mathscr{B}$ and $M' \subset M$ entail $M' \in \mathscr{B}$. If X is a group, one says that $\mathscr{B}$ is* ***compatible with the group law*** *of X, or makes X a* ***bornological group****, if $M, M' \in \mathscr{B}$ entails $M^{-1}M' \in \mathscr{B}$.*

Example **(3.1.2)**. — a) The set of the relatively compact subsets of a Hausdorff topological group X makes X a bornological group.

b) Let E be a metric space. The group $\operatorname{Isom} E$ of the isometries of E possesses a natural bornology formed of the sets M possessing the following equivalent properties:

(i) there exists a point $x \in E$ such that the set of the $g(x)$ for $g \in M$ is bounded in E;

(ii) for every bounded subset F of E, the set of the $g(x)$ for $g \in M$ and $x \in F$ is bounded.

c) Let X' be a bornological group and $\varphi : X \to X'$ a homomorphism of groups. The set of the subsets of X whose image is a bounded subset of X' forms a bornology on X, compatible with the group law of X, and called the *inverse image of the bornology of X' by φ*.

Proposition **(3.1.3)**. — *Let $\mathscr{B}$ be the set of the subsets of G whose canonical image in $B\backslash G/B$ is finite. Then $\mathscr{B}$ is a bornology on G compatible with the group law of G.*

That $\mathscr{B}$ be a bornology is evident. On the other hand it is clear that $M \in \mathscr{B}$ entails $M^{-1} \in \mathscr{B}$, and it suffices to show that $M, M' \in \mathscr{B}$ entails $MM' \in \mathscr{B}$, which is immediate since the product of two double cosets modulo B belongs to $\mathscr{B}$ ([5], chap. IV, § 2, n^o 1, lemma 1).

Definition **(3.1.4)**. — *The bornology $\mathscr{B}$ introduced in (3.1.3) is said to be* ***defined by the Tits system*** (G, B, N).

Since $M \in \mathscr{B}$ is equivalent to $gMg^{-1} \in \mathscr{B}$ (whatever $g \in G$), it is clear that $\mathscr{B}$ depends only on the conjugacy class of B. *Two equivalent Tits systems define the same bornology*: we shall see later the converse (3.5.1).

Proposition **(3.1.5)**. — *Let $\mathscr{B}$ be a bornology on G, compatible with the group law and such that $B \in \mathscr{B}$. Let $\mathscr{B}_{\mathbf{W}}$ be the set of the subsets M of $\mathbf{W}$ such that $BMB \in \mathscr{B}$. Then $\mathscr{B}_{\mathbf{W}}$ makes $\mathbf{W}$ a bornological group. If X is a subset of $\mathbf{W}$ such that the union of the $BxBx^{-1}B$, for $x \in X$, is bounded for $\mathscr{B}$, the set of the reflections associated with the various elements of X is bounded in $\mathbf{W}$.* p. 62

The first assertion is evident. The second follows from the fact that, if t belongs to the set T_w of the reflections associated with $w \in \mathbf{W}$ (2.3.10), one has $BtB \subset BwBw^{-1}B$ ([5], chap. IV, § 2, n^o 4, cor. 2 of th. 2).

Corollary **(3.1.6)**. — *Let us endow G with the bornology $\mathscr{B}$ defined by the Tits system (G, B, N) and let M be a subset of G. In order that M be bounded, it is necessary and sufficient that the union of the gBg^{-1} for $g \in M$ be so.*

Set $M' = \bigcup_{g \in M} gBg^{-1}$. If M is bounded, one has $M' \subset MBM^{-1} \in \mathscr{B}$. Conversely, suppose M' bounded and let $X \subset \mathbf{W}$ be such that $BXB = BMB$. In view of (3.1.5), the set of the reflections associated with the various elements of X is bounded in $\mathbf{W}$, that is to say finite. Since every $w \in \mathbf{W}$ is a product of the reflections associated with it, taken once and only once in a suitable order, X itself is finite and M is bounded.

Remark **(3.1.7)**. — The preceding results are valid for an arbitrary Tits system, not necessarily of affine type.

Proposition **(3.1.8)**. — *Let $\operatorname{Isom} \mathscr{I}$ be the group of the isometries of the building $\mathscr{I}$ of (G, B, N). The bornology of G defined by the Tits system (G, B, N) is the inverse image of the natural bornology of $\operatorname{Isom} \mathscr{I}$ by the canonical homomorphism of G into $\operatorname{Isom} \mathscr{I}$.*

Let $M \subset G$ and let X be the canonical image of M in $B\backslash G/B$. To say that the image of M in $\operatorname{Isom} \mathscr{I}$ is bounded amounts to saying that

$$\sup_{g\in M} d\big(F(B), F(gBg^{-1})\big) < +\infty.$$

Now, if $g \in BxB$ with $x \in \mathbf{W}$, one has $d(F(B), F(gBg^{-1})) = d(\mathbf{C}, x(\mathbf{C}))$. Since $\mathbf{W}$ operates properly on $\mathbf{A}$, one has $\sup_{x\in X} d(\mathbf{C}, x(\mathbf{C})) < +\infty$ if and only if X is finite, whence the proposition.

(3.1.9) Let $\varphi : G \to \widehat{G}$ be a B-adapted homomorphism and let $B_1 = \operatorname{Stab} B$ be the normalizer of $\varphi(B)$ in $\widehat{G}$. We defined in 2.7 a homomorphism of $\widehat{G}$ into $\operatorname{Isom} \mathscr{I}$ "prolonging" the canonical homomorphism of G into $\operatorname{Isom} \mathscr{I}$. We shall again call *bornology defined by the Tits system* (G, B, N) in $\widehat{G}$ the inverse image by this homomorphism of the natural bornology of $\operatorname{Isom} \mathscr{I}$.

Proposition. — Let us endow G and $\widehat{G}$ with the bornologies defined by the Tits system (G, B, N). Let $M \subset \widehat{G}$; the following conditions are equivalent:

(i) M is bounded in $\widehat{G}$;

(ii) $\varphi^{-1}(MB_1)$ is bounded in G;

(iii) the image of M in $B_1\backslash\widehat{G}/B_1$ is finite.

Let us show that (i) entails (ii). Since B_1 is bounded in $\widehat{G}$, as stabilizer of the chamber $F(B)$, p. 63
MB_1 is bounded as soon as M is, and the image of MB_1 is bounded in $\operatorname{Isom} \mathscr{I}$, whence (ii).

If $\varphi^{-1}(MB_1) \subset BXB$ with $X \subset \mathbf{W}$, one has $M \subset B_1XB_1$; one deduces that (ii) entails (iii).

Finally, (iii) entails (i) since B_1 is bounded.

3.2. A fixed point lemma. —

(3.2.1) Let $x, y \in \mathscr{I}$. By (2.5.4) there exists a unique point $m \in \mathscr{I}$ such that $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$: it is called the *midpoint* of $\{x, y\}$.

Lemma. — Let $x, y, z \in \mathscr{I}$ and let m be the midpoint of $\{x, y\}$. One has

$$d(x,z)^2 + d(y,z)^2 \ \geq\ 2d(m,z)^2 + \tfrac{1}{2}d(x,y)^2. \tag{7}$$

Let A be an apartment of $\mathscr{I}$ containing x and y, let C be a chamber of A whose closure contains m, and set $z' = \rho_{A;C}(z)$ (2.3.5). In view of (2.5.3) one has $d(x, z) \geq d(x, z')$, $d(y, z) \geq d(y, z')$ and $d(m, z) = d(m, z')$. Relation (1) then follows from the equality

$$d(x,z')^2 + d(y,z')^2 = 2d(m,z')^2 + \tfrac{1}{2}d(x,y)^2$$

valid for every point z' of the Euclidean space A ([31], book VII, prop. 122).

Remark **(3.2.2)**. — Let E be a metric space and let x, y, m be three points of E such that relation (1) is satisfied for every point $z \in E$. Making successively $z = x$ and $z = y$, one sees that $d(m, x) = d(m, y) = \frac{1}{2}d(x, y)$. Moreover, if $m' \in E$ is such that $d(m', x) = d(m', y) = \frac{1}{2}d(x, y)$, one has $m = m'$, as one sees on making $z = m'$ in (1). In particular, for $x, y \in \mathscr{I}$, the midpoint m of $\{x, y\}$ is the unique point of $\mathscr{I}$ satisfying (1) for every $z \in \mathscr{I}$.

Lemma **(3.2.3)**. — *Let E be a complete metric space and E' a subset of E possessing the following property:*

(CN) whatever the points x and y of E', there exists a point $m \in E'$ such that relation (1) of (3.2.1) is satisfied for every point z of E'.

If M is a nonempty bounded subset of E', the stabilizer of M in $\operatorname{Isom} E$ *possesses a fixed point in the closure of E' in E.*

If X and Y are two subsets of E, we shall set

$$\operatorname{diam}(X, Y) = \sup_{x\in X,\, y\in Y} d(x, y), \qquad \operatorname{diam} X = \operatorname{diam}(X, X).$$

Let k be a real number such that $0 < k < 1$. For every subset X of E', let us denote by $f(X)$ the set of the points $m \in E'$ for which there exist $x, y \in X$ such that relation (1) is satisfied for every $z \in E'$ and that $d(x, y) \geq k \operatorname{diam} X$; in view of (CN), one has $f(X) \neq \emptyset$ if $X \neq \emptyset$. p. 64

For points x, y, m satisfying these conditions, and for $z \in E'$, one has:

$$(8) \qquad d(m, z)^2 \leq \tfrac{1}{2}\big(d(x, z)^2 + d(y, z)^2\big) - \tfrac{1}{4}d(x, y)^2 \leq \operatorname{diam}(X, \{z\})^2 - \tfrac{k^2}{4}(\operatorname{diam} X)^2.$$

For $z \in X$ one draws from this:

$$(9) \qquad \operatorname{diam}(f(X), X) \leq k_1 \operatorname{diam} X \qquad \text{with } k_1 = \Big(1 - \tfrac{k^2}{4}\Big)^{1/2} < 1.$$

Moreover, taking $z \in f(X)$ in (2), one draws from (2) and (3):

$$(10) \qquad \operatorname{diam} f(X) \leq k_2 \operatorname{diam} X \qquad \text{with } k_2 = \Big(1 - \tfrac{k^2}{2}\Big)^{1/2} < 1.$$

Let M be a nonempty bounded subset of E'; by (4) one has

$$(11) \qquad \operatorname{diam} f^q(M) \leq k_2^q \operatorname{diam} M \qquad \text{for every integer } q \geq 1.$$

Let us then choose a point $x_q \in f^q(M)$ for $q \in \mathbf{N}^*$, which is permissible since $f^q(M)$ is not empty. It follows from (3) and (5) that $d(x_q, x_{q+1}) \leq k_1 k_2^q \operatorname{diam} M$, and the Cauchy sequence (x_q) converges to a point $x \in \overline{E'}$, independent of the choice of the x_q by (5). Finally, it is clear that the stabilizer of M in $\operatorname{Isom} E$ leaves invariant each of the $f^q(M)$, and hence also x. This completes the proof.

One will note that a priori the point x thus constructed depends on the choice of the constant k.

Proposition **(3.2.4)**. — *Let M be a nonempty bounded subset of the building $\mathscr{I}$ of G. The stabilizer of M in* Isom $\mathscr{I}$ *possesses a fixed point belonging to the closure of the convex hull of M.*

This follows from lemmas (3.2.1) and (3.2.3), taking in the latter for E the building $\mathscr{I}$ and for E' the convex hull of M.

Remark **(3.2.5)**. — Let E be a metric space and x, y, m three points of E. The proof of lemma (3.2.1) shows that if there exists a map ρ from E into a Euclidean space, decreasing distances, such that $d(\rho(m), \rho(z)) = d(m, z)$ for every $z \in E$ and such that $\rho(m)$ is the midpoint of the segment $[\rho(x)\rho(y)]$, then relation (1) is satisfied for every $z \in E$. This applies in particular to the case where E is a *simply connected Riemannian space of negative curvature* and where m is the midpoint of the geodesic segment $[xy]$: it suffices to take for ρ the map of E onto its tangent space at m, inverse of the exponential map ([24], chap. I, § 13). Lemma (3.2.3) then furnishes in this case a simple proof of the classical result on the existence of a fixed point for a compact group of isometries of such a Riemannian manifold. It is rather curious that this same lemma, which will serve us to determine the conjugacy classes of maximal bounded subgroups of semisimple algebraic groups over a local field, can thus be used to prove the conjugacy of maximal compact subgroups of real Lie groups.

Moreover, the building $\mathscr{I}$ plays for G a role somewhat analogous to the one played, for a real semisimple Lie group, by the Riemannian symmetric space obtained as the quotient by a maximal compact subgroup: it is a complete contractible metric space on which the group acts, the stabilizers of each point being bounded, and lemma (3.2.1) expresses a kind of "negative curvature" property. For other examples of this analogy, see [35].

3.3. Maximal bounded subgroups. —

Let $\varphi : G \to \widehat{G}$ be a B-adapted homomorphism; endow $\widehat{G}$ with the bornology defined by the Tits system (G, B, N, S) as in (3.1.9).

Theorem (3.3.1). — *In order that a subgroup of $\widehat{G}$ be bounded, it is necessary and sufficient that it stabilize a parahoric subgroup of G.*

If a subgroup Γ of $\widehat{G}$ stabilizes a parahoric subgroup P of G, it fixes the facet $F(P)$ of $\mathscr{I}$ and is consequently bounded, by the very definition of the bornology of $\widehat{G}$.

Conversely, if Γ is bounded, every orbit of Γ in $\mathscr{I}$ is bounded, and (3.2.4) implies that Γ possesses a fixed point x in $\mathscr{I}$. Let P be the parahoric subgroup of G such that $x \in F(P)$. Then Γ preserves $F(P)$ and one has ${}^{g}P = P$ for all $g \in \Gamma$ (2.7.1), which completes the proof.

Corollary (3.3.2). — *Every bounded subgroup of $\widehat{G}$ is contained in a maximal bounded subgroup. The maximal bounded subgroups are the maximal elements of the set of stabilizers of parahoric subgroups of G.*

This follows immediately from the theorem and from (1.2.19), which shows that the stabilizer of a parahoric subgroup is contained in only finitely many stabilizers of parahoric subgroups.

Corollary (3.3.3). — *The maximal bounded subgroups of G are the maximal parahoric subgroups. In particular, they form $\prod_{1\leqslant i\leqslant m}(\ell_i + 1)$ conjugacy classes (where the integers ℓ_i, for $1 \leqslant i \leqslant m$, are the dimensions of the irreducible components of* $\mathbf{A}$*).*

It suffices to apply (3.3.2) to the case $\varphi = \mathrm{Id}_G$.

(3.3.4) Recall that in (1.2.16) we defined a homomorphism ξ from $\widehat{G}$ into the group of automorphisms of the Coxeter graph of $\mathbf{W}$, such that, if P is a parahoric subgroup of type X of G, then ${}^{g}P$ is of type $\xi(g)(X)$ for all $g \in \widehat{G}$. Put $\Xi = \xi(\widehat{G})$. The following proposition is an "abstract" version of a result of Iwahori–Matsumoto:

Proposition. — *Let P and P' be two parahoric subgroups of G, let X and X' be their types and let Ξ_X be the stabilizer of X in Ξ. In order that* $\operatorname{Stab} P$ *be a maximal bounded subgroup of $\widehat{G}$, it is necessary and sufficient that X be maximal among the types of parahoric subgroups invariant under Ξ_X. In order that* $\operatorname{Stab} P$ *and* $\operatorname{Stab} P'$ *be conjugate in $\widehat{G}$, it is necessary and sufficient that there exist $t \in \Xi$ such that $X' = t(X)$.*

One may assume $P = B_X$. We recalled (1.2.19) that $\operatorname{Stab} P \subset \operatorname{Stab} P'$ is equivalent to $P' = B_{X'}$ with $X \subset X'$ and $\Xi_X \subset \Xi_{X'}$. This last condition means that X' is invariant under Ξ_X, whence the first assertion.

If there exists $g \in \widehat{G}$ such that $\operatorname{Stab} P' = g(\operatorname{Stab} P)g^{-1}$, one has $P' = {}^{g}P$, whence $X' = \xi(g)(X)$. Conversely, if there exists $t \in \Xi$ such that $X' = t(X)$, and if $g \in \xi^{-1}(t)$, the subgroups ${}^{g}P$ and P' are of the same type, hence conjugate, and the same holds for $\operatorname{Stab} P'$ and $\operatorname{Stab} P = g^{-1}(\operatorname{Stab} {}^{g}P)g$.

(3.3.5) Suppose the Tits system (G, B, N) irreducible. Then $\operatorname{Stab} P$ *is maximal if and only if the type X of P is the complement of an orbit of a subgroup of Ξ in* $\mathbf{S}$.

If moreover Ξ is cyclic of order q (which is frequently the case in the applications to simple algebraic groups), the number c of conjugacy classes of maximal bounded subgroups of G satisfies the inequality $c < \ell + 1$ (where ℓ is the dimension of $\mathbf{A}$), except when $q = 1$ or 2, in which case one has $c = \ell + 1$. Indeed, one sees easily that if Ξ permutes cyclically a subset Y of $\mathbf{S}$, the number of conjugacy classes of maximal bounded subgroups of $\widehat{G}$ of the form $\operatorname{Stab} B_X$ with $X \supset \mathbf{S} - Y$ is equal to the number of divisors of $\operatorname{Card} Y$.

(3.3.6) For examples of the explicit determination of the conjugacy classes of maximal subgroups $\operatorname{Stab} P$, see [28].

3.4. Characterization of the bornology defined by a Tits system of affine type. —

We resume the notation of 2.6.

Theorem (3.4.1). — *Let $\mathscr{B}$ be a bornology on G, compatible with the group law of G and such that B is bounded. There exists one and only one subset $J \subset \{1, \ldots, m\}$ such that $\mathscr{B}$ is the bornology defined by the Tits system (G, B^J, N).*

We shall first establish two lemmas.

Lemma (3.4.2). — *Suppose $\mathbf{W}$ irreducible and let $\mathbf{W}^0$ be the stabilizer in $\mathbf{W}$ of a special point $a \in \overline{\mathbf{C}}$. If X is an infinite subset of $\mathbf{W}$, every reflection of $\mathbf{W}$ is associated with at least one element of $\mathbf{W}^0 \cdot X$.*

Make $\mathbf{A}$ into a vector space by taking a as origin. Let r be a reflection of $\mathbf{W}$ and let φ be a linear form on $\mathbf{A}$ and $k \in \mathbf{R}$ be such that the equation $\varphi(x) = k$ defines the wall L of $\mathbf{A}$, the set of fixed points of r, and such that $\mathbf{C}$ is contained in $\{x \in \mathbf{A} \mid \varphi(x) < k\}$. In view of the irreducibility of $\mathbf{W}$, the set Φ of the transforms of φ by ${}^v\mathbf{W}$ generates the dual of $\mathbf{A}$, and $\Phi = -\Phi$, since the reflection in the wall $\operatorname{Ker}\varphi$ transforms φ into $-\varphi$. It follows that the set

$$E = \bigcap_{w \in {}^v\mathbf{W}} w \cdot \{x \in \mathbf{A} \mid \varphi(x) < k\}$$

is bounded, and that there exists $x \in X$ such that $x(\mathbf{C}) \not\subset E$. There then exists $w \in \mathbf{W}^0$ such that $wx(\mathbf{C}) \subset \{x \in \mathbf{A} \mid \varphi(x) > k\}$, and the reflection r is associated with wx by (2.3.10).

Lemma (3.4.3). — *There exists an integer N such that every element of $\mathbf{W}$ is a product of at most N reflections of $\mathbf{W}$.*

Make $\mathbf{A}$ into a vector space by taking a special point as origin, and resume the notation of (1.3.8). Let $\{a_1, \ldots, a_\ell\}$ be a basis of the root system associated with $\mathbf{W}$. For every root $a \in {}^v\boldsymbol{\Sigma}$ and every integer $n \in \mathbf{Z}$, the composite $r_{a,n} \circ r_{a,0}$ is the translation by the vector $-n \cdot a^\vee$; since the translations by the vectors $a_i^\vee$ for $1 \leqslant i \leqslant \ell$ form a basis of the group $\mathbf{V}$ of translations of $\mathbf{W}$, and since $\mathbf{W}$ is the semidirect product of the stabilizer $\mathbf{W}^0$ of the origin by $\mathbf{V}$ (1.3.7), one deduces that one may take $N = N' + 2\ell$, where N' is the smallest integer such that every element of $\mathbf{W}^0$ (which is finite and isomorphic to ${}^v\mathbf{W}$) can be written as a product of at most N' reflections.

Remark (3.4.4). — Put $M = \bigcup_{g \in G} gBg^{-1}$ and let N be an integer satisfying the conditions of (3.4.3). One then has $G = M^{2N}$. Indeed, for every reflection $r \in \mathbf{W}$ there exists $x \in \mathbf{W}$ such that $r \in T_x$ (for example r itself). One then has $BrB \subset BxBx^{-1}B \subset B \cdot M \subset M^2$. If $w \in \mathbf{W}$ is the product of $k \leqslant N$ reflections r_j, one therefore has $BwB \subset \prod_j Br_jB \subset M^{2N}$.

(3.4.5) Let us now prove theorem (3.4.1). Let $\mathscr{B}$ be a bornology on G, compatible with the group law and such that B is bounded. For each $j = 1, \ldots, m$, consider the bornology $\mathscr{B}_j$ induced by $\mathscr{B}$ on $B_j = B\mathbf{W}_jB$, and let $\mathscr{B}_\mathbf{W}$ (resp. $\mathscr{B}_{\mathbf{W}_j}$) be the bornology on $\mathbf{W}$ (resp. $\mathbf{W}_j$) deduced from $\mathscr{B}$ (resp. $\mathscr{B}_j$) as in (3.1.5). If there exists a subset X of $\mathbf{W}_j$ which is infinite and bounded for $\mathscr{B}_{\mathbf{W}_j}$, lemma (3.4.2) together with (3.1.5) shows that the set of reflections of $\mathbf{W}_j$ is bounded, hence, by (3.4.3), that $\mathbf{W}_j$ itself is bounded for $\mathscr{B}_j$. In other words, either $B_j \in \mathscr{B}$, or $\mathscr{B}_j$ is the bornology defined by the Tits system $(B_j, B, B_j \cap N)$.

Let then $J = \{ j \in \{1, \ldots, m\} \mid B_j \in \mathscr{B} \}$. We shall show that $\mathscr{B}$ is the bornology defined by (G, B^J, N). Replacing B by B^J, we may suppose that $J = \varnothing$. Let then $X \in \mathscr{B}_\mathbf{W}$ and let Y be the set of reflections associated with the various points of X. We have seen (3.1.5) that Y is bounded for $\mathscr{B}_{\mathbf{W}}$; consequently, by the first part of the proof, $Y \cap \mathbf{W}_i$ is *finite* for every i. Since every reflection of $\mathbf{W}$ is contained in one of the $\mathbf{W}_i$, it follows that Y itself is finite. Hence X is finite, whence our assertion.

Finally the uniqueness of J is clear: it is the largest subset of $\{1, \ldots, m\}$ such that B^J is bounded.

Corollary (3.4.6). — *The bornology defined by (G, B, N) is contained in every bornology compatible with the group law of G and for which B is bounded. The bornologies defined by the irreducible components of (G, B, N) are the maximal bornologies among the bornologies compatible with the group law of G for which B is bounded and G is not. If the system (G, B, N) is irreducible, the bornology it defines is the unique bornology compatible with the group law of G for which B is bounded and G is not.*

Corollary (3.4.7). — *Let $\varphi : G \to \widehat{G}$ be a B-adapted homomorphism. The bornology defined on $\widehat{G}$ by the Tits system (G, B, N) is the unique bornology on $\widehat{G}$ which is compatible with the group structure of $\widehat{G}$ and for which $\widehat{B}$ is bounded and $\varphi(B^J)$ is unbounded for every non-empty subset J of $\{1, \ldots, m\}$.*

If in particular the Tits system (G, B, N) is irreducible, the bornology it defines in $\widehat{G}$ is the unique bornology compatible with the group structure of $\widehat{G}$ for which $\widehat{B}$ is bounded and $\widehat{G}$ is not.

If a bornology on $\widehat{G}$ satisfies the conditions of the first part of the statement, it induces on the subgroup $\widehat{G}_0 = \operatorname{Ker} \xi$ (with the notation of (1.2.17)) the bornology defined by the Tits system $(\widehat{B}, \varphi(N))$, which is obviously the same as the bornology defined in $\widehat{G}_0$ by (G, B, N) and the homomorphism $\varphi : G \to \widehat{G}_0$. Now $\widehat{G}_0$ is of finite index in $\widehat{G}$, whence the first assertion of the corollary. The second follows.

3.5. Characterization of a Tits system of affine type by its bornology. —

Theorem (3.3.1) shows that the maximal parahoric subgroups of the Tits system (G, B, N) are well determined by the datum of the bornology defined by this Tits system. More precisely:

Theorem (3.5.1). — *Two Tits systems of affine type in the same group are equivalent if and only if they define the same bornology.*

Let us begin by establishing two lemmas.

Lemma (3.5.2). — *Suppose $\mathbf{W}$ irreducible and let P, P' be two distinct maximal parahoric subgroups. In order that $P \cap P'$ be a parahoric subgroup, it is necessary and sufficient that it be a maximal subgroup of P.*

If $P \cap P'$ is a parahoric subgroup, one may assume that $B \subset P \cap P'$. Since $\mathbf{W}$ is irreducible, there exist two distinct elements $s, s' \in \mathbf{S}$ such that $P = B_{\mathbf{S}-\{s\}}$ and $P' = B_{\mathbf{S}-\{s'\}}$, and $P \cap P' = B_{\mathbf{S}-\{s,s'\}}$ is maximal in P.

Conversely, suppose $P \cap P'$ maximal in P. Let D be a ball with center $x = a_P$ (2.1.2) possessing the properties of lemma (2.5.11) and let $a \in D \cap [a_P a_{P'}]$ with $a \neq a_P$. Let Q be the parahoric subgroup such that $a \in F(Q)$. One has $P \cap P' \subset Q$ by (2.5.4)(iii) and $a_P \in \overline{F(Q)}$ by (2.5.11), whence $Q \subset P$. One deduces $P \cap P' = Q$.

Lemma (3.5.3). — *Let $P_1, \ldots, P_q$ be parahoric subgroups. The intersection $P_1 \cap \cdots \cap P_q$ is a parahoric subgroup if and only if $P_h \cap P_k$ is a parahoric subgroup for $1 \leqslant h, k \leqslant q$.*

That the condition is necessary is evident. Let us show that it is sufficient by induction on q. Suppose that $Q = P_1 \cap \cdots \cap P_{q-1}$ is a parahoric subgroup and let A be an apartment containing

$F(Q)$ and $F(P_q)$. Let Γ be the convex hull of the union of the $F(P_h)$ for $1 \leqslant h \leqslant q$ and let L be a wall of A. Since $F(P_h)$ and $F(P_k)$ are two facets of A situated on the same side of L (for $1 \leqslant h, k \leqslant q$), the set Γ lies entirely on one side of L. This being true for every wall, Γ is therefore contained in the closure of a facet $F(Q')$, and one has $P_h \supset Q'$ for all h, which entails that the intersection of the P_h is indeed a parahoric subgroup.

(3.5.4) Let us now pass to the *proof of theorem* (3.5.1). We have already seen that the condition is necessary (3.1.4). Let us show that it is sufficient. Let (B, N) and (B', N') be two Tits systems of affine type in the same group G, defining the same bornology. By (3.4.6), there exists a bijection from the set of irreducible components of (G, B, N) onto that of the irreducible components of (G, B', N') such that the bornologies defined by two irreducible components corresponding under this bijection are the same. Taking (2.6.4) into account, we see that it suffices to prove our assertion in the case where (B, N) and (B', N') are irreducible, that is, in view of (3.4.6), when the bornology they define is maximal among the bornologies compatible with the group law of G for which G is not bounded. Since (B, N) and (B', N') have the same maximal parahoric subgroups, lemmas (3.5.2) and (3.5.3) show that they have the same parahoric subgroups, that is, they are equivalent.

Remark (3.5.5). — The bornology defined by a Tits system of affine type is completely determined by the datum of the maximal parahoric subgroups, and even by the datum of a single parahoric subgroup, since the bounded subsets are those which are contained in the union of finitely many double cosets modulo a given parahoric subgroup. We thus see that two Tits systems of affine type in the same group having the same maximal parahoric subgroups, or having a common parahoric subgroup, are equivalent.

Let us point out that one can prove that two (arbitrary) Tits systems in the same group having the same maximal *parabolic* subgroups are equivalent.

Proposition (3.5.6). — *Let $\varphi : G \to \widehat{G}$ be a B-adapted homomorphism. Let $\widehat{G}_0$ be the kernel of the corresponding homomorphism $\xi : \widehat{G} \to$* $\operatorname{Aut}\operatorname{Cox}(\mathbf{W}, \mathbf{S})$ *(1.2.16). The group $\widehat{G}_0$ is the intersection of the subgroups of finite index of $\widehat{G}$ which contain a maximal bounded subgroup of $\widehat{G}$ (for the bornology defined by φ).*

One may suppose $G = \widehat{G}_0$ and $\varphi = \mathrm{id}$ (1.2.17). If L is a subgroup of finite index of $\widehat{G}$ containing a maximal bounded subgroup, then $L \cap G$ contains a parahoric subgroup of G (3.3.4) and consequently $L \cap G = G$, since a parabolic subgroup of the Tits system (G, B, N) is not of finite index in G unless it is equal to G.

Now let $g \in \widehat{G} - G$ and let X be an orbit of $\xi(g)$ in $\operatorname{Cox}(\mathbf{W}, \mathbf{S})$ not reduced to a point. There exists a subset $Y \subset \mathbf{S}$ which is a type of maximal parahoric subgroup of G and which is such that $X \not\subset Y$ and $X \not\subset \mathbf{S} - Y$: if $\operatorname{Card}(X \cap \mathbf{S}_i) \leqslant 1$ for every irreducible component $\mathbf{S}_i$ of $\operatorname{Cox}(\mathbf{W}, \mathbf{S})$ — which implies that $(\mathbf{W}, \mathbf{S})$ is not irreducible — it suffices to choose a point x_i in each $\mathbf{S}_i$ in such a way that for one index i_1 one has $x_{i_1} \in X$ and for an index i_2 one has $x_{i_2} \notin X$ (recall that $\operatorname{Card}\mathbf{S}_i \geqslant 2$ for all i), and to take for Y the complement in $\mathbf{S}$ of the set of the x_i; if there exists an index i_0 such that $\operatorname{Card}(X \cap \mathbf{S}_{i_0}) \geqslant 2$, it suffices to take Y such that $\mathbf{S}_{i_0} - (\mathbf{S}_{i_0} \cap Y)$ is reduced to a point of X. Let then Ξ_Y be the stabilizer of Y in $\Xi = \xi(\widehat{G})$ and let $L = \xi^{-1}(\Xi_Y)$. Then L contains the maximal bounded subgroup $\operatorname{Stab} B_Y$, does not contain g, and is of finite index in $\widehat{G}$.

(3.5.7) Proposition (3.5.6) shows that the datum of the bornology of $\widehat{G}$ determines $\widehat{G}_0$ and also, by (3.5.1), the Tits system of $\widehat{G}_0$ up to equivalence, hence the building of $\widehat{G}$, etc., that is to say, everything that the B-adapted homomorphism φ brings in substance to the study of $\widehat{G}$.

4. Iwasawa and Cartan decompositions

We keep the notation of the preceding sections.

We denote by $\varphi : G \to \widehat{G}$ a B-N-adapted homomorphism. Recall that in 2.7 we defined an action of $\widehat{G}$ on the building $\mathscr{I}$, for which the operations of $\widehat{G}$ are isometric automorphisms of polysimplicial complex and permute among themselves the apartments (resp. half-apartments, walls, sectors) of $\mathscr{I}$.

4.1. Fixers and stabilizers. —

(4.1.1) Let Ω be a subset of $\mathscr{I}$; we set

$$\begin{aligned}
\widehat{P}_\Omega &= \{ g \in \widehat{G} \mid g.x = x \text{ for all } x \in \Omega \} \\
\widehat{P}^\dagger_\Omega &= \{ g \in \widehat{G} \mid g.\Omega = \Omega \} \\
P_\Omega &= \varphi^{-1}(\widehat{P}_\Omega) = \{ g \in G \mid g.x = x \text{ for all } x \in \Omega \} \\
P^\dagger_\Omega &= \varphi^{-1}(\widehat{P}^\dagger_\Omega) = \{ g \in G \mid g.\Omega = \Omega \}.
\end{aligned}$$

When Ω reduces to a point x, we write P_x, etc., instead of $P_{\{x\}}$, etc. When $\Omega = \mathbf{C}$ (identified with $j(\mathbf{C})$), one has $P_\mathbf{C} = P^\dagger_\mathbf{C} = B$, the subgroup $\widehat{P}_\mathbf{C}$ is none other than the subgroup $\widehat{B}$ introduced in (2.7.1), and $\widehat{P}^\dagger_\mathbf{C} = \operatorname{Stab} B$.

The subgroups P_Ω and $\widehat{P}_\Omega$ do not change if Ω is replaced by its closed convex hull ((2.5.4)(iii)).

It is clear that $g\widehat{P}_\Omega g^{-1} = \widehat{P}_{g.\Omega}$, etc., for $\Omega \subset \mathscr{I}$ and $g \in \widehat{G}$.

Since the set of fixed points of an element $g \in G$ in $\mathbf{A}$ is a convex union of closures of facets, one always has

$$P_\Omega = P_{\mathrm{cl}(\Omega)}. \tag{1}$$

If Ω contains a chamber C, one also has

$$\widehat{P}_\Omega = \widehat{P}_{\mathrm{cl}(\Omega)}. \tag{2}$$

Indeed, one may suppose $C = \mathbf{C}$. One then has $\widehat{P}_\Omega \subset \widehat{B} \subset \widehat{G}_0 = \operatorname{Ker} \xi$, and (2) follows from (1) applied to the Tits system $(\widehat{G}_0, \widehat{B}, \varphi(N))$ (1.2.17).

(4.1.2) We denote by $\widehat{N}$ the stabilizer $\widehat{P}^\dagger_\mathbf{A}$ of the apartment $\mathbf{A}$ (identified with $j(\mathbf{A})$) in $\widehat{G}$. One has, by (2.2.5), $N = P^\dagger_\mathbf{A} = \varphi^{-1}(\widehat{N})$ and $\widehat{G} = \widehat{N} \cdot \varphi(G)$ (2.7.2). We denote by $\widehat{\nu}$ the obvious homomorphism from $\widehat{N}$ into the group of automorphisms of the Euclidean affine space $\mathbf{A}$, and we set $\widehat{\mathbf{W}} = \widehat{\nu}(\widehat{N})$. We write $\widehat{\mathbf{V}}$ for the subgroup of translations of $\widehat{\mathbf{W}}$. The group $\widehat{\mathbf{W}}$ is contained in the normalizer $\widetilde{\mathbf{W}}$ of $\mathbf{W}$ in the group of automorphisms of $\mathbf{A}$. Consequently, the subgroups $\mathbf{W}$, $\mathbf{V}$ and $\widehat{\mathbf{V}}$ are normal in $\widehat{\mathbf{W}}$.

We set $\widehat{H} = \operatorname{Ker} \widehat{\nu}$; it is clear that $\widehat{H} = \widehat{P}_\mathbf{A}$ and that $H = \varphi^{-1}(\widehat{H})$. For every chamber C contained in $\mathbf{A}$, one has $\widehat{N} \cap \widehat{P}_C = \widehat{H}$.

Definition (4.1.3). — *We shall say that the B-N-adapted homomorphism φ is* of connected type *if the images ${}^v\mathbf{W}$ and ${}^v\widehat{\mathbf{W}}$ of $\mathbf{W}$ and $\widehat{\mathbf{W}}$ in the group of automorphisms of ${}^v\mathbf{A}$ coincide, in other words if $\widehat{\mathbf{W}} \subset \widetilde{\mathbf{W}}_{\mathrm{int}}$ ((1.3.18)(6)).*

If φ is of connected type and if x is a special point of $\mathbf{A}$, one has $\widehat{\mathbf{W}}_x = \mathbf{W}_x$ and $\widehat{\mathbf{W}} = \mathbf{W}_x \cdot \widehat{\mathbf{V}}$.

Proposition (4.1.4). — *For every subset Ω contained in* $\mathbf{A}$, *put* $\widehat{N}^{\dagger}_{\Omega} = \widehat{P}^{\dagger}_{\Omega} \cap \widehat{N}$ *and* $\widehat{N}_{\Omega} = \widehat{P}_{\Omega} \cap \widehat{N}$. *One has*

$$\widehat{P}^{\dagger}_{\Omega} = \widehat{N}^{\dagger}_{\Omega} \cdot \varphi(P^{\dagger}_{\Omega}) \qquad \textit{and} \qquad \widehat{P}_{\Omega} = \widehat{N}_{\Omega} \cdot \varphi(P_{\Omega}).$$

If Ω contains a non-empty open set, one has $\widehat{P}_{\Omega} = \widehat{H} \cdot \varphi(P_{\Omega})$.

Let $g \in \widehat{P}^{\dagger}_{\Omega}$; by (2.5.8), there exists $g' \in P^{\dagger}_{\Omega}$ such that $g'.\mathbf{A} = g.\mathbf{A}$. One then has $\varphi(g'^{-1})g \in \widehat{N}^{\dagger}_{\Omega}$, and even $\varphi(g'^{-1})g \in \widehat{N}_{\Omega}$ if $g \in \widehat{P}_{\Omega}$; whence the first assertion. If Ω contains a non-empty open set, one has $\widehat{\nu}(n) = 1$ for all $n \in \widehat{N}_{\Omega}$, which completes the proof of the proposition.

(4.1.5) If $\mathbf{D}$ is a vector chamber (1.3.10), we shall denote by $\widehat{\mathbf{V}}_{\mathbf{D}}$ the set of elements of $\widehat{\mathbf{V}}$ belonging to $\overline{\mathbf{D}}$, and by $E_{\mathbf{D}}$ the set of sectors of $\mathbf{A}$ of direction $\mathbf{D}$ (1.3.11). If $\mathfrak{C}_1, \mathfrak{C}_2 \in E_{\mathbf{D}}$, one also has $\mathfrak{C}_1 \cap \mathfrak{C}_2 \in E_{\mathbf{D}}$: this follows immediately from the fact that $\mathbf{D}$ is a simplicial cone, the set of points with strictly positive coordinates for a suitable basis of ${}^{v}\mathbf{A}$. Since $\widehat{P}_{\mathfrak{C}_1} \cup \widehat{P}_{\mathfrak{C}_2} \subset \widehat{P}_{\mathfrak{C}_1 \cap \mathfrak{C}_2}$, the subgroups $\widehat{P}_{\mathfrak{C}}$ for $\mathfrak{C} \in E_{\mathbf{D}}$ form an increasing filtered family and their *union* is a subgroup of G which we shall denote by $\widehat{\mathfrak{B}}^0_{\mathbf{D}}$.

For $g \in \widehat{\nu}^{-1}(\widehat{\mathbf{V}})$ and $\mathfrak{C} \in E_{\mathbf{D}}$, one has $\widehat{\nu}(g) \cdot \mathfrak{C} \in E_{\mathbf{D}}$ and $g\widehat{P}_{\mathfrak{C}}g^{-1} = \widehat{P}_{\widehat{\nu}(g).\mathfrak{C}}$. Consequently, the subgroup $\widehat{\nu}^{-1}(\widehat{\mathbf{V}})$ normalizes $\widehat{\mathfrak{B}}^0_{\mathbf{D}}$ and $\widehat{\nu}^{-1}(\widehat{\mathbf{V}}) \cdot \widehat{\mathfrak{B}}^0_{\mathbf{D}}$ is a *subgroup* of $\widehat{G}$, which we shall denote by $\widehat{\mathfrak{B}}_{\mathbf{D}}$. We set $\mathfrak{B}^0_{\mathbf{D}} = \varphi^{-1}(\widehat{\mathfrak{B}}^0_{\mathbf{D}})$ and $\mathfrak{B}_{\mathbf{D}} = \varphi^{-1}(\widehat{\mathfrak{B}}_{\mathbf{D}})$. It is immediate that $\mathfrak{B}_{\mathbf{D}} = \nu^{-1}(\mathbf{V}) \cdot \mathfrak{B}^0_{\mathbf{D}}$, that $\widehat{\mathfrak{B}}_{\mathbf{D}} = \widehat{\nu}^{-1}(\widehat{\mathbf{V}}) \cdot \varphi(\mathfrak{B}_{\mathbf{D}})$ and that $\widehat{\mathfrak{B}}^0_{\mathbf{D}} = \widehat{H} \cdot \varphi(\mathfrak{B}^0_{\mathbf{D}})$.

In the remainder of this section, we fix a vector chamber $\mathbf{D}$ *and we shall in general omit the index* $\mathbf{D}$ *in the notation* $\widehat{\mathfrak{B}}^0_{\mathbf{D}}$, $\widehat{\mathfrak{B}}_{\mathbf{D}}$, $\mathfrak{B}^0_{\mathbf{D}}$ *and* $\mathfrak{B}_{\mathbf{D}}$.

4.2. Double cosets. —

Let Q and Q' be two subgroups of $\widehat{G}$ containing $\widehat{H}$. Put

$$\widehat{\mathbf{W}}_Q = \widehat{\nu}(\widehat{N} \cap Q), \qquad \widehat{\mathbf{W}}_{Q'} = \widehat{\nu}(\widehat{N} \cap Q').$$

Since $\widehat{H} = \operatorname{Ker}\widehat{\nu} \subset Q \cap Q' \cap \widehat{N}$, the inverse image under $\widehat{\nu}$ of a double coset $\widehat{\mathbf{W}}_Q w \widehat{\mathbf{W}}_{Q'}$ in $\widehat{\mathbf{W}}$ is a double coset $(\widehat{N} \cap Q)n(\widehat{N} \cap Q')$ in $\widehat{N}$ (where n is any element of $\widehat{\nu}^{-1}(w)$) and is therefore contained in a single double coset QnQ' of G, contained in $Q\widehat{N}Q'$. One thus obtains a map, called canonical and denoted $\lambda_{Q,Q'}$, from $\widehat{\mathbf{W}}_Q \backslash \widehat{\mathbf{W}} / \widehat{\mathbf{W}}_{Q'}$ into $Q \backslash Q\widehat{N}Q'/Q'$, a map which is obviously surjective.

Proposition (4.2.1). — *Let Ω and Ω' be two subsets of* $\mathbf{A}$ *and let Q and Q' be two subgroups of $\widehat{G}$ such that*

$$\widehat{H} \cdot \varphi(P_{\Omega}) \subset Q \subset \widehat{P}^{\dagger}_{\Omega} \qquad \textit{and} \qquad \widehat{H} \cdot \varphi(P_{\Omega'}) \subset Q' \subset \widehat{P}^{\dagger}_{\Omega'}.$$

The map $\lambda_{Q,Q'}$ from $\widehat{\mathbf{W}}_Q \backslash \widehat{\mathbf{W}} / \widehat{\mathbf{W}}_{Q'}$ into $Q \backslash Q\widehat{N}Q'/Q'$ is bijective.

It suffices to show that if one has $qn = n'q'$, with $n, n' \in \widehat{N}$, $q \in Q$ and $q' \in Q'$, then $n' \in (\widehat{N} \cap Q)n(\widehat{N} \cap Q')$. Now, one has $\Omega \subset \mathbf{A} \cap q(\mathbf{A})$ and $qn(\Omega') = n'(\Omega') \subset \mathbf{A} \cap q(\mathbf{A})$. Moreover, there exists an element $g_0 \in G$ such that $q(\mathbf{A}) = g_0(\mathbf{A})$ and $g_0.x = x$ for all $x \in \mathbf{A} \cap q(\mathbf{A})$ (2.5.8). One has $g_0 \in P_{\Omega} \cap P_{n'(\Omega')}$ and $g = \varphi(g_0) \in Q \cap n'Q'n'^{-1}$, whence

$$m = g^{-1}qn = g^{-1}n'q' \in Qn \cap n'Q' \cap \widehat{N}.$$

Consequently, $n \in (\widehat{N} \cap Q)m$ and $n' \in m(\widehat{N} \cap Q')$, q.e.d.

(4.2.2) Let us now take for the subgroup Q' one of the subgroups $\widehat{\mathfrak{B}}^0$ or $\widehat{\mathfrak{B}}$. One checks at once that $\widehat{\mathbf{W}}_{\widehat{\mathfrak{B}}^0} = \{e\}$ and $\widehat{\mathbf{W}}_{\widehat{\mathfrak{B}}} = \widehat{\mathbf{V}}$.

Corollary. — *Let Ω be a subset of $\mathbf{A}$ and Q a subgroup of $\widehat{G}$ such that $\widehat{H}\cdot\varphi(P_\Omega)\subset Q\subset\widehat{P}^\dagger_\Omega$. The following canonical maps are bijective:*

$$\begin{array}{ll} (i) & \widehat{\mathbf{W}}_Q\backslash\widehat{\mathbf{W}}\to Q\backslash Q\widehat{N}\widehat{\mathfrak{B}}^0/\widehat{\mathfrak{B}}^0; \\ (ii) & \widehat{\mathbf{W}}_Q\backslash\widehat{\mathbf{W}}/\widehat{\mathbf{V}}\to Q\backslash Q\widehat{N}\widehat{\mathfrak{B}}/\widehat{\mathfrak{B}}; \\ (iii) & \widehat{\mathbf{W}}\to\widehat{\mathfrak{B}}^0\backslash\widehat{\mathfrak{B}}^0\widehat{N}\widehat{\mathfrak{B}}^0/\widehat{\mathfrak{B}}^0; \\ (iv) & \widehat{\mathbf{W}}/\widehat{\mathbf{V}}\to\widehat{\mathfrak{B}}\backslash\widehat{\mathfrak{B}}\widehat{N}\widehat{\mathfrak{B}}/\widehat{\mathfrak{B}}. \end{array}$$

Here again, only the injectivity of these maps has to be proved. Let us prove that of (i); if $q_1n_1b_1=q_2n_2b_2$ (with $q_i\in Q$, $n_i\in\widehat{N}$ and $b_i\in\widehat{\mathfrak{B}}^0$), there exists a sector $\mathfrak{C}\in E_{\mathbf{D}}$ such that $b_i\in\widehat{P}_{\mathfrak{C}}$. One then has $Qn_1\widehat{P}_{\mathfrak{C}}=Qn_2\widehat{P}_{\mathfrak{C}}$. Since $\widehat{\mathbf{W}}_{\widehat{P}_{\mathfrak{C}}}=\{e\}$, one deduces from proposition (4.2.1) that $\widehat{\mathbf{W}}_Q\widehat{\nu}(n_1)=\widehat{\mathbf{W}}_Q\widehat{\nu}(n_2)$, whence the injectivity of the map (i).

Since $\widehat{\mathfrak{B}}=\widehat{\nu}^{-1}(\widehat{\mathbf{V}})\cdot\widehat{\mathfrak{B}}^0$, the injectivity of (ii) follows from that of (i). One proves in the same way the injectivity of (iii) and (iv) (taking into account, for the definition of the map (iv), that $\widehat{\mathbf{V}}$ is normal in $\widehat{\mathbf{W}}$, whence $\widehat{\mathbf{V}}\backslash\widehat{\mathbf{W}}/\widehat{\mathbf{V}}=\widehat{\mathbf{W}}/\widehat{\mathbf{V}}$).

Corollary (4.2.3). — *Keep the hypotheses of corollary (4.2.2) and suppose moreover that Q contains the subgroup $\widehat{B}$. Then the following two conditions are equivalent:*

(i) $\widehat{B}\widehat{N}\widehat{\mathfrak{B}}\subset Q\widehat{\mathfrak{B}}$;
(ii) $\widehat{\mathbf{W}}=\widehat{\mathbf{W}}_Q\cdot\widehat{\mathbf{V}}$.

Indeed, condition (i) amounts to $Q\widehat{N}\widehat{\mathfrak{B}}=Q\widehat{\mathfrak{B}}$ and is therefore equivalent to (ii) by corollary (4.2.2)(ii).

(4.2.4) Proposition (4.2.1) will interest us above all when $\widehat{G}=Q\widehat{N}Q'$. Let us note moreover that one has

$$(1)\qquad Q\widehat{N}Q'=\varphi(P_\Omega)\widehat{N}\varphi(P_{\Omega'})=\widehat{P}_\Omega\widehat{N}\widehat{P}_{\Omega'}=\widehat{P}^\dagger_\Omega\widehat{N}\widehat{P}^\dagger_{\Omega'}.$$

Indeed, let $q\in Q$; one has $\Omega\subset\mathbf{A}\cap q(\mathbf{A})$ and there exists an element $g\in P_\Omega$ such that $\varphi(g)q.\mathbf{A}=\mathbf{A}$ (2.5.8), whence $\varphi(g)q\in\widehat{N}\cap Q$. Consequently, one has $Q=\varphi(P_\Omega)(\widehat{N}\cap Q)$, whence the first of the equalities (1). The other two follow, taking for example $Q=\widehat{P}_\Omega$ or $Q=\widehat{P}^\dagger_\Omega$.

The relation $\widehat{G}=Q\widehat{N}Q'$ has a simple geometric interpretation: it means that *whatever $g,g'\in\widehat{G}$, there exists an apartment containing $g(\Omega)$ and $g'(\Omega')$*. Indeed, suppose $\widehat{G}=Q\widehat{N}Q'$ and put $g^{-1}g'=qnq'$ (with $g,g'\in\widehat{G}$, $q\in Q$, $q'\in Q'$ and $n\in\widehat{N}$). One then has $g'(\Omega')=gqn(\Omega')\subset gq(\mathbf{A})$ and $g(\Omega)=gq(\Omega)\subset gq(\mathbf{A})$. Conversely, if $g(\Omega)\cup\Omega'\subset h(\mathbf{A})$ (with $g,h\in\widehat{G}$), there exists, by (2.5.8), an element $q'\in Q'$ such that $q'h(\mathbf{A})=\mathbf{A}$, whence $q'g(\Omega)\subset\mathbf{A}$ and $\Omega\subset\mathbf{A}\cap g^{-1}q'^{-1}(\mathbf{A})$. Again by (2.5.8), there exists an element $q\in Q$ such that $qg^{-1}q'^{-1}(\mathbf{A})=\mathbf{A}$, that is such that $qg^{-1}q'^{-1}\in\widehat{N}$, whence finally $g^{-1}\in Q\widehat{N}Q'$.

(4.2.5) In an analogous way, the relation $\widehat{G}=\widehat{\mathfrak{B}}\widehat{N}\widehat{\mathfrak{B}}$ (which is equivalent to $\widehat{G}=\widehat{\mathfrak{B}}^0\widehat{N}\widehat{\mathfrak{B}}^0$) *means that, whatever the sectors $\mathfrak{C}$ and $\mathfrak{C}'$ of the building $\mathscr{I}$, there exist sectors $\mathfrak{C}_1\subset\mathfrak{C}$ and $\mathfrak{C}'_1\subset\mathfrak{C}'$ contained in one and the same apartment.* Indeed, suppose first that $\widehat{G}=\widehat{\mathfrak{B}}^0\widehat{N}\widehat{\mathfrak{B}}^0$ and let $\mathfrak{C}$ and $\mathfrak{C}'$ be two sectors of $\mathscr{I}$. In view of the transitivity of ${}^v\mathbf{W}$ on the vector chambers and in view of the very definition of the sectors of $\mathscr{I}$ (2.2.2), there exist elements $g,g'\in\widehat{G}$ such that $\mathfrak{C}=g.\mathfrak{C}_0$ and $\mathfrak{C}'=g'.\mathfrak{C}'_0$, where $\mathfrak{C}_0$, $\mathfrak{C}'_0$ are sectors of $\mathbf{A}$ of direction $\mathbf{D}$. Put then $g^{-1}g'=bnb'$ with $n\in\widehat{N}$, $b,b'\in\widehat{\mathfrak{B}}^0$. Replacing $\mathfrak{C}$ and $\mathfrak{C}'$ by smaller sectors, we may suppose $\mathfrak{C}'_0=\mathfrak{C}_0$ and $b,b'\in\widehat{P}_{\mathfrak{C}_0}$. One then has $\mathfrak{C}'=gbnb'.\mathfrak{C}_0\subset gb(\mathbf{A})$ and $\mathfrak{C}=gb.\mathfrak{C}_0\subset gb(\mathbf{A})$.

Conversely, let $g \in \widehat{G}$; let $\mathfrak{C} \in E_{\mathbf{D}}$ and suppose that there exist sectors $\mathfrak{C}_1 \subset \mathfrak{C}$ and $\mathfrak{C}'_1 \subset g(\mathfrak{C})$ contained in one and the same apartment. Replacing $\mathfrak{C}$ by a smaller sector, we may suppose $\mathfrak{C}_1 = \mathfrak{C}$ and $\mathfrak{C}'_1 = g(\mathfrak{C})$. Let then $h \in \widehat{G}$ be such that $\mathfrak{C} \cup g(\mathfrak{C}) \subset h(\mathbf{A})$. By (2.5.8), there exists $b \in \widehat{P}_{\mathfrak{C}} \subset \widehat{\mathfrak{B}}^0$ such that $bh(\mathbf{A}) = \mathbf{A}$, and one then has $\mathfrak{C} \subset \mathbf{A} \cap g^{-1}b^{-1}(\mathbf{A})$, whence the existence of $b' \in \widehat{\mathfrak{B}}^0$ such that $b'g^{-1}b^{-1}(\mathbf{A}) = \mathbf{A}$, which entails $g \in \widehat{\mathfrak{B}}^0\widehat{N}\widehat{\mathfrak{B}}^0$.

Entirely analogous reasoning shows that the relation $\widehat{G} = Q\widehat{N}\widehat{\mathfrak{B}}$ (or $\widehat{G} = Q\widehat{N}\widehat{\mathfrak{B}}^0$) *means that, whatever the sector $\mathfrak{C}$ of $\mathscr{I}$, there exist a sector $\mathfrak{C}' \subset \mathfrak{C}$ and an apartment containing Ω and $\mathfrak{C}'$.* In particular, the relation $G = BN\mathfrak{B}$ *means that for every chamber C and every sector $\mathfrak{C}$ of $\mathscr{I}$, there exist a sector $\mathfrak{C}' \subset \mathfrak{C}$ and an apartment containing C and $\mathfrak{C}'$.*

4.3. Relations between double cosets. —

Proposition (4.3.1). — *Let Ω, Ω' and Ω'' be three subsets of $\mathbf{A}$. Suppose that one of them has non-empty interior and that one of the two following conditions is satisfied:*

a) Ω, Ω' and Ω'' are bounded;

b) G is complete for a topology satisfying the conditions of (2.8.2).

Then one has

$$\widehat{P}_{\Omega'} \cdot \widehat{P}_{\Omega} \,\cap\, \widehat{P}_{\Omega''} \cdot \widehat{P}_{\Omega} = (\widehat{P}_{\Omega'} \cap \widehat{P}_{\Omega''}) \cdot \widehat{P}_{\Omega}.$$

Let $g = p'p = p''q$ with $p' \in \widehat{P}_{\Omega'}$, $p'' \in \widehat{P}_{\Omega''}$, $p, q \in \widehat{P}_{\Omega}$. Put $M = \Omega' \cup \Omega'' \cup g.\Omega$ and let f be the map from $\Omega' \cup \Omega'' \cup \Omega$ onto M which is the identity on $\Omega' \cup \Omega''$ and coincides with the action of g on Ω (note that $g.x = x$ for $x \in \Omega \cap (\Omega' \cup \Omega'')$). Then f is an isometry: if $x \in \Omega$ and $y \in \Omega'$ for example, one has $d(g.x, y) = d(p'.x, p'.y) = d(x, y)$. Since M has non-empty interior, there exists an apartment A containing M ((2.8.1) and (2.8.4)) and there exists $h \in \widehat{P}_{\Omega'} \cap \widehat{P}_{\Omega''}$ with $h.A = \mathbf{A}$ (2.5.8), whence $hg.\Omega \subset \mathbf{A}$, and, in view of (2.5.9) combined with the relation $\widehat{G} = \varphi(G) \cdot \widehat{N}$, there exists $n \in \widehat{N}$ such that $hg.x = n.x$ for all $x \in \Omega$. For $x \in \Omega$ and $y \in \Omega' \cup \Omega''$, one then has

$$d(n^{-1}y, x) = d(y, n.x) = d(y, hg.x) = d(y, g.x) = d(y, x).$$

If Ω (resp. $\Omega' \cup \Omega''$) contains a chamber, one deduces $n^{-1}.y = y$ and $n \in \widehat{P}_{\Omega'} \cap \widehat{P}_{\Omega''}$ (resp. $n.x = x$ and $n \in \widehat{P}_{\Omega}$). In both cases, one has $g \in h^{-1}n\widehat{P}_{\Omega} \subset (\widehat{P}_{\Omega'} \cap \widehat{P}_{\Omega''}) \cdot \widehat{P}_{\Omega}$.

Corollary (4.3.2). — *Under the hypotheses of (4.3.1), one has*

$$\widehat{P}_{\Omega} \cap (\widehat{P}_{\Omega'} \cdot \widehat{P}_{\Omega''}) = (\widehat{P}_{\Omega} \cap \widehat{P}_{\Omega'}) \cdot (\widehat{P}_{\Omega} \cap \widehat{P}_{\Omega''}).$$

Indeed, if $g = p'p'' \in \widehat{P}_{\Omega}$ with $p' \in \widehat{P}_{\Omega'}$ and $p'' \in \widehat{P}_{\Omega''}$, proposition (4.3.1) shows that $p'' = p'^{-1}g \in (\widehat{P}_{\Omega'} \cap \widehat{P}_{\Omega''}) \cdot (\widehat{P}_{\Omega} \cap \widehat{P}_{\Omega''})$. One may therefore suppose $p'' \in \widehat{P}_{\Omega}$, and one then has $p' \in \widehat{P}_{\Omega}$.

(4.3.3) It is known that proposition (4.3.1) and corollary (4.3.2) are also true when Ω, Ω' and Ω'' are three facets of one and the same chamber ([5], chap. IV, § 2, exercise 9).

Corollary (4.3.4). — *Let $g \in \widehat{G}$ and $x \in \mathbf{A}$. Suppose that the set Y of the points y of $\mathbf{A}$ such that $g \in \widehat{P}_y\widehat{P}_x$ contains a chamber. Then Y is closed.*

It suffices to show that, if C is a chamber contained in Y and $y \in Y$, one has $\mathrm{cl}(C \cup \{y\}) \subset Y$. Now, one has

$$g \in \widehat{P}_C\widehat{P}_x \cap \widehat{P}_y\widehat{P}_x = (\widehat{P}_C \cap \widehat{P}_y) \cdot \widehat{P}_x = \widehat{P}_{\mathrm{cl}(C\cup\{y\})} \cdot \widehat{P}_x.$$

Corollary (4.3.5). — *Let Ω, Ω' and Ω'' be three non-empty subsets of* $\mathbf{A}$. *Suppose that $\Omega \cup \Omega' \cup \Omega''$ (resp. $\Omega' \cup \Omega''$) contains a chamber and that Ω is contained in the enclosure of $\Omega' \cup \Omega''$. Then one has*

$$P_{\Omega'}P_\Omega \cap P_{\Omega''}P_\Omega = P_\Omega \qquad (\textit{resp. } \widehat{P}_{\Omega'}\widehat{P}_\Omega \cap \widehat{P}_{\Omega''}\widehat{P}_\Omega = \widehat{P}_\Omega).$$

Indeed, one can write Ω, Ω' and Ω'' as unions of bounded subsets Ω_n, Ω'_n and Ω''_n in such a way that $\Omega_n \cup \Omega'_n \cup \Omega''_n$ (resp. $\Omega'_n \cup \Omega''_n$) contains a chamber and $\Omega_n \subset \mathrm{cl}(\Omega'_n \cup \Omega''_n)$. One then has, by (4.1.1)(1),

$$P_{\Omega'}P_\Omega \cap P_{\Omega''}P_\Omega \subset P_{\Omega'_n}P_{\Omega_n} \cap P_{\Omega''_n}P_{\Omega_n} = P_{\mathrm{cl}(\Omega'_n\cup\Omega''_n)}P_{\Omega_n} = P_{\Omega_n}$$

for all n, whence the first equality. The second is proved in the same way, using (4.1.1)(2) instead of (4.1.1)(1).

Corollary (4.3.6). — *Let Ω and Ω' be two non-empty convex subsets of* $\mathbf{A}$ *and let $n \in \widehat{N}$. One then has*

$$\widehat{P}_{\Omega'}n\widehat{P}_\Omega \cap \widehat{\mathfrak{B}}^0 n\widehat{P}_\Omega = n\widehat{P}_\Omega$$

whenever $n.\Omega \subset \mathrm{cl}(\Omega' + \mathbf{D})$, and in particular when $\Omega' = \Omega$ and $\widehat{\nu}(n) \in \widehat{\mathbf{V}}_{\mathbf{D}}$.

Replacing Ω by $n.\Omega$, hence $\widehat{P}_\Omega$ by $n\widehat{P}_\Omega n^{-1}$, we may suppose that $n = 1$. One then has

$$\widehat{P}_{\Omega'}\widehat{P}_\Omega \cap \widehat{\mathfrak{B}}^0\widehat{P}_\Omega = \bigcup_{\mathfrak{C}\in E_{\mathbf{D}}} \left(\widehat{P}_{\Omega'}\widehat{P}_\Omega \cap \widehat{P}_{\mathfrak{C}}\widehat{P}_\Omega\right).$$

But, for every sector $\mathfrak{C} \in E_{\mathbf{D}}$, the convex hull of $\mathfrak{C} \cup \Omega'$ contains $\Omega' + \mathbf{D}$, and the relation $\Omega \subset \mathrm{cl}(\Omega' + \mathbf{D})$ entails $\Omega \subset \mathrm{cl}(\Omega' \cup \mathfrak{C})$.

Corollary (4.3.7). — *Suppose G complete for a topology satisfying the conditions of (2.8.2). Let $\Omega \subset$* $\mathbf{A}$ *and $n \in \widehat{N}_\Omega$. Then one has*

$$\widehat{\mathfrak{B}}^0 n\widehat{\mathfrak{B}}^0 \cap \widehat{P}_\Omega = \widehat{P}_{\Omega+\mathbf{D}}\, n\, \widehat{P}_{\Omega+\mathbf{D}}.$$

It suffices to show that, for $\mathfrak{C} \in E_{\mathbf{D}}$, one has

$$\widehat{P}_{\mathfrak{C}}n\widehat{P}_{\mathfrak{C}} \cap \widehat{P}_\Omega \subset \widehat{P}_{\Omega+\mathbf{D}}\, n\, \widehat{P}_{\Omega+\mathbf{D}}$$

or again that

$$(\widehat{P}_{\mathfrak{C}} \cdot \widehat{P}_{n.\mathfrak{C}}) \cap \widehat{P}_\Omega \subset \widehat{P}_{\Omega+\mathbf{D}} \cdot \widehat{P}_{\Omega+n\mathbf{D}}$$

which follows from (4.3.2).

Proposition (4.3.8). — *Let x, y, z, u be four points of* $\mathbf{A}$ *such that y and z belong to the segment $[xu]$ and $d(x,y) \leqslant d(x,z)$. One has*

$$\widehat{P}_x\widehat{P}_u \cap \widehat{P}_y\widehat{P}_z = (\widehat{P}_x \cap \widehat{P}_y)\cdot(\widehat{P}_z \cap \widehat{P}_u).$$

More precisely, if $p \in \widehat{P}_x$ and $p' \in \widehat{P}_u$ are such that $pp' \in \widehat{P}_y\widehat{P}_z$, then $p \in \widehat{P}_y$ and $p' \in \widehat{P}_z$.

Put $g = pp' = qq'$ with $q \in \widehat{P}_y$ and $q' \in \widehat{P}_z$. One has $z' = g.z = q.z$ and $u' = g.u = p.u$, whence $d(y,z') = d(y,z)$ and $d(x,u') = d(x,u)$. One deduces:

(12) $$d(y,u') \leqslant d(y,g.z) + d(g.z,g.u) = d(y,z) + d(z,u) = d(y,u)$$

(13) $$d(x,u') \leqslant d(x,y) + d(y,u') \leqslant d(x,y) + d(y,u) = d(x,u) = d(x,u')$$

(14) $$d(x,z') \leqslant d(x,y) + d(y,z') = d(x,y) + d(y,z) = d(x,z) = d(x,p.z)$$

(15) $$d(x,u') \leqslant d(x,z') + d(z',u') \leqslant d(x,z) + d(z,u) = d(x,u) = d(x,u')$$

and one sees that all the inequalities appearing in (1) to (4) are in fact equalities. From (2) one then deduces that $y \in [xu'] = [x(p.u)]$, hence, in view of (1), that $y = p.y$, that is $p \in \widehat{P}_y$.

Similarly, (4) shows that $z' \in [xu'] = [x(p.u)]$ and, in view of (3), one has $z' = p.z$, whence $p' = p^{-1}g \in \widehat{P}_z$.

Corollary (4.3.9). — *Let $y, z \in \mathbf{A}$ and $t \in \widehat{\nu}^{-1}(\widehat{\mathbf{V}})$ be such that $t.z \in y + \mathbf{D}$. One has*

$$\widehat{\mathfrak{B}}^0_{-\mathbf{D}}\, t\, \widehat{\mathfrak{B}}^0_{\mathbf{D}} \cap \widehat{P}_y t \widehat{P}_z = \widehat{P}_{y-\mathbf{D}}\, t\, \widehat{P}_{z+\mathbf{D}}.$$

Multiplying on the right by t^{-1}, one reduces to the case $t = 1$. If $pp' \in \widehat{P}_y\widehat{P}_z$, with $p \in \widehat{\mathfrak{B}}^0_{-\mathbf{D}}$ and $p' \in \widehat{\mathfrak{B}}^0_{\mathbf{D}}$, there exists a sector $\mathfrak{C}'$ (resp. $\mathfrak{C}$) of direction $\mathbf{D}$ (resp. $-\mathbf{D}$) such that $p' \in \widehat{P}_{\mathfrak{C}'}$ (resp. $p \in \widehat{P}_{\mathfrak{C}}$), and there exist $x \in \mathfrak{C}$ and $u \in \mathfrak{C}'$ such that the hypotheses of (4.3.8) are satisfied. One then has $p \in \widehat{P}_y \cap \widehat{P}_{\mathfrak{C}} = \widehat{P}_{y-\mathbf{D}}$ and $p' \in \widehat{P}_z \cap \widehat{P}_{\mathfrak{C}'} = \widehat{P}_{z+\mathbf{D}}$.

Remark (4.3.10). — One can show that the preceding corollary remains valid if one supposes only that $t.z \in y + \overline{\mathbf{D}}$.

Lemma (4.3.11). — *Suppose $\varphi : G \to \widehat{G}$ of connected type. Let $x, y \in \mathbf{A}$ be such that $y \in x + \overline{\mathbf{D}}$. Let ρ be the retraction of $\mathscr{I}$ onto $\mathbf{A}$ having as center a chamber of $\mathbf{A}$. For every $g \in \widehat{G}$, one has $\rho(g.y) - \rho(g.x) \leqslant_{\mathbf{D}} y - x$ (cf. (1.3.16)).*

Let $(x_0 = x, \dots, x_k = y)$ be a monotone sequence of points of the segment $[xy]$, such that $[x_i x_{i+1}]$ is contained in the closure of a chamber C_i $(0 \leqslant i < k)$. There exists $h_i \in \widehat{G}$ such that the restriction of ρ to $g.\overline{C}_i$ coincides with the action of h_i, and consequently there exists $n_i \in \widehat{N}$ such that the restriction to $\overline{C}_i$ of the map $z \mapsto \rho(g.z)$ coincides with the action of n_i. Put $w_i = {}^v\widehat{\nu}(n_i)$. By (1.3.16), one has

$$\rho(g.x_{i+1}) - \rho(g.x_i) = w_i(x_{i+1} - x_i) \leqslant_{\mathbf{D}} x_{i+1} - x_i$$

since $x_{i+1} - x_i \in \overline{\mathbf{D}}$ and $w_i \in {}^v\mathbf{W}$. Adding these inequalities, one obtains the lemma.

Proposition (4.3.12). — *Let $x, y, z, u \in \mathbf{A}$ be such that $y - x$, $z - y$ and $u - z$ belong to $\overline{\mathbf{D}}$. If φ is of connected type, one has*

$$\widehat{P}_x\widehat{P}_y \cap \widehat{P}_u\widehat{P}_z = \widehat{P}_y \cap \widehat{P}_z.$$

Let $g \in p\widehat{P}_y \cap q\widehat{P}_z$, with $p \in \widehat{P}_x$ and $q \in \widehat{P}_u$. By (4.3.11), one has, for every retraction ρ of $\mathscr{I}$ onto $\mathbf{A}$ with center an arbitrary chamber C of $\mathbf{A}$:

$$\rho(g.y) - x = \rho(p.y) - \rho(p.x) \leqslant_{\mathbf{D}} y - x \tag{16}$$

$$\rho(g.z) - \rho(g.y) \leqslant_{\mathbf{D}} z - y \tag{17}$$

$$u - \rho(g.z) = \rho(q.u) - \rho(q.z) \leqslant_{\mathbf{D}} u - z. \tag{18}$$

Adding, one sees that these inequalities are in fact equalities. One therefore has $\rho(g.y) = y$ and $\rho(g.z) = z$. Choosing C in such a way that $y \in \overline{C}$ (resp. $z \in \overline{C}$), one obtains $g.y = y$ (resp. $g.z = z$).

Corollary (4.3.13). — *Let Ω, Ω' and Ω'' be three non-empty convex subsets of $\mathbf{A}$, such that Ω is contained in the union of the intersections $(\Omega' + \overline{D}) \cap (\Omega'' - \overline{D})$, for D running over the set of vector chambers of ${}^v\mathbf{A}$. One has*

(i) $\widehat{P}_{\Omega'}\widehat{P}_\Omega \cap \widehat{P}_{\Omega''}\widehat{P}_\Omega = \widehat{P}_\Omega$

(ii) $\widehat{P}_\Omega \cap \widehat{P}_{\Omega'}\widehat{P}_{\Omega''} = (\widehat{P}_\Omega \cap \widehat{P}_{\Omega'}) \cdot (\widehat{P}_\Omega \cap \widehat{P}_{\Omega''})$.

Let $g \in \widehat{P}_{\Omega'}\widehat{P}_{\Omega} \cap \widehat{P}_{\Omega''}\widehat{P}_{\Omega}$ and let $a \in \Omega$; there exist a vector chamber D and elements $b \in \Omega'$ and $c \in \Omega''$ such that $a - b$ and $c - a$ belong to $\overline{D}$. Applying then (4.3.12) with $\mathbf{D}$ replaced by D, x by b, y and z by a and u by c, one obtains $g \in \widehat{P}_a$, whence (i). Relation (ii) is deduced as in (4.3.2).

Remark (4.3.14). — The hypothesis of (4.3.13) is satisfied when Ω is contained in the convex hull of $\Omega' \cup \Omega''$.

Proposition (4.3.15). — *Let $x, y, z, u \in \mathbf{A}$, $n \in \widehat{N}$ and $w = \widehat{\nu}(n)$. If $\widehat{P}_x\widehat{P}_u \cap \widehat{P}_z n\widehat{P}_y \neq \varnothing$, there exists $w' \in \widehat{\mathbf{W}}_z$ such that $d(x, w'w.y) \leqslant d(x,u) + d(u,y)$. If moreover φ is of connected type and if $y \in u + \overline{\mathbf{D}} \subset x + \overline{\mathbf{D}}$, one may choose w' such that $w'w.y \leqslant_{\mathbf{D}} y$. If in addition $x \in z + \overline{\mathbf{D}}$, then $w.y \leqslant_{\mathbf{D}} y$.*

Let $g = g'g'' \in hn\widehat{P}_y$, with $g' \in \widehat{P}_x$, $g'' \in \widehat{P}_u$ and $h \in \widehat{P}_z$. Let C be a chamber of $\mathbf{A}$ such that $z \in \overline{C}$ and let ρ be the retraction of $\mathscr{I}$ onto $\mathbf{A}$ with center C. There exists $h' \in \widehat{P}_C$ such that $\rho(g.y) = h'g.y = h'hn.y$. On the other hand, since z and $n.y$ belong to $\mathbf{A} \cap (h'h)^{-1}.\mathbf{A}$, there exists $h'' \in \widehat{G}$ with $\mathbf{A} = h''(h'h)^{-1}.\mathbf{A}$, $h''.z = z$ and $h''n.y = y$ (2.5.8). Put $n' = h'hh''^{-1}$: one has $n' \in \widehat{N}$ and $w' = \widehat{\nu}(n') \in \widehat{\mathbf{W}}_z$. Moreover, $\rho(g.y) = h'hn.y = h'hh''^{-1}n.y = n'n.y = w'w.y$. Consequently,

$$\begin{aligned} d(w'w.y, x) = d(\rho(g.y), x) &\leqslant d(x, g.y) = d(x, g''.y) \\ &\leqslant d(x,u) + d(u, g''.y) = d(x,u) + d(u,y). \end{aligned}$$

Suppose now ${}^v\widehat{\mathbf{W}} = {}^v\mathbf{W}$ and $y \in u + \overline{\mathbf{D}} \subset x + \overline{\mathbf{D}}$. In view of (4.3.11), one has

$$\begin{aligned} w'w.y - x = \rho(g.y) - \rho(g.u) + \rho(g.u) - \rho(x) &= \rho(g.y) - \rho(g.u) + \rho(g'.u) - \rho(g'.x) \\ \leqslant_{\mathbf{D}} y - u + u - x = y - x. \end{aligned}$$

Finally, if $x \in z + \overline{\mathbf{D}}$, one has, again by (4.3.11),

$$\begin{aligned} wy - z &= \rho(n'^{-1}g.y) - \rho(n'^{-1}g.u) + \rho(n'^{-1}g'.u) - \rho(n'^{-1}g'.x) + \rho(n'^{-1}.x) - \rho(n'^{-1}.z) \\ &\leqslant_{\mathbf{D}} y - u + u - x + x - z = y - z. \end{aligned}$$

Corollary (4.3.16). — *Let $x, y \in \mathbf{A}$, let C be a chamber of $\mathbf{A}$ and let $n \in \widehat{N}$. If $\widehat{P}_x\widehat{P}_y \cap \widehat{P}_C n\widehat{P}_y \neq \varnothing$, then $d(x, n.y) \leqslant d(x,y)$. If moreover φ is of connected type and $y \in x + \overline{\mathbf{D}}$, one has $n.y \leqslant_{\mathbf{D}} y$.*

It suffices to apply the preceding result, taking a point $z \in C$ such that $\widehat{P}_C = \widehat{P}_z$ and noting that $\widehat{\mathbf{W}}_z$ is then reduced to $\{1\}$.

Corollary (4.3.17). — *Suppose φ of connected type. Let $x \in \mathbf{A}$ and $n, n' \in \widehat{N}$ be such that $n.x \in x + \overline{\mathbf{D}}$. Let D be a vector chamber. If $\widehat{P}_x n\widehat{P}_x \cap \widehat{\mathfrak{B}}^0_D n'\widehat{P}_x \neq \varnothing$, then one has $n'n^{-1}.x \leqslant_D x$.*

Indeed, there then exists a chamber C such that

$$\widehat{P}_x n\widehat{P}_x n^{-1} \cap \widehat{P}_C n'n^{-1} \cdot n\widehat{P}_x n^{-1} \neq \varnothing$$

and it suffices to apply (4.3.16) taking $y = n.x$.

4.4. The good maximal bounded subgroups. —

In this section, we shall make explicit some particular cases of (4.2.3), (4.3.6) and (4.3.17). We keep the hypotheses and notation of the preceding numbers and we assume moreover that

(1) $$G = \mathfrak{B}.N.B.$$

We shall see in §§ 5 and 7 sufficient conditions for this relation to hold. Recall (2.9.4) that it holds as soon as G is equipped with a topology satisfying the conditions of (2.8.2); it will also hold in all the applications to the case of algebraic groups. Finally, note that relation (1) is equivalent to

(1 bis) $$\widehat{G} = \widehat{\mathfrak{B}}.\widehat{N}.\widehat{B}$$

as follows from the geometric translation of (1) or of (1 bis) given in (4.2.5).

(4.4.1) We saw (3.3.2) that every maximal bounded subgroup K of $\widehat{G}$ is the stabilizer of a well-determined facet F of the building $\mathscr{I}$. We shall say that K is a *special* maximal bounded subgroup if this facet F reduces to a special vertex of $\mathscr{I}$.

Definition. — We say that a maximal bounded subgroup K of $\widehat{G}$ is good *if one has $\widehat{G} = \widehat{\mathfrak{B}}.K$.*

Let K be a good maximal bounded subgroup and let $g \in \widehat{G}$. Put $g^{-1} = bk$ with $b \in \widehat{\mathfrak{B}}$ and $k \in K$. One then has

$$g\widehat{\mathfrak{B}}g^{-1}K = g\widehat{\mathfrak{B}}bkK = g\widehat{\mathfrak{B}}K = \widehat{G}.$$

We thus see that the notion of good maximal bounded subgroup is independent of the choice of $\widehat{\mathfrak{B}}$ in its conjugacy class (i.e. of the choice of the apartment $\mathbf{A}$ and of the vector chamber $\mathbf{D}$) and that *every conjugate of a good maximal bounded subgroup is a good maximal bounded subgroup*.

Proposition (4.4.2). — *Let K be a bounded subgroup of $\widehat{G}$ containing $\widehat{B}$. The following conditions are equivalent:*

(i) K is a good maximal bounded subgroup;
(ii) one has $\widehat{\mathbf{W}} = \widehat{\mathbf{W}}_K \cdot \widehat{\mathbf{V}}$;
(iii) the canonical map from $\widehat{\mathbf{W}}_K$ into ${}^{v}\mathbf{W}$ is surjective;
(iv) $\operatorname{Card} \widehat{\mathbf{W}}_K \geqslant \operatorname{Card} {}^{v}\mathbf{W}$.

It is immediate that (ii), (iii) and (iv) are equivalent. The equivalence of (i) and (ii) follows from (4.2.3) and from the fact that, if (ii) is satisfied, then K is a maximal bounded subgroup of $\widehat{G}$. To establish this assertion, consider a bounded subgroup K' containing K. One has $\widehat{\mathbf{W}}_{K'} \supset \widehat{\mathbf{W}}_K$; since the restriction of the map $w \mapsto {}^{v}w$ to $\widehat{\mathbf{W}}_{K'}$ is injective, one deduces from (iii) (equivalent to (ii)) that $\widehat{\mathbf{W}}_{K'} = \widehat{\mathbf{W}}_K$, whence $K' = B.\widehat{\mathbf{W}}_{K'}.B = K$.

Proposition (4.4.3). — *Let K be a good maximal bounded subgroup of $\widehat{G}$ containing $\widehat{B}$. One has*

(1) $$\widehat{G} = \widehat{\mathfrak{B}}^0 \cdot \widehat{\mathbf{V}} \cdot K \qquad (\text{“Iwasawa decomposition of } \widehat{G}\text{”})$$

and the canonical map from $\widehat{\mathbf{V}}$ into $\widehat{\mathfrak{B}}^0\backslash\widehat{G}/K$ is bijective. Moreover one has

(2) $$\widehat{G} = K \cdot \widehat{\mathbf{V}}_{\mathbf{D}} \cdot K \qquad (\text{“Cartan decomposition of } \widehat{G}\text{”})$$

and the canonical map from $\widehat{\mathbf{V}}_{\mathbf{D}}$ into $K\backslash\widehat{G}/K$ is bijective when the homomorphism $\varphi : G \to \widehat{G}$ is of connected type.

The first assertion is only a particular case of (4.2.2)(i). Relation (2) follows from (4.2.1), taking into account that $\widehat{\mathbf{V}}_{\mathbf{D}}$ is a fundamental domain for ${}^{v}\mathbf{W}$ acting on $\widehat{\mathbf{V}}$: this indeed entails that the canonical map from $\widehat{\mathbf{V}}_{\mathbf{D}}$ into $\widehat{\mathbf{W}}_K \backslash \widehat{\mathbf{W}} / \widehat{\mathbf{W}}_K$ is surjective, and is bijective when ${}^v\widehat{\mathbf{W}} = {}^v\mathbf{W}$, that is when φ is of connected type.

(4.4.4) Let us now make explicit some of the relations of 4.3 in the case of interest to us, which will give us relations between the Iwasawa and Cartan decompositions of $\widehat{G}$:

Proposition. — Suppose φ of connected type. Let K be a good maximal bounded subgroup of $\widehat{G}$ containing $\widehat{B}$ and let $t \in \widehat{\mathbf{V}}_{\mathbf{D}}$, $t' \in \widehat{\mathbf{V}}$, $t'' \in \widehat{\mathbf{V}}_{\mathbf{D}}$.

(i) If $K.t.K \cap \widehat{\mathfrak{B}}^0.t'.K \neq \varnothing$, one has $t' \leqslant_{\mathbf{D}} t$ (in other words $p(t-t') \geqslant 0$ for every dominant weight p (relative to $\mathbf{D}$) of ${}^v\mathbf{\Sigma}$).

(ii) One has $K.t.K \cap \widehat{\mathfrak{B}}^0.t.K = t.K$.

(iii) If $t' \in \widehat{\mathbf{V}}_{\mathbf{D}}$ and if $KtKt'K \cap Kt''K \neq \varnothing$, then one has $t'' \leqslant_{\mathbf{D}} t + t'$ (writing here additively the composition law of $\widehat{\mathbf{V}}$ identified with a subgroup of ${}^v\mathbf{A}$).

(iv) If $t' \in \widehat{\mathbf{V}}_{\mathbf{D}}$, one has

$$t^{-1}t''^{-1}Kt''Kt \cap t'Kt'^{-1}K = t^{-1}Kt \cap K.$$

In particular, one has

$$t^{-1}t''^{-1}Kt''Kt \cap K = t^{-1}Kt \cap K.$$

Assertion (i) follows from (4.3.17) and (ii) from (4.3.7). The hypothesis of (iii) entails $K \cap t^{-1}Kt \cdot t^{-1}t''t'^{-1} \cdot t'Kt'^{-1} \neq \varnothing$; setting $K = \widehat{P}_x$, $y = u = t.x$ and $z = t^{-1}.x$ and applying (4.3.15), one deduces (iii). Finally, (iv) follows from (4.3.12), taking $K = \widehat{P}_z$, $x = (t''t)^{-1}.z$, $y = t^{-1}.z$ and $u = t'.z$.

Remark (4.4.5). — Suppose φ of connected type and let G_1 be a normal subgroup of $\widehat{G}$ containing $\varphi(G)$. Then $\varphi : G \to G_1$ is B-N-adapted of connected type. Let F be a facet of $\mathbf{C}$ whose stabilizer K_1 in G_1 is a good maximal bounded subgroup of G_1. It is then immediate that the stabilizer K of F in $\widehat{G}$ is a good maximal bounded subgroup of $\widehat{G}$. If $g \in K$, there exists $h \in K_1$ such that $g.\mathbf{A} = h.\mathbf{A}$ and, since ${}^v\widehat{\nu}(K_1 \cap \widehat{N}) = {}^v\widehat{\mathbf{W}} = {}^v\mathbf{W}$, one sees that

$$K = K_1 \cdot (\widehat{\nu}^{-1}(\widehat{\mathbf{V}}) \cap K).$$

One then deduces from (4.4.3) that

$$\widehat{G} = \widehat{\mathfrak{B}}^0 \cdot \widehat{\nu}^{-1}(\widehat{\mathbf{V}}) \cdot K_1 \tag{19}$$

$$\widehat{G} = K_1 \cdot \widehat{\nu}^{-1}(\widehat{\mathbf{V}}) \cdot K_1. \tag{20}$$

Let us now put on $\widehat{G}$ the bornology $\mathscr{B}$ for which the bounded sets are the finite unions of translates of bounded subsets of G_1. Since the bornology of G_1 is invariant under $\widehat{G}$ acting on G_1 by inner automorphisms, this bornology is compatible with the group law of $\widehat{G}$. If one supposes moreover that $\widehat{G}/G_1$ has no elements of finite order, a subgroup of $\widehat{G}$ is bounded for $\mathscr{B}$ if and only if it is contained and bounded in G_1. Consequently, K_1 is a maximal bounded subgroup of $\widehat{G}$ (for $\mathscr{B}$), giving rise to an "Iwasawa decomposition" (1) and to a "Cartan decomposition" (2). We shall see later an example of this situation: $\widehat{G}$ will be the group of rational points of a connected reductive algebraic group $\mathscr{G}$ defined over a local field k (for a valuation ω), G the group of rational points over k of the derived group of $\mathscr{G}$, φ the natural homomorphism and G_1 the set of $g \in \widehat{G}$ such that $\omega(\chi(g)) = 0$ for every character χ of $\mathscr{G}$ rational over k, the bornology $\mathscr{B}$ of $\widehat{G}$ being none other than the natural bornology of $\widehat{G}$.

(4.4.6) Let us now study the existence of good maximal bounded subgroups:

Proposition. — (i) If φ is of connected type, every special maximal bounded subgroup is a good maximal bounded subgroup.

(ii) If φ is the identity, the only good maximal bounded subgroups are the special maximal bounded subgroups.

Taking (4.4.2) into account, the first assertion is evident, since, if x is a special point of $\mathbf{A}$, one has ${}^{v}\widehat{\mathbf{W}}_x \supset {}^{v}\mathbf{W}_x = {}^{v}\mathbf{W} = {}^{v}\widehat{\mathbf{W}}$; the second is evident as well, in view of the very definition of special points.

Recall that the special points are given by the table of no. (1.3.12).

(4.4.7) Suppose still φ *of connected type*. If the homomorphism ξ from $\widehat{G}$ into $\operatorname{Aut}\operatorname{Cox}\mathbf{W}$ is not trivial, there may exist good maximal bounded subgroups which are not special. To determine them, one reduces at once to the irreducible case, for, φ being of connected type, $\xi(\widehat{G})$ fixes each connected component of $\operatorname{Cox}\mathbf{W}$. In the irreducible case, one shows easily, by counting the orders of the two groups $\widehat{\mathbf{W}}_K$ and ${}^{v}\widehat{\mathbf{W}}$ and using (4.4.2)(iv), that the only cases of non-special good maximal bounded subgroups are the three following ones:

- G of type A_1, $\quad \xi(\widehat{G}) = \mathbf{Z}/2\mathbf{Z}$, $\quad K = \operatorname{Stab} B$;
- G of type C_2, $\quad \xi(\widehat{G}) = \mathbf{Z}/2\mathbf{Z}$, $\quad K = \operatorname{Stab} B_X$, where X is the set of the two special points of $\operatorname{Cox}\mathbf{W}$ (which is $\circ\overset{4}{-\!\!-}\circ\overset{4}{-\!\!-}\circ$, the two extreme vertices being the special ones);
- G of type B_n ($n \geqslant 3$), $\quad \xi(\widehat{G}) = \mathbf{Z}/2\mathbf{Z}$, $\quad K = \operatorname{Stab} B_X$, where X is the set of points of $\mathbf{S}$ distinct from the non-special extremal vertex of $\operatorname{Cox}\mathbf{W}$ (the diagram being the one with a fork of two vertices at one end and a bond marked 4 at the other end).

(4.4.8) On the other hand, when φ is not of connected type, there may be no good maximal bounded subgroups. This is the case for example when $\mathbf{W}$ is of type A_{2n+1}, with $\xi(\widehat{G}) = \mathbf{Z}/2\mathbf{Z}$ acting by symmetry without fixed points on $\operatorname{Cox}\mathbf{W}$, or when $\mathbf{W}$ is of type D_{2n+1}, with $\xi(\widehat{G}) = \mathbf{Z}/2\mathbf{Z}$ acting without fixed points on $\operatorname{Cox}\mathbf{W}$.

On the other hand, one shows easily that, if φ is not of connected type and if $\mathbf{W}$ is irreducible, a good maximal bounded subgroup is always special.

(4.4.9) The preceding relations have interesting consequences when $\widehat{G}$ is a locally compact topological group, the subgroup $\widehat{B}$ being an open and closed subgroup of $\widehat{G}$, which happens for example when $\widehat{G}$ is a semisimple algebraic group over a locally compact local field. The bounded subsets of $\widehat{G}$ are then the relatively compact subsets, and the maximal bounded subgroups are the maximal compact subgroups. Propositions (4.4.3) and (4.4.4) then imply for example that, if K is a good maximal compact subgroup of $\widehat{G}$, the convolution algebra of continuous complex functions with compact support on $\widehat{G}$, invariant on the left and on the right under K, is *commutative*: the proof given in [7] adapts immediately; one may also consult [29], [30] and [32].

5. Double Tits systems

In this section, we begin by giving necessary and sufficient conditions in order that, with the notation of § 4, one have $G = \mathfrak{B}N\mathfrak{B}$. We shall see that this relation is equivalent to saying that $(G, \mathfrak{B}, N)$ is a Tits system with finite Weyl group isomorphic to ${}^{v}\mathbf{W}$. We shall then say that (G, B, N, S) is a *double Tits system*. It is systems of this type that we shall construct later in the group of rational points of a simply connected semisimple algebraic group over a local field.

In 5.2, we shall give necessary and sufficient conditions in order that a family of subgroups P_α (for $\alpha \in \mathbf{\Sigma}$) be the family of subgroups P_α associated with a double Tits system with Weyl group $\mathbf{W}$. Along the way, we shall establish results on the structure of the subgroups P_Ω of a double Tits system.

Recall that the results of this section are not used in the remainder of the chapter.

5.1. Double Tits systems. —

Throughout 5.1, we keep the notation of § 4. In particular, we assume chosen a vector chamber $\mathbf{D}$, and the letters $\mathfrak{B}$ and $\mathfrak{B}^0$ denote the subgroups introduced in (4.1.5).

(5.1.1) Since $\mathfrak{B} = \mathfrak{B}^0 \cdot \nu^{-1}(\mathbf{V})$ and $\mathfrak{B}^0 \cap N = H \subset \nu^{-1}(\mathbf{V})$, one has $\mathfrak{B} \cap N = \nu^{-1}(\mathbf{V})$. Consequently, $\mathfrak{B} \cap N$ is a normal subgroup of N and the quotient group $N/\mathfrak{B} \cap N$ is canonically identified with $\mathbf{W}/\mathbf{V}$, or again with ${}^v\mathbf{W}$. This last group is generated by the set R of the reflections with respect to the walls of the vector chamber $\mathbf{D}$ (that is, with respect to the hyperplanes $\operatorname{Ker} a$, for a running over the basis of ${}^v\mathbf{\Sigma}$ associated with $\mathbf{D}$). Consequently, *the quadruple* $(G, \mathfrak{B}, N, R)$ *satisfies axiom* (T 2) *of Tits systems* (1.2.6).

Definition. — We say that the Tits system of affine type (G, B, N, S) *is a* double Tits system *if the quadruple* $(G, \mathfrak{B}, N, R)$ *is a Tits system.*

Definition (5.1.2). — *We say that two closed subsets* Ω' *and* Ω'' *of* $\mathbf{A}$ *are* transverse *if, for every affine root* α *of* $\mathbf{A}$ *containing the intersection* $\Omega' \cap \Omega''$*, one has either* $\alpha \supset \Omega'$*, or* $\alpha \supset \Omega''$.

Theorem (5.1.3). — *The following conditions are equivalent:*

(i) (G, B, N, S) *is a double Tits system;*

(ii) one has $G = \mathfrak{B}.N.\mathfrak{B}$*;*

(ii bis) whatever the sectors $\mathfrak{C}$ *and* $\mathfrak{C}'$ *of* $\mathscr{I}$*, there exist sectors* $\mathfrak{C}_1 \subset \mathfrak{C}$ *and* $\mathfrak{C}'_1 \subset \mathfrak{C}'$ *contained in one and the same apartment;*

(iii) whatever the transverse closed subsets Ω' *and* Ω'' *of* $\mathbf{A}$ *such that the intersection* $\Omega = \Omega' \cap \Omega''$ *contains a chamber, one has*

$$P_\Omega = P_{\Omega'} \cdot P_{\Omega''};$$

(iv) whatever the chamber C *of* $\mathbf{A}$ *and the vector chamber* D*, one has*

$$P_C = P_{C+D} \cdot P_{C-D};$$

(v) there exist a chamber C *of* $\mathbf{A}$ *and a vector chamber* D *such that* $P_C = P_{C+D} \cdot P_{C-D}$.

Moreover, these conditions imply:

(vi) $G = BN\mathfrak{B}$*;*

(vi bis) whatever the chamber C *and the sector* $\mathfrak{C}$ *of* $\mathscr{I}$*, there exist a sector* $\mathfrak{C}' \subset \mathfrak{C}$ *and an apartment* A *containing* C *and* $\mathfrak{C}'$*;*

(vii) whatever the affine roots α *and* β *of* $\mathbf{A}$ *such that* $\beta^* \subsetneq \alpha$*, one has*

$$P_{\alpha\cap\beta} = P_\alpha \cdot P_\beta;$$

(viii) for every affine root α *of* $\mathbf{A}$*, one has*

$$P_{\partial\alpha} = (P_\alpha r_\alpha P_\alpha) \cup (P_\alpha P_{(\alpha^*)_+}).$$

(5.1.4) The proof of theorem (5.1.3) will occupy the following numbers, up to (5.1.21). We already know that (i) implies (ii) (1.2.7) and that (ii) is equivalent to (ii bis) (4.2.5).

(5.1.5) *Proof of the implication* (ii bis) $\Rightarrow$ (iii).

Suppose condition (ii bis) realized. Let Ω' and Ω'' be two transverse closed subsets of $\mathbf{A}$, such that $\Omega = \Omega' \cap \Omega''$ contains a chamber. We wish to prove that

$$P_\Omega = P_{\Omega'} \cdot P_{\Omega''}. \tag{1}$$

It is clear that we may suppose that Ω contains the chamber $\mathbf{C}$. Let us first prove the following lemma:

Lemma (5.1.6). — *Let M be a subset of Ω'. Then:*

a) Ω' and $\mathrm{cl}(\Omega'' \cup M)$ are transverse and $\mathrm{cl}(\Omega \cup M) = \Omega' \cap \mathrm{cl}(\Omega'' \cup M)$;
b) Ω'' and $\mathrm{cl}(\Omega \cup M)$ are transverse and $\Omega = \Omega'' \cap \mathrm{cl}(\Omega \cup M)$.

Since the intersection of two closed subsets is closed, one has $\mathrm{cl}(M \cup \Omega) \subset \Omega' \cap \mathrm{cl}(M \cup \Omega'')$. On the other hand, if an affine root α contains $\Omega \cup M$, then, since Ω' and Ω'' are transverse, either $\alpha \supset \Omega'$, or $\alpha \supset \Omega''$ and $\alpha \supset \mathrm{cl}(M \cup \Omega'')$. Since $\mathrm{cl}(M \cup \Omega)$ is the intersection of the affine roots containing $M \cup \Omega$ (2.4.5), one has $\mathrm{cl}(M \cup \Omega) \supset \Omega' \cap \mathrm{cl}(M \cup \Omega'')$, whence a). Similarly, one has $\Omega \subset \Omega'' \cap \mathrm{cl}(M \cup \Omega)$ and if an affine root α contains Ω, either $\alpha \supset \Omega''$, or $\alpha \supset \Omega' \supset M$ and $\alpha \supset \mathrm{cl}(M \cup \Omega)$. One deduces b) at once.

Lemma (5.1.7). — *Suppose moreover that Ω' contains a sector of direction D'. One then has $P_\Omega \subset P_{\mathrm{cl}(\Omega + D')} \cdot P_{\Omega''}$.*

Let $g \in P_\Omega$. Let $\mathfrak{C}'$ be a sector of direction D' contained in Ω' and let $\mathfrak{C}''$ be a sector of $\mathbf{A}$ opposite to $\mathfrak{C}'$. By (ii bis) we may assume, after shrinking $\mathfrak{C}'$ and $\mathfrak{C}''$ if necessary, that the two sectors $\mathfrak{C}'$ and $g.\mathfrak{C}''$ are contained in one and the same apartment A. Let us show that $\Omega \subset \mathrm{A}$. It suffices to show that every chamber $\mathrm{C}_0 \subset \Omega$ is contained in A. Let $x \in \mathfrak{C}'$ and $y \in \mathfrak{C}''$ be such that $\mathrm{C}_0 \subset \mathrm{cl}(\{x, y\})$ (2.4.9). It follows from (2.8.8) that $\mathrm{C}_0 \subset \mathrm{cl}(\{x, g.y\}) \subset \mathrm{A}$, whence our assertion. p. 85

Let moreover $h \in G$ be such that $\mathbf{A} = h.\mathrm{A}$ and $h.z = z$ for every $z \in \mathrm{A} \cap \mathbf{A}$. The restrictions of h and of $\rho_{\mathbf{A};\mathbf{C}}$ (resp. of g^{-1} and of $\rho_{\mathbf{A};\mathbf{C}}$) to A (resp. to $g.\mathbf{A}$) coincide, since these are isometries and they agree on $\mathbf{C}$. Consequently the restrictions of h and of g^{-1} to $g.\mathfrak{C}''$ are equal, and one has $hg \in P_{\mathfrak{C}''}$.

On the other hand, one has $\Omega'' \subset \mathrm{cl}(\Omega \cup \mathfrak{C}'')$. Indeed, if an affine root α contains Ω and $\mathfrak{C}''$, it cannot contain $\mathfrak{C}'$, hence a fortiori it does not contain Ω'; since Ω' and Ω'' are transverse, it contains Ω'', whence our assertion (2.4.5). As h and g belong to P_Ω, we conclude that $hg \in P_{\mathrm{cl}(\Omega \cup \mathfrak{C}'')} \subset P_{\Omega''}$.

We have thus shown that $g = h^{-1}hg$ belongs to $P_{\mathrm{cl}(\Omega \cup \mathfrak{C}')} \cdot P_{\Omega''}$. To finish the proof it suffices to remark that $\Omega + \mathrm{D}' \subset \mathrm{cl}(\Omega \cup \mathfrak{C}')$.

Lemma **(5.1.8)**. — *Let $\mathrm{D}_1, \dots, \mathrm{D}_k$ be vector chambers such that $\Omega' + \mathrm{D}_i \subset \Omega'$ for $1 \leqslant i \leqslant k$. Then*

$$P_\Omega \subset P_{\mathrm{cl}(\Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_k)} \cdot P_{\Omega''}.$$

Lemma (5.1.6) (applied to $\mathrm{M} = \Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_i$) shows that, for $0 \leqslant i < k$, the closed sets Ω' and $\Omega''_i = \mathrm{cl}(\Omega'' + \mathrm{D}_1 + \cdots + \mathrm{D}_i)$ are transverse and that their intersection is $\mathrm{cl}(\Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_i)$. Lemma (5.1.7) then shows, since Ω' contains a sector of direction D_{i+1}, that

$$\begin{aligned} P_{\mathrm{cl}(\Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_i)} &\subset P_{\mathrm{cl}(\Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_{i+1})} \cdot P_{\mathrm{cl}(\Omega'' + \mathrm{D}_1 + \cdots + \mathrm{D}_i)} \\ &\subset P_{\mathrm{cl}(\Omega + \mathrm{D}_1 + \cdots + \mathrm{D}_{i+1})} \cdot P_{\Omega''} \end{aligned}$$

whence the lemma.

Proposition **(5.1.9)**. — *Assume that condition* (ii bis) *still holds. Let α be a half-apartment and* C *a chamber possessing a panel* F *contained in the wall $\partial\alpha$ of α. Then there exists an apartment containing α and* C.

We may assume $\alpha \subset \mathbf{A}$ and $\mathrm{C} \not\subset \mathbf{A}$. Let C' (resp. C'') be the chamber of $\mathbf{A}$ admitting F as a panel and contained in α (resp. not contained in α). Take $\Omega' = \alpha$, $\Omega'' = \overline{\mathrm{C}' \cup \mathrm{C}''}$, whence $\Omega = \overline{\mathrm{C}'}$. One sees at once that Ω' and Ω'' are closed and transverse, since α is the unique affine root containing C' and not C''. Let $\mathrm{D}_1, \ldots, \mathrm{D}_k$ be the vector chambers on which the root ${}^v\alpha$ (1.3.8) takes negative values. One checks easily that $\alpha = \mathrm{C}' + \mathrm{D}_1 + \cdots + \mathrm{D}_k$, and Lemma (5.1.8) shows that

(1) $$P_{\mathrm{C}'} = P_\alpha \cdot P_{\mathrm{C}' \cup \mathrm{C}''}.$$

Now the proposition is nothing but the geometric translation of (1). Indeed, there exists an element $g \in P_{\mathrm{C}'}$ such that $\mathrm{C} = g.\mathrm{C}''$ (2.3.7). Write $g = h_1 h_2$, with $h_1 \in P_\alpha$ and $h_2 \in P_{\mathrm{C}' \cup \mathrm{C}''}$. *p. 86* One has $\mathrm{C} = g.\mathrm{C}'' = h_1.\mathrm{C}''$, and C, like α, is contained in the apartment $h_1.\mathbf{A}$. We leave it to the reader to deduce (1) from (5.1.9).

(5.1.10) One may also formulate (5.1.9) as follows: let α be a half-apartment and let F be a panel contained in the wall of α; then *the group P_α is transitive on the set of chambers not contained in α and admitting* F *as a panel.*

If, with the notation of (5.1.9), one assumes moreover that $\mathrm{C}' = \mathbf{C}$, then (1) reads $B = P_\alpha \cdot (B \cap r_\alpha B r_\alpha)$, from which one draws $B r_\alpha B \subset P_\alpha r_\alpha B r_\alpha r_\alpha B$ and finally

(1 bis) $$B r_\alpha B = P_\alpha r_\alpha B.$$

Conversely, (1 bis) implies $B r_\alpha \subset P_\alpha r_\alpha B$, whence $B \subset P_\alpha r_\alpha B r_\alpha$ and $B = P_\alpha \cdot (B \cap r_\alpha B r_\alpha)$, that is to say (5.1.9) (1).

Corollary **(5.1.11)**. — *Assume* (ii bis) *holds. For every closed subset Ω_1 of $\mathbf{A}$, the set of points of $\mathbf{A}$ left fixed by all the elements of P_{Ω_1} reduces to Ω_1. If Ω_1 and Ω_1' are two closed subsets of $\mathbf{A}$, one has $P_{\Omega_1} \subset P_{\Omega_1'}$ if and only if $\Omega_1' \subset \Omega_1$.*

Let $\alpha \in \mathbf{\Sigma}$ and $x \in \mathbf{A} - \alpha$. Let F be a panel contained in $\partial\alpha$ and let $\Gamma = (\mathrm{C}, \ldots, \mathrm{C}')$ be a taut gallery between F and x. One has $\mathrm{C} \subset \alpha^*$. If x were left fixed by all the elements of P_α, the same would hold for C (2.4.13), which would contradict (5.1.10).

Now let Ω_1 be a closed subset of $\mathbf{A}$ and let $x \in \mathbf{A} - \Omega_1$. By definition (2.4.6) there exists $\alpha \in \mathbf{\Sigma}$ with $\alpha \supset \Omega_1$ and $x \notin \alpha$. Then x is not left fixed by all the elements of P_α, hence a fortiori not by all those of P_{Ω_1}, which establishes the first assertion of the corollary. The second follows at once.

One should not believe that in general Ω is the set of points of the building $\mathscr{I}$ left fixed by all the elements of P_Ω: this may already fail when Ω is an affine root $\alpha \in \mathbf{\Sigma}$, or when Ω equals the whole of $\mathbf{A}$. We shall see an example of this later (the case of $SL_2(\mathbf{Q}_2)$).

Lemma **(5.1.12)**. — *Let* C *be a chamber of Ω'. Then*

$$P_\Omega \subset P_{\mathrm{cl}(\Omega \cup \mathrm{C})} \cdot P_{\Omega''}.$$

We shall argue by induction on the length m of a minimal gallery $\Gamma = (\mathrm{C}_0, \mathrm{C}_1, \ldots, \mathrm{C}_m = \mathrm{C})$ of smallest possible length among the galleries with end C and origin contained in Ω. The lemma is obvious if $m = 0$. Since $\mathrm{C}_i \subset \Omega'$ for $0 \leqslant i \leqslant m$, Ω' being closed, the induction hypothesis implies that $P_\Omega \subset P_{\mathrm{cl}(\Omega \cup \mathrm{C}_{m-1})} \cdot P_{\Omega''}$, and Lemma (5.1.6) shows that one may assume $m = 1$. Let then α be the affine root containing C_0 and not containing C_1. One has $\Omega'' \subset \alpha$; otherwise Ω'' would contain a chamber $\mathrm{C}' \subset \mathbf{A} - \alpha$ and one would have $\mathrm{C}_1 \subset \mathrm{cl}(\mathrm{C}_0 \cup \mathrm{C}') \subset \Omega''$, whence $\mathrm{C}_1 \subset \Omega$ and $m = 0$. Let then $g \in P_\Omega$; the chamber $g^{-1}.\mathrm{C}_1$ admits as a panel the common panel of C_0 and C_1, and we may apply Proposition (5.1.9). There therefore exists an apartment A' containing α and $g^{-1}.\mathrm{C}_1$, and there exists an element $h \in P_\alpha \subset P_{\Omega''}$ such that $h.\mathrm{A}' = \mathbf{A}$. One then has $hg^{-1} \in P_\Omega \cap P_{\mathrm{C}_1}$ and $g = gh^{-1}.h \in P_{\mathrm{cl}(\Omega \cup \mathrm{C}_1)} \cdot P_{\Omega''}$, Q.E.D.

(5.1.13) *Let us now prove* (5.1.5) (1) *in the case where Ω' contains a sector.* *p. 87*

One knows (2.4.5) that Ω' is the convex hull of finitely many points x_i and half-lines X_j. Now one sees easily that, if X is a half-line and D′ a vector chamber, the enclosure of $X + D'$ is the closure of a finite union of sectors. One deduces that there exist chambers $C_1, \ldots, C_k$ and vector chambers $D_1, \ldots, D_k$ such that

$$\Omega' = \mathrm{cl}\Big(\bigcup_{1 \leqslant i \leqslant k} (C_i + D_i)\Big).$$

We already know that $P_\Omega \subset P_{\mathrm{cl}(\Omega + D_1 + \cdots + D_k)} \cdot P_{\Omega''}$ (Lemma (5.1.8)). It then suffices to apply Lemma (5.1.12) a certain number of times in order to obtain the desired relation:

$$P_\Omega \subset P_{\mathrm{cl}((\Omega + D_1 + \cdots + D_k) \cup C_1 \cup \cdots \cup C_k)} \cdot P_{\Omega''} \subset P_{\Omega'} \cdot P_{\Omega''}.$$

(5.1.14) Let us now complete the proof of (5.1.5) (1) in the general case. Let M be a segment or a closed half-line with origin $x \in \Omega$, contained in Ω'. Let X be the set of affine roots containing Ω and not containing M, and put $\Omega''' = \bigcap_{\alpha \in X} \alpha$. Every affine root $\alpha \in X$ contains in its interior the open half-line with origin x opposite to M. One deduces at once that Ω''' *contains a sector*. Moreover, an affine root $\alpha \in X$ contains Ω and does not contain Ω', hence contains Ω''. Consequently one has $\Omega''' \supset \Omega''$.

Let us show that $\Omega = \Omega''' \cap \mathrm{cl}(\Omega \cup M)$ and that Ω''' and $\mathrm{cl}(\Omega \cup M)$ are transverse. Indeed, if an affine root contains Ω, then either it contains M and hence $\mathrm{cl}(\Omega \cup M)$, or it belongs to X and contains Ω'''. Since Ω''' contains a sector, one deduces from (5.1.13) that

$$P_\Omega \subset P_{\mathrm{cl}(\Omega \cup M)} \cdot P_{\Omega'''} \subset P_{\mathrm{cl}(\Omega \cup M)} \cdot P_{\Omega''}.$$

Taking Lemma (5.1.6) into account, we are reduced to proving (5.1.5) (1) with Ω'' replaced by $\mathrm{cl}(\Omega'' \cup M)$. Now Ω' is the convex hull of finitely many points x_i (for $1 \leqslant i \leqslant a$) and closed half-lines X_j with origin y_j (for $1 \leqslant j \leqslant b$). Let x be a point of Ω; put $M_i = [x x_i]$ for $1 \leqslant i \leqslant a$, $M_i = [x y_{i-a}]$ for $a < i \leqslant a + b$, and $M_i = X_{i-(a+b)}$ for $a + b < i \leqslant a + 2b$. Applying the preceding argument successively to the pairs $\Omega', \Omega''_i = \mathrm{cl}(\Omega \cup M_1 \cup \cdots \cup M_i)$ (for $1 \leqslant i \leqslant a+2b$), everything finally reduces to proving (5.1.5) (1) in the case where $\Omega' \subset \Omega''$, which is trivial.

The proof of the implication (ii bis) ⇒ (iii) is complete.

(5.1.15) *Proof of the implications* (iii) ⇒ (iv) *and* (iv) ⇒ (v).

It suffices to remark that, for every chamber C of **A**, one has $C = \mathrm{cl}(C + D) \cap \mathrm{cl}(C - D)$ and that every affine root containing C contains either $C + D$ or $C - D$, i.e. that $\mathrm{cl}(C + D)$ and $\mathrm{cl}(C - D)$ are transverse.

(5.1.16) *Throughout the rest of* 5.1, *except in* (5.1.18), *we assume that condition* (v) *is satisfied, and we denote by* C *(resp.* D*) a chamber (resp. a vector chamber) such that* $P_C = P_{C+D} \cdot P_{C-D}$.

Lemma. — For every sector $\mathfrak{C}$ *of* **A**, *of direction* D′, *there exists a chamber* $C' \subset \mathfrak{C}$ *such that* $P_{C'} = P_{C'+D'} \cdot P_{C'-D'}$. p. 88

Indeed, there exists $w \in \mathbf{W}$ such that $D' = {}^v w(D)$. On the other hand, since **V** is a lattice in ${}^v\mathbf{A}$ and $\mathfrak{C}$ is an open cone of **A**, there is a translation $t \in \mathbf{V}$ such that $tw(C) \subset \mathfrak{C}$. Take then $C' = tw(C)$: one indeed has $P_{C'} = P_{tw(C+D)} \cdot P_{tw(C-D)} = P_{C'+D'} \cdot P_{C'-D'}$.

(5.1.17) *Proof of* (vi) *and* (vi bis).

We have already seen (4.2.5) that (vi) and (vi bis) are equivalent. Let $\mathfrak{C}$ be a sector of **A**, of direction D′, and let C_1 be a chamber of $\mathscr{I}$. Let a be a point of **A**, put $d = \sup_{x \in C_1} d(a, x)$ and let S be the ball of radius d and center a in **A**. Let b be a point of **A** such that $S \subset b - D'$ and let C′ be a chamber contained in the sector $(b + D') \cap \mathfrak{C}$ such that $P_{C'} = P_{C'+D'} \cdot P_{C'-D'}$. Let ρ be the retraction of $\mathscr{I}$ onto **A** with center C′ (2.3.5). Since ρ is the identity on **A** and decreases distances (2.5.3), one has $\rho(C_1) \subset S \subset b - D' \subset C' - D'$. On the other hand, there exists an element $g \in P_{C'}$ such that $\rho(C_1) = g(C_1)$. Write then $g = xy$, with $x \in P_{C'-D'}$ and $y \in P_{C'+D'}$. One has $x^{-1} g(C_1) = g(C_1)$ since $g(C_1) \subset C' - D'$, and consequently $C_1 \subset g^{-1}(\mathbf{A}) = y^{-1}(\mathbf{A})$.

Since $y \in P_{\mathrm{C}'+\mathrm{D}'}$, this implies that the chamber C_1 and the sector $c + \mathrm{D}' \subset \mathfrak{C}$ (where c is any point of C') are both contained in the apartment $y^{-1}(\mathbf{A})$. One passes from this to the general case of (vi bis) by transforming $\mathfrak{C}$ by an element of G.

Remark **(5.1.18)**. — On examining the preceding proof one sees easily that condition (vi) $G = BN\mathfrak{B}$ is implied by the following condition, weaker than (v):

(v bis) *there exist a chamber* C *and a vector chamber* D *such that* $P_{\mathrm{C}} = P_{\mathrm{C}+\mathrm{D}} \cdot P_{\Omega}$ *for every bounded subset* Ω *of* $\mathrm{C} - \mathrm{D}$.

Moreover, one can prove that conversely (vi) implies (v bis), and even implies

(iii bis) *Let* Ω' *and* Ω'' *be two transverse closed subsets such that* $\Omega = \Omega' \cap \Omega''$ *contains a chamber. If* Ω' *is bounded and if* Ω'' *is contained in a sector, then* $P_{\Omega} = P_{\Omega'} \cdot P_{\Omega''}$.

Finally, one can prove that, without making any supplementary hypothesis on the Tits system (G, B, N, S), the condition (iii ter), obtained by adding to (iii) or to (iii bis) the hypothesis that Ω' and Ω'' are both *bounded*, is always satisfied. For this it suffices to prove the analogue of (5.1.12) when Ω' and Ω'' are bounded. This is done as in (5.1.12), but replacing the use of Proposition (5.1.9) by that of Corollary (2.4.12).

(5.1.19) *Proof of* (vii).

Let α and β be two affine roots of $\mathbf{A}$ such that $\beta^* \subsetneq \alpha$. There exists $w \in \mathbf{W}$ such that $w(\mathrm{C}) \subset \alpha \cap \beta$. Let a be a special point of $\overline{w(\mathrm{C})}$. There exists an element w' of the stabilizer $\mathbf{W}_a$ of a in $\mathbf{W}$ such that one of the faces of the vector chamber $\mathrm{D}' = {}^v(w'w)(\mathrm{D})$ is parallel to $\partial\alpha$ and to $\partial\beta$, and one may assume that $\mathrm{C}' = w'w(\mathrm{C})$ is contained in $\alpha \cap \beta$: indeed, if for instance p. 89
$w'w(\mathrm{C}) \not\subset \alpha$, one necessarily has $a \in \partial\alpha$, $w'w(\mathrm{C}) \subset \alpha^*$, whence $r_\alpha w'w(\mathrm{C}) \subset \alpha$. Moreover one has $P_{\mathrm{C}'} = P_{\mathrm{C}'+\mathrm{D}'} \cdot P_{\mathrm{C}'-\mathrm{D}'}$. On the other hand, either the affine root α contains $\mathrm{C}' + \mathrm{D}'$ and $\beta \supset \mathrm{C}' - \mathrm{D}'$, or $\alpha \supset \mathrm{C}' - \mathrm{D}'$ and $\beta \supset \mathrm{C}' + \mathrm{D}'$. Interchanging α and β if necessary, we may assume that α contains $\mathrm{C}' + \mathrm{D}'$ and that β contains $\mathrm{C}' - \mathrm{D}'$.

Let $g \in P_{\alpha\cap\beta} \subset P_{\mathrm{C}'}$ and write $g = hk$, with $h \in P_{\mathrm{C}'+\mathrm{D}'}$ and $k \in P_{\mathrm{C}'-\mathrm{D}'}$. One has $h = gk^{-1} \in P_{\partial\alpha\cap(\mathrm{C}'-\mathrm{D}')}$. Now the closure of the convex hull of $\partial\alpha \cap (\mathrm{C}' - \mathrm{D}')$ and of $\mathrm{C}' + \mathrm{D}'$ is the whole affine root α. Consequently one has $h \in P_\alpha$ and likewise $k \in P_\beta$, which proves (vii).

(5.1.20) *Proof of* (viii).

Let us resume the notation of (5.1.19), with $\beta = (\alpha^*)_+$. Let x be a point of C' and let δ_0 be a half-line with origin x, contained in the open cone $x - \mathrm{D}'$. Since $\partial\alpha$ is parallel to a face of D' and since $\alpha \supset x + \mathrm{D}'$, hence $\alpha \not\supset x - \mathrm{D}'$, this half-line meets $\partial\alpha$ in a point y, which we may assume to be interior to a panel $\mathrm{F} \subset \partial\alpha$. Let δ be the half-line carried by δ_0 with origin y and let C_1 (resp. C_2) be the chamber contained in α (resp. α^*) admitting F as a panel. For $z \in \delta$, let $(\mathrm{X}_0, \ldots, \mathrm{X}_k = \mathrm{C}_2)$ be a taut gallery between z and C_2. Since C_1 meets $[yx]$, one has $\mathrm{C}_1 \subset \mathrm{cl}(\mathrm{C}_2 \cup \mathrm{C}')$ and there exists a minimal gallery $(\mathrm{X}_k = \mathrm{C}_2, \mathrm{X}_{k+1} = \mathrm{C}_1, \ldots, \mathrm{X}_n = \mathrm{C}')$ (2.4.4). Since $\mathrm{C}_2 \subset \mathrm{cl}(\{z\} \cup \mathrm{C}')$, the gallery $\Gamma = (\mathrm{X}_0, \ldots, \mathrm{X}_n)$ is minimal, again by (2.4.4).

Let $g \in P_{\partial\alpha}$. Suppose first that $g(\mathrm{C}_1) = \mathrm{C}_1$. Since the enclosure of $\mathrm{C}_1 \cup \partial\alpha$ is the strip $\alpha \cap (\alpha^*)_+$, it follows from (vii) that $g \in P_\alpha P_{(\alpha^*)_+}$.

Suppose now that $\mathrm{C}_2' = g(\mathrm{C}_1) \neq \mathrm{C}_1$. Put $\delta' = gr_\alpha(\delta)$ and $\mathfrak{C} = gr_\alpha(y - \mathrm{D}')$. By (vi), there exist a sector $\mathfrak{C}' \subset \mathfrak{C}$ and an apartment A containing $\mathfrak{C}'$ and C'. Since δ' is contained in $\mathfrak{C}$, one has $\delta' \cap \mathfrak{C}' \neq \emptyset$. Take then a point $z \in \delta$ contained in a chamber and such that $gr_\alpha(z) \in \mathfrak{C}'$, and resume the minimal gallery Γ considered above. The sequence

$$\Gamma' = (gr_\alpha(\mathrm{X}_0), \ldots, gr_\alpha(\mathrm{X}_k) = gr_\alpha(\mathrm{C}_2), \mathrm{X}_{k+1} = \mathrm{C}_1, \ldots, \mathrm{X}_n = \mathrm{C}')$$

is a gallery non-stuttering, since $gr_\alpha(\mathrm{C}_2) = \mathrm{C}_2' \neq \mathrm{C}_1$, and is of the same type as Γ. It is therefore also a minimal gallery. Since $gr_\alpha(\mathrm{X}_0)$ contains $gr_\alpha(z)$, hence is contained in A as is C', one deduces that the whole of Γ' is contained in A, and in particular $\mathrm{C}_1 \cup \mathrm{C}_2' \subset \mathrm{A}$. Consequently

A contains the convex hull of $\{y\} \cup \mathfrak{C}'$, a convex hull which is nothing but $\mathfrak{C}$. In particular, A contains the cone $\mathrm{Y} = (y - \overline{\mathrm{D}'}) \cap \partial\alpha$.

Let then h be an element of G such that $h(\mathbf{A}) = \mathrm{A}$ and such that h leaves fixed all the points of $\mathbf{A} \cap \mathrm{A}$. One has $h \in P_{\mathrm{C}'} \cap P_{\mathrm{Y}} \cap P_{\mathrm{C}_1}$ and $h(\mathrm{C}_2) = \mathrm{C}'_2$. Write $h = pq$, with $p \in P_{\mathrm{C}'+\mathrm{D}'}$ and $q \in P_{\mathrm{C}'-\mathrm{D}'}$. Since $\mathrm{C}' - \overline{\mathrm{D}'}$ meets C_1 and C_2 and contains Y, one has $\mathrm{C}'_2 = p(\mathrm{C}_2)$ and $p \in P_{\mathrm{C}'+\mathrm{D}'} \cap P_{\mathrm{Y}}$. But the closure of the convex hull of $\mathrm{Y} \cup (\mathrm{C}' + \mathrm{D}')$ is equal to α, hence $p \in P_\alpha$. Finally, one has $r_\alpha p^{-1} g(\mathrm{C}_1) = \mathrm{C}_1$ and $r_\alpha p^{-1} g \in P_{\partial\alpha}$. By what we saw above, this implies $r_\alpha p^{-1} g \in P_{(\alpha^*)_+} P_\alpha$. One deduces

$$g \in p r_\alpha P_{(\alpha^*)_+} P_\alpha \subset P_\alpha r_\alpha P_{(\alpha^*)_+} r_\alpha^{-1} r_\alpha P_\alpha = P_\alpha r_\alpha P_\alpha$$

(since $r_\alpha P_{(\alpha^*)_+} r_\alpha^{-1} \subset P_\alpha$).

We have indeed shown that $P_{\partial\alpha} \subset P_\alpha P_{(\alpha^*)_+} \cup P_\alpha r_\alpha P_\alpha$, whence (viii), the opposite inclusion p. 90
being obvious.

(5.1.21) *Proof of* (i).

In view of (5.1.16) we may assume $\mathrm{D} = \mathbf{D}$. On the other hand, we have already seen (5.1.1) that axiom (T 2) of Tits systems was satisfied.

Proof of (T 1). One knows that there exists an element $n \in N$ such that ${}^v\nu(n)(\mathbf{D}) = -\mathbf{D}$. One then has $P_{\mathrm{C}+\mathbf{D}} \subset \mathfrak{B}$ and $P_{\mathrm{C}-\mathbf{D}} \subset n\mathfrak{B}n^{-1}$. This, together with (v), shows that the group generated by $\mathfrak{B}$ and N contains P_{C}, hence contains B, which proves (T 1).

Proof of (T 3). Let us make $\mathbf{A}$ into a vector space by choosing as origin a special point o, and let $\mathfrak{C}$ be the sector $o + \mathbf{D}$. This is a simplicial cone in $\mathbf{A}$. Let $\mathscr{L}$ be the set of hyperplanes of $\mathbf{A}$ which contain a face of $\mathfrak{C}$; the images of the reflections s_{L} for $\mathrm{L} \in \mathscr{L}$ form exactly the subset R of ${}^v\mathbf{W}$. Let $\mathrm{L} \in \mathscr{L}$ and let $r \in N$ be such that $\nu(r) = s_{\mathrm{L}}$. We have to show that

(1) $$\mathfrak{B} r \mathfrak{B} w \mathfrak{B} \subset (\mathfrak{B} r w \mathfrak{B}) \cup (\mathfrak{B} w \mathfrak{B})$$

for every $w \in N$. We shall first show that

(2) $$\mathfrak{B} \cup \mathfrak{B} r \mathfrak{B} \text{ is a group.}$$

Let $b \in \mathfrak{B}^0$. Since b leaves fixed all the points of a sufficiently small sector contained in $\mathfrak{C}$, there exists a point $x \in r.\mathfrak{C}$ such that $rbr^{-1} \in P_{x+r.\overline{\mathfrak{C}}}$. Replacing x by a point of $x + r.\mathfrak{C}$ if necessary, we may assume that x is the transform of a point of C by an element t of $\nu^{-1}(\mathbf{V})$. One then has

$$P_{x+r.\overline{\mathfrak{C}}} \subset P_{t.\mathrm{C}} = t P_{\mathrm{C}} t^{-1} = t P_{\mathrm{C}+\overline{\mathfrak{C}}} t^{-1} \cdot t P_{\mathrm{C}-\overline{\mathfrak{C}}} t^{-1} = P_{x+\overline{\mathfrak{C}}} \cdot P_{x-\overline{\mathfrak{C}}}.$$

There therefore exist $b_1 \in P_{x+\overline{\mathfrak{C}}} \subset \mathfrak{B}^0$ and $c \in P_{x-\overline{\mathfrak{C}}}$ such that

(3) $$rbr^{-1} = b_1 c.$$

Since rbr^{-1} and b_1 belong to $P_{x+(\overline{\mathfrak{C}} \cap r.\overline{\mathfrak{C}})}$, one also has

(4) $$c \in P_{x-\overline{\mathfrak{C}}} \cap P_{x+(\overline{\mathfrak{C}} \cap r.\overline{\mathfrak{C}})}.$$

Now $\overline{\mathfrak{C}} \cap r.\overline{\mathfrak{C}}$ is a cone with nonempty interior in the hyperplane L, and L is consequently the convex hull of the union of $\overline{\mathfrak{C}} \cap r.\overline{\mathfrak{C}}$ and of $-(\overline{\mathfrak{C}} \cap r.\overline{\mathfrak{C}}) = (-\overline{\mathfrak{C}}) \cap \mathrm{L}$. One therefore draws from (4) that $c \in P_{x+\mathrm{L}} \cap P_{x-\mathfrak{C}}$. But the closed half-space E of $\mathbf{A}$ with wall $x + \mathrm{L}$ and not containing $x + \mathfrak{C}$ is the convex hull of the union of $x + \mathrm{L}$ and of $x - \mathfrak{C}$, and one has $c \in P_{\mathrm{E}}$. Let α be the affine root with boundary L containing C. What precedes shows that there exist affine roots γ equipollent to α such that

(5) $$rbr^{-1} \in \mathfrak{B}^0 \cdot P_{\gamma^*}.$$

It suffices indeed to choose γ such that $\mathrm{E} \supset \gamma^*$.

We shall now distinguish two cases:

1st case: (5) *is satisfied for every affine root γ equipollent to α.*

Put then, for every affine root γ contained in α,

$$rbr^{-1} = b_\gamma c_\gamma \quad \text{with } b_\gamma \in \mathfrak{B}^0 \text{ and } c_\gamma \in P_{\gamma^*}.$$

One has $c_\gamma c_\alpha^{-1} \in P_{\alpha^*} \cap \mathfrak{B}^0$. But the convex hull of a sector of direction $\mathbf{D}$ and of α^* is the whole p. 91
of $\mathbf{A}$; consequently one has $P_{\alpha^*} \cap \mathfrak{B}^0 = H$ and $c_\alpha \in HP_{\gamma^*} = P_{\gamma^*}$. Since the intersection of the subgroups P_{γ^*} for $\gamma \subset \alpha$ reduces to H, one draws $c_\alpha \in H$ and finally $rbr^{-1} \in \mathfrak{B}^0$.

2nd case: there exists a smallest affine root γ equipollent to α such that (5) *is satisfied.*

Let us then use (viii). One has

$$P_{\gamma^*} = r_\gamma P_\gamma r_\gamma \subset (P_\gamma r_\gamma P_\gamma) \cup (P_\gamma P_{(\gamma^*)_+}). \tag{6}$$

Since $P_\gamma \subset \mathfrak{B}^0$ and since $rbr^{-1} \notin \mathfrak{B}^0 P_{(\gamma^*)_+}$, one sees that $c_\gamma \in P_\gamma r_\gamma P_\gamma$, whence

$$rbr^{-1} \in \mathfrak{B}^0 P_\gamma r_\gamma P_\gamma \subset \mathfrak{B}^0 r_\gamma \mathfrak{B}^0 = \mathfrak{B}^0 r r^{-1} r_\gamma \mathfrak{B}^0 \subset \mathfrak{B}^0 r \mathfrak{B} \tag{7}$$

since $r^{-1} r_\gamma \in \nu^{-1}(\mathbf{V}) \subset \mathfrak{B}$.

In both cases we have therefore shown that $rbr^{-1} \in \mathfrak{B}^0 \cup \mathfrak{B}^0 r \mathfrak{B}$. One thus has $r\mathfrak{B}^0 r^{-1} \subset \mathfrak{B}^0 \cup \mathfrak{B}^0 r \mathfrak{B}$ and

$$r\mathfrak{B}r = r\mathfrak{B}^0 \nu^{-1}(\mathbf{V}) r = r\mathfrak{B}^0 r^{-1} r \nu^{-1}(\mathbf{V}) r \subset (\mathfrak{B}^0 \cup \mathfrak{B}^0 r \mathfrak{B}) \nu^{-1}(\mathbf{V}) = \mathfrak{B} \cup \mathfrak{B} r \mathfrak{B}$$

which implies at once that $\mathfrak{B} \cup \mathfrak{B} r \mathfrak{B}$ is a group.

Let now $w \in N$ and suppose first that *the length of ${}^v\nu(rw)$ (with respect to the generating system R of ${}^v\mathbf{W}$) is greater than that of ${}^v\nu(w)$*. By ([5], chap. V, § 3, n° 2, th. 1), this means that the vector chambers $\mathbf{D}$ and ${}^v\nu(rw).\mathbf{D}$ lie on either side of the hyperplane of ${}^v\mathbf{A}$ parallel to L; consequently, for every affine root γ equipollent to α, there exists a sector $\mathfrak{C}' \subset \mathfrak{C}$ such that $rw.\mathfrak{C}' \subset \gamma^*$, whence $P_{\gamma^*} \subset P_{rw.\mathfrak{C}'}$ and

$$(rw)^{-1} P_{\gamma^*} rw \subset P_{\mathfrak{C}'} \subset \mathfrak{B}^0.$$

Let then $b \in \mathfrak{B}^0$. By (5), there exist an affine root γ equipollent to α, an element $b_1 \in \mathfrak{B}$ and an element $c \in P_{\gamma^*}$ such that $rbw = rbr^{-1}rw = b_1 crw$. One deduces

$$rbw = b_1 rw (rw)^{-1} crw \in \mathfrak{B} rw \cdot (rw)^{-1} P_{\gamma^*} rw \subset \mathfrak{B} rw \mathfrak{B}.$$

Since $\nu^{-1}(\mathbf{V})$ is normal in N, it follows that

$$\mathfrak{B} r \mathfrak{B} w \mathfrak{B} \subset \mathfrak{B} rw \mathfrak{B} \quad \text{when } \ell({}^v\nu(rw)) > \ell({}^v\nu(w)). \tag{8}$$

Suppose now that the length of ${}^v\nu(rw)$ is smaller than that of ${}^v\nu(w)$. Substituting rw for w in (8), one draws $\mathfrak{B} r \mathfrak{B} rw \mathfrak{B} \subset \mathfrak{B} w \mathfrak{B}$, whence, using (2),

$$\mathfrak{B} r \mathfrak{B} w \mathfrak{B} \subset \mathfrak{B} r \mathfrak{B} r^{-1} \mathfrak{B} rw \mathfrak{B} \subset (\mathfrak{B} \cup \mathfrak{B} r \mathfrak{B}) \mathfrak{B} rw \mathfrak{B} = \mathfrak{B} rw \mathfrak{B} \cup \mathfrak{B} r \mathfrak{B} rw \mathfrak{B} \subset \mathfrak{B} rw \mathfrak{B} \cup \mathfrak{B} w \mathfrak{B}. \tag{9}$$

We have thus indeed proved (1), that is to say (T 3).

The quadruple $(G, \mathfrak{B}, N, R)$ satisfying axioms (T 1), (T 2) and (T 3) of Tits systems, one knows ([5], chap. IV, § 2, n° 3, remark) that $G = \mathfrak{B} N \mathfrak{B}$, that is to say that (ii) is satisfied. It follows that *condition* (iv) *is satisfied*.

Let us finally prove axiom (T 4):

$$r\mathfrak{B}r^{-1} \neq \mathfrak{B} \qquad (\text{for } r = s_{\mathrm{L}} \text{ with } \mathrm{L} \in \mathscr{L}).$$

Suppose there exists a hyperplane $\mathrm{L} \in \mathscr{L}$ such that $r\mathfrak{B}r^{-1} = \mathfrak{B}$ (with $r = s_{\mathrm{L}}$). If $\mathfrak{C} = x + \mathbf{D}$ is a sector of direction $\mathbf{D}$, one has

$$P_{\mathfrak{C}} \subset \mathfrak{B}^0 \subset r\mathfrak{B}r^{-1} = \nu^{-1}(\mathbf{V}) r \mathfrak{B}^0 r^{-1}.$$

For every element $g \in P_{\mathfrak{C}}$ there therefore exist a sector $\mathfrak{C}'$ of direction $\mathrm{D}' = {}^v\nu(r).\mathbf{D}$, an element p. 92
$h \in P_{\mathfrak{C}'}$ and an element $t \in \nu^{-1}(\mathbf{V})$ such that $g = th$. Let E (resp. E′) be the enclosure of $\mathfrak{C} \cup \mathfrak{C}'$

(resp. $\mathfrak{C} \cup t.\mathfrak{C}'$); it is a convex subset of $\mathbf{A}$ and one has $g.\mathrm{E} = \mathrm{E}'$. Moreover, the map $y \mapsto g.y$ from E onto E′ is affine, reduces to the identity on $\mathfrak{C}$ and to the translation $y \mapsto t.y$ on $\mathfrak{C}'$. One concludes that $\nu(t) = 1$ and that $g \in P_{\mathfrak{C}'}$. In other words, *every element which leaves fixed the points of a sector of direction* $\mathbf{D}$ *also leaves fixed the points of a sector of direction* D′.

Let then C′ be the chamber contained in $\mathfrak{C}$ to which o is adherent. What precedes implies that $P_{\mathrm{C}'+\mathbf{D}} \subset P_{\mathrm{C}'+\mathrm{D}'}$. Using (iv), one draws

$$P_{\mathrm{C}'} = P_{\mathrm{C}'+\mathbf{D}} \cdot P_{\mathrm{C}'-\mathbf{D}} \subset P_{\mathrm{C}'+\mathrm{D}'} \cdot P_{\mathrm{C}'-\mathbf{D}} \subset P_{(\mathrm{C}'+\mathrm{D}')\cap(\mathrm{C}'-\mathbf{D})}. \tag{10}$$

Now the only affine root which contains C′ and does not contain $r(\mathrm{C}')$ is the affine root α with boundary L containing C′. It follows that the chamber $r(\mathrm{C}')$ is contained in the intersection $(\mathrm{C}'+\mathrm{D}')\cap(\mathrm{C}'-\mathbf{D})$, and one deduces from (10) that $P_{\mathrm{C}'} \subset P_{r(\mathrm{C}')}$, which is impossible since the stabilizers of two distinct chambers are two distinct parahoric subgroups. We have thus reached a contradiction and axiom (T 4) is satisfied.

This completes the proof of (i) and consequently of Theorem (5.1.3).

Remark **(5.1.22)**. — Condition (viii) implies the following condition:

(viii bis) *The union of two half-apartments with the same wall having no chamber in common is an apartment.*

Indeed, suppose (viii) satisfied and let M and M′ be two half-apartments with the same wall, having no chamber in common. We may obviously restrict ourselves to the case where M′ is an affine root α of $\mathbf{A}$. There then exists an element $g \in P_{\partial\alpha}$ such that $g(\mathbf{A})$ is an apartment containing M (2.5.8). Replacing g by gr_α if necessary, we may assume that $g(\alpha) = \mathrm{M}$. Since $\mathrm{M}\cap\alpha$ contains no chamber, one has $g \notin P_\alpha \cdot P_{(\alpha^*)_+}$, whence, by (viii), $g = hr_\alpha h'$, with $h, h' \in P_\alpha$. Then $\mathrm{M} = hr_\alpha(\alpha) = h(\alpha^*)$ and $\mathrm{M} \cup \alpha = h(\mathbf{A})$, which proves (viii bis).

Conversely, one can show that condition (viii bis) implies (viii).

(5.1.23) *Until the end of* 5.1 *we assume that* (G, B, N, S) *is a double Tits system.* For every affine root α, the subgroup $P_{\partial\alpha}$ is then generated by P_α and P_{α^*}. Indeed, by (5.1.10) there exists an element $u \in P_{\alpha^*}$ which does not leave fixed a chamber C contained in α and bordered by $\partial\alpha$. One then has $u \notin P_\alpha \cdot P_{(\alpha^*)_+}$, whence $P_\alpha u P_\alpha = P_\alpha r_\alpha P_\alpha$ by (viii).

On the other hand, *the set of points of* $\mathscr{I}$ *left fixed by every element of* $P_{\partial\alpha}$ *reduces to* $\partial\alpha$. Indeed, let $x \in \mathscr{I}$ not belonging to $\partial\alpha$ and let y be a point of a panel F contained in $\partial\alpha$. If x is invariant under $P_{\partial\alpha}$, the segment $[xy]$ is likewise left fixed pointwise by $P_{\partial\alpha}$. Now there is a chamber C admitting F as a panel and containing a point of $[xy]$. This chamber would then be invariant under $P_{\partial\alpha}$, which contradicts (5.1.10).

Lemma **(5.1.24)**. — *Let* α *and* β *be two affine roots of* $\mathbf{A}$. *In order that* $\alpha^* = \beta$ *it is necessary and sufficient that the group generated by* P_α *and* P_β *leave fixed no chamber, but leave fixed a panel of* $\mathscr{I}$.

That the condition is necessary follows from (5.1.23). Conversely, if the affine root β is not equipollent to α^*, the group generated by P_α and P_β leaves fixed a chamber, since $\alpha \cap \beta$ has nonempty interior. The same holds when $\beta \supsetneq \alpha^*$. If $\beta \subset \alpha^*$, the fixed points of the group generated by P_α and P_β are contained in the set of fixed points of the group generated by P_α and P_{α^*} (resp. P_β and P_{β^*}), that is to say, by (5.1.23), in $\partial\alpha$ (resp. $\partial\beta$). If moreover $\beta \neq \alpha^*$, one has $\partial\alpha \cap \partial\beta = \emptyset$, which proves that the condition is sufficient. *p. 93*

(5.1.25) It follows from (5.1.24) that the datum of the set of subgroups P_α, for $\alpha \in \mathbf{\Sigma}$, allows one to determine the pairs (P_α, P_{α^*}), hence also the walls $\partial\alpha$ (5.1.23), and consequently the apartment $\mathbf{A}$, which is the convex hull of the union of the $\partial\alpha$. Consequently:

Proposition. —

(i) In order that an element g of G belong to N, it is necessary and sufficient that the inner automorphism defined by g permute among themselves the subgroups P_α for $\alpha \in \mathbf{\Sigma}$.

(ii) One has $H = \bigcap_{\alpha\in\mathbf{\Sigma}} P_\alpha = \bigcap_{\alpha\in\mathbf{\Sigma}} \operatorname{Norm} P_\alpha$.

(5.1.26) For every affine root α of $\mathbf{A}$, let us give ourselves a subset U_α of P_α such that $P_\alpha = H.U_\alpha$. One then sees easily, using (5.1.10), that for every chamber C bordered by $\partial\alpha$ and not contained in α, there exists an element $u \in U_\alpha$ such that $u(\mathrm{C}) \neq \mathrm{C}$. One deduces, as in (5.1.23) and (5.1.24), that the set of points of $\mathscr{I}$ left fixed by $U_\alpha \cup U_{\alpha^*}$ reduces to $\partial\alpha$, and that, if α and β are two affine roots, one has $\beta = \alpha^*$ if and only if $U_\alpha \cup U_\beta$ leaves fixed a panel and leaves fixed no chamber of $\mathscr{I}$. One may then generalize Proposition (5.1.25) as follows: *in order that an element $g \in G$ belong to the stabilizer N of the apartment* $\mathbf{A}$*, it is necessary and sufficient that for every affine root α of* $\mathbf{A}$ *there exist an affine root β of* $\mathbf{A}$ *such that $P_\beta = HgU_\alpha g^{-1}$.*

(5.1.27) Let Ω be a nonempty closed subset of $\mathbf{A}$ and let L be the affine subspace of $\mathbf{A}$ generated by Ω. Since Ω is a convex union of facets, one may choose a facet $\mathrm{F} \subset \Omega$ which is *open* in L. Let Δ be a connected component of the complement X in $\mathbf{A}$ of the union of the walls of $\mathbf{A}$ containing L. The orthogonal symmetry with respect to L transforms Δ into another connected component of X, denoted Δ'. One denotes by C_0 (resp. C'_0) the unique chamber contained in Δ (resp. Δ') and containing F in its closure.

Lemma. — Let C *be a chamber of $\mathscr{I}$ containing* F *in its closure. There exists an apartment containing* C *and* Δ.

Let $\Gamma = (\mathrm{C}_0, \mathrm{C}_1, \dots, \mathrm{C}_m = \mathrm{C})$ be a taut gallery between C_0 and C. Let us first show, by induction on i, that $\mathrm{F} \subset \overline{\mathrm{C}}_i$ for $0 \leqslant i \leqslant m$. This is true by hypothesis for $i = 0$. Let A be an apartment containing C_0 and C, hence Γ. Suppose $i > 0$ and let M be the wall of A separating C_{i-1} and C_i. Since M separates C_0 and C (2.3.10) and since $\mathrm{F} \subset \overline{\mathrm{C}}_0 \cap \overline{\mathrm{C}}$, one has $\mathrm{F} \subset \mathrm{M}$. By the induction hypothesis, $\mathrm{F} \subset \overline{\mathrm{C}}_{i-1}$, hence $\mathrm{F} \subset \mathrm{M} \cap \overline{\mathrm{C}}_{i-1} = \overline{\mathrm{C}}_{i-1} \cap \overline{\mathrm{C}}_i$ and $\mathrm{F} \subset \overline{\mathrm{C}}_i$.

Let us now prove the lemma by induction on m. If $m = 0$, there is nothing to prove. If $m \geqslant 1$, there exists by the induction hypothesis an apartment A con-taining Δ and C_{m-1}. By (2.5.8), there exists $g \in P_\Delta$ with $g.\mathrm{A} = \mathbf{A}$. Replacing C by g.C, one sees that one may assume $\mathrm{C}_{m-1} \subset \mathbf{A}$. Let then α be the affine root containing C_{m-1} and whose boundary $\partial\alpha$ contains $\overline{\mathrm{C}}_{m-1} \cap \overline{\mathrm{C}}$. Then $\partial\alpha$ separates C_0 and C (2.3.10) and one has $\mathrm{C}_0 \subset \alpha$. Since $\partial\alpha \supset \mathrm{F}$, one has $\Delta \subset \alpha$. But (5.1.10) implies that there exists an apartment containing C and α, whence the lemma. p. 94

Proposition **(5.1.28)**. — *Let us keep the notation of* (5.1.27)*. Put $N_\Omega = P_\Omega \cap N$ and let $\widetilde{\Omega}$ be the intersection of the affine roots α such that $\alpha \supset \Omega$ and $\Omega \not\subset \alpha^*$. Then $\widetilde{\Omega}$ is a closed subset with nonempty interior and one has*

$$P_\Omega = P_\Delta N_\Omega P_\Delta P_{\widetilde{\Omega}} = P_{\mathrm{L}} \cdot P_{\widetilde{\Omega}}.$$

The affine roots α satisfying $\Omega \subset \alpha$ and $\Omega \not\subset \alpha^*$ are those satisfying $\Omega \subset \alpha$ and $\Omega \not\subset \partial\alpha$, or again $\Omega \subset \alpha$ and $\mathrm{F} \not\subset \partial\alpha$. Let us show that

$$\widetilde{\Omega} = \mathrm{cl}(\Omega \cup \mathrm{C}_0 \cup \mathrm{C}'_0). \tag{1}$$

Indeed, if $\alpha \supset \Omega$ and $\partial\alpha \not\supset \mathrm{F}$, the facet F is interior to α; since $\mathrm{F} \subset \overline{\mathrm{C}}_0 \cap \overline{\mathrm{C}}'_0$, one concludes that $\mathrm{C}_0 \cup \mathrm{C}'_0 \subset \alpha$. Conversely, if $\alpha \supset \Omega \cup \mathrm{C}_0 \cup \mathrm{C}'_0$, then $\partial\alpha \not\supset \mathrm{F}$, since every wall containing F separates C_0 and C'_0. Whence (1). It is then clear that $\widetilde{\Omega}$ is a closed subset with nonempty interior.

Let now $g \in P_\Omega$. Put $\mathrm{C} = g.\mathrm{C}_0$. By Lemma (5.1.27), there exists an apartment A containing C and Δ, and consequently an element $u \in P_\Delta$ such that $u.\mathrm{A} = \mathbf{A}$ (2.5.8), whence $ug.\mathrm{C}_0 \subset \mathbf{A}$.

There then exists $n \in N$ with $nug.\mathrm{C}_0 = \mathrm{C}_0$, and n is the identity on $\overline{\mathrm{C}}_0 \cap ug.\overline{\mathrm{C}}_0 \supset \mathrm{F}$. Hence n is the identity on L and $n \in N_\Omega$.

Put $\mathrm{C}' = nug.\mathrm{C}'_0$. Again by Lemma (5.1.27), there exists an apartment A′ containing Δ and C', whence an element $u' \in P_\Delta$ such that $g'.\mathrm{C}'_0 \subset \mathbf{A}$, on putting $g' = u'nug$. Since $g'.\mathrm{C}_0 = \mathrm{C}_0$, one has $w(\mathrm{C}_0, \mathrm{C}'_0) = w(\mathrm{C}_0, g'.\mathrm{C}'_0)$ (with the notation of (2.1.7)), whence $g'.\mathrm{C}'_0 = \mathrm{C}'_0$ by (2.3.8). Hence $g' \in P_{\Omega\cup\mathrm{C}_0\cup\mathrm{C}'_0}$ and $g' \in P_{\widetilde{\Omega}}$ by (1). Finally one has indeed $g = u^{-1}n^{-1}u'^{-1}g' \in P_\Delta N_\Omega P_\Delta P_{\widetilde{\Omega}}$, whence the proposition.

Corollary **(5.1.29)**. — *Let $\varphi : G \to \widehat{G}$ be a B-N-adapted homomorphism. Let us resume the notation of* (4.1.1) *and put $\widehat{N}_\Omega = \widehat{N} \cap \widehat{P}_\Omega$ and $\widehat{N}^\dagger_\Omega = \widehat{N} \cap \widehat{P}^\dagger_\Omega$. One has*

$$a) \qquad \widehat{P}_\Omega = (\widehat{N}^\dagger_\Delta \cap \widehat{N}_\Omega)\varphi(P_\Omega) = \varphi(P_\Delta)\widehat{N}_\Omega\varphi(P_\Delta P_{\widetilde{\Omega}}) = \widehat{P}_\Delta\widehat{N}_\Omega\widehat{P}_\Delta\widehat{P}_{\widetilde{\Omega}};$$
$$b) \qquad \widehat{P}^\dagger_\Omega = (\widehat{N}^\dagger_\Delta \cap \widehat{N}^\dagger_\Omega)\varphi(P_\Omega) = \varphi(P_\Delta)\widehat{N}^\dagger_\Omega\varphi(P_\Delta P_{\widetilde{\Omega}}) = \widehat{P}_\Delta\widehat{N}^\dagger_\Omega\widehat{P}_\Delta\widehat{P}_{\widetilde{\Omega}}.$$

Let $g \in \widehat{P}^\dagger_\Omega$. By (2.5.8), there exists $h \in P_\Omega$ with $h.g^{-1}.\mathbf{A} = \mathbf{A}$, whence $\varphi(h)g^{-1} \in \widehat{N} \cap \widehat{P}^\dagger_\Omega = \widehat{N}^\dagger_\Omega$. Hence $\widehat{P}^\dagger_\Omega = \widehat{N}^\dagger_\Omega\varphi(P_\Omega)$. But, if $n \in \widehat{N}_\Omega$, $n.\Delta$ is a connected component of X and there exists $n' \in N_\Omega$ such that $n'.n.\Delta = \Delta$, whence $\widehat{N}^\dagger_\Omega = (\widehat{N}^\dagger_\Delta \cap \widehat{N}^\dagger_\Omega)\varphi(N_\Omega)$. This proves the first of the equalities b). On the other hand, if $n \in \widehat{N}^\dagger_\Delta$, one has $n\widehat{P}_\Delta n^{-1} = \widehat{P}_\Delta$, whence $n\varphi(P_\Delta)n^{-1} = \varphi(P_\Delta)$ since $\varphi(G)$ is normal in $\widehat{G}$ and since $\varphi(P_\Delta) = \varphi(G) \cap \widehat{P}_\Delta$. One deduces the other equalities b), and a) follows at once from b).

Remark **(5.1.30)**. — It is clear that $\widehat{N}_\Omega = \widehat{N}_\mathrm{L}$. On the other hand, one may have $\widehat{N}^\dagger_\Omega \neq \widehat{N}^\dagger_\mathrm{L}$. *p. 95*
Moreover, one has

$$(1) \qquad \widehat{N}^\dagger_\Delta \cap \widehat{N}_\Omega = \hat{\nu}^{-1}(\widehat{\mathbf{W}}^\dagger_{\mathrm{C}_0} \cap \widehat{\mathbf{W}}_\mathrm{F})$$

since C_0 is the only chamber contained in Δ and admitting F as a facet. One concludes that *$\widehat{H}\varphi(P_\Omega)$ is a normal subgroup of finite index of $\widehat{P}_\Omega$*, the quotient $\widehat{P}_\Omega/\widehat{H}\varphi(P_\Omega)$ being isomorphic to $\widehat{\mathbf{W}}^\dagger_{\mathrm{C}_0} \cap \widehat{\mathbf{W}}_\mathrm{F}$.

Proposition **(5.1.31)**. — *Every B-N-adapted homomorphism is also $\mathfrak{B}$-N-adapted.*

Let $\varphi : G \to \widehat{G}$ be a B-N-adapted homomorphism. One knows that $\widehat{G} = \varphi(G).\widehat{N}$. Since $\widehat{N}$ normalizes $\varphi(N)$ and $\varphi(\nu^{-1}(\mathbf{V}))$, it suffices to show that, for every $n \in \widehat{N}$, there exists $n' \in N$ such that $\varphi(n')n$ normalizes $\varphi(\mathfrak{B}^0)$. One may write $w = \hat{\nu}(n) = w'w''t$, with $w' \in \mathbf{W}_x$, $w'' \in \widetilde{\mathbf{W}}^\dagger_{x+\mathbf{D}}$ and $t \in \widetilde{\mathbf{V}}$ (where x is a special point of $\mathbf{A}$). Let $n' \in \nu^{-1}(w')$. For every sector $\mathfrak{C}$ of direction $\mathbf{D}$ of $\mathbf{A}$, one then has

$$n\varphi(P_\mathfrak{C})n^{-1} = \varphi(P_{w'w''t(\mathfrak{C})}) = \varphi(n'P_{\mathfrak{C}'}n'^{-1})$$

where $\mathfrak{C}' = w''t(\mathfrak{C})$ is a sector of $\mathbf{A}$, of direction $\mathbf{D}$. One therefore has $n\varphi(\mathfrak{B}^0)n^{-1} = \varphi(n'\mathfrak{B}^0n'^{-1})$, whence the proposition.

Let moreover ${}^v\omega$ be the homomorphism from $\widehat{G}$ into $\mathrm{Aut}({}^v\mathbf{W}, R)$ associated with φ (regarded as a $\mathfrak{B}$-N-adapted homomorphism). One knows that ${}^v\omega$ is trivial on $\varphi(G)$ and we have just shown that $\widehat{G} = \varphi(G).\mathrm{E}$, with $\mathrm{E} = \{g \in \widehat{N} \mid {}^v\hat{\nu}(g).\mathbf{D} = \mathbf{D}\}$. If $g \in \mathrm{E}$ and $n \in N$, one has (1.2.20)

$${}^v\omega(g)({}^v\nu(n)) = {}^v\nu(g\varphi(n)g^{-1}) = {}^v\hat{\nu}(g)\,{}^v\nu(n)\,{}^v\hat{\nu}(g)^{-1}$$

which shows that ${}^v\omega$ restricted to E is identified with the composite of ${}^v\hat{\nu}$ and the canonical isomorphism of ${}^v\widehat{\mathbf{W}}^\dagger_\mathbf{D}$ onto $\mathrm{Aut}({}^v\mathbf{W}, R)$ ((1.3.18) (4)). It follows that *the kernel ${}^v\widehat{G}_0$ of ${}^v\omega$ is equal to $\varphi(G)\hat{\nu}^{-1}(\widehat{\mathbf{V}})$*. Moreover, since $\hat{\nu}^{-1}(\widehat{\mathbf{V}})$ normalizes $\varphi(\mathfrak{B})$ and since $\varphi(\mathfrak{B})$ is its own normalizer in $\varphi(G)$, one deduces that $\mathrm{Stab}\,\mathfrak{B} \cap {}^v\widehat{G}_0 = \varphi(\mathfrak{B}) \cdot \hat{\nu}^{-1}(\widehat{\mathbf{V}}) = \widehat{\mathfrak{B}}$, which shows that the notation $\widehat{\mathfrak{B}}$ is coherent with (1.2.17).

Corollary **(5.1.32)**. — *If $\varphi : G \to \widehat{G}$ is a B-N-adapted homomorphism of connected type* (4.1.3), *then $\widehat{G} = {}^v\widehat{G}_0$, one has $\widehat{N} \cap \widehat{\mathfrak{B}} = \hat{\nu}^{-1}(\widehat{\mathbf{V}})$, the quotient group $\widehat{N}/\widehat{\mathfrak{B}} \cap \widehat{N}$ is identified with ${}^v\mathbf{W}$, and the quadruple $(\widehat{G}, \widehat{\mathfrak{B}}, \widehat{N}, R)$ is a Tits system with Weyl group ${}^v\mathbf{W}$.*

Since ${}^v\widehat{\mathbf{W}} = {}^v\mathbf{W}$, one has $\mathrm{E} = \hat{\nu}^{-1}(\widehat{\mathbf{V}})$, whence $\widehat{G} = {}^v\widehat{G}_0$, and the corollary follows from (1.2.17).

(5.1.33) One knows ([5], [39]) that to every Tits system one may associate an abstract simplicial complex, which we shall here call the *combinatorial building* of the system, whose simplices represent the parabolic subgroups, the face relation corresponding to the inverse of the inclusion relation. In what follows we shall briefly indicate the relations between the building $\mathscr{I}$ and the combinatorial building of the Tits system $(G, \mathfrak{B}, N, R)$ with Weyl group ${}^v\mathbf{W}$, which we shall p. 96 denote by ${}^v\mathscr{I}$. By definition, the chambers of ${}^v\mathscr{I}$ are in bijective correspondence with the minimal parabolic subgroups, that is to say with the conjugates of $\mathfrak{B}$, this correspondence being compatible with the action of G on ${}^v\mathscr{I}$ on the one hand and on the set of conjugates of $\mathfrak{B}$ (by inner automorphisms) on the other.

Moreover, the group $\mathfrak{B}$ is the stabilizer of the sector germ $\widetilde{\mathfrak{C}}_{\mathfrak{B}}$ (2.9.2) containing the sectors $x + \mathbf{D}$ for $x \in \mathbf{A}$, and it is easy to see that G operates transitively on the set of sector germs of $\mathscr{I}$, the stabilizer of $g.\widetilde{\mathfrak{C}}_{\mathfrak{B}}$ being $g\mathfrak{B}g^{-1}$. Since $\mathfrak{B}$ is its own normalizer, one deduces a bijection compatible with the action of G between the set of sector germs of $\mathscr{I}$ and the set of minimal parabolic subgroups, or again the set of chambers of ${}^v\mathscr{I}$: the chamber $f(\widetilde{\mathfrak{C}})$ associated with the sector germ $\widetilde{\mathfrak{C}}$ is the one whose stabilizer coincides with that of $\widetilde{\mathfrak{C}}$.

Moreover, *sector germs $\widetilde{\mathfrak{C}}_i$ $(1 \leqslant i \leqslant m)$ are contained in one and the same apartment of $\mathscr{I}$ if and only if the chambers $f(\widetilde{\mathfrak{C}}_i)$ are contained in one and the same apartment of ${}^v\mathscr{I}$* (recall that an apartment of ${}^v\mathscr{I}$ is the transform by an element of G of the subcomplex ${}^v\mathbf{A}$ of ${}^v\mathscr{I}$ whose simplices are those associated with the parabolic subgroups $n\mathfrak{P}n^{-1}$ for $\mathfrak{P} \supset \mathfrak{B}$ and $n \in N$). To show this, it suffices to see that $\widetilde{\mathfrak{C}} \subset \mathbf{A}$ is equivalent to $f(\widetilde{\mathfrak{C}}) \subset {}^v\mathbf{A}$, which is obvious, since $\widetilde{\mathfrak{C}} \subset \mathbf{A}$ is equivalent to the existence of an $n \in N$ with $\widetilde{\mathfrak{C}} = n.\widetilde{\mathfrak{C}}_{\mathfrak{B}}$. One deduces at once that *there exists one and only one bijection, again denoted f, from the set of apartments of $\mathscr{I}$ onto the set of apartments of ${}^v\mathscr{I}$, such that a sector germ $\widetilde{\mathfrak{C}}$ is contained in an apartment* A *if and only if the chamber $f(\widetilde{\mathfrak{C}})$ is contained in the apartment $f(\mathrm{A})$*. It is clear that this bijection is compatible with the action of G on $\mathscr{I}$ and on ${}^v\mathscr{I}$.

It follows in particular from what precedes that to every automorphism θ of the building $\mathscr{I}$ there corresponds one and only one automorphism $\nu(\theta)$ of ${}^v\mathscr{I}$ such that $\nu(\theta).f(\widetilde{\mathfrak{C}}) = f(\theta.\widetilde{\mathfrak{C}})$ for every sector germ $\widetilde{\mathfrak{C}}$ of $\mathscr{I}$. One thus obtains a homomorphism ν from $\operatorname{Aut}\mathscr{I}$ into $\operatorname{Aut}{}^v\mathscr{I}$, a homomorphism which is *injective*. Indeed, if $\nu(\theta) = 1$, then θ preserves all the apartments of $\mathscr{I}$. Now one sees easily that each facet F of $\mathscr{I}$ has as closure the intersection of the apartments of $\mathscr{I}$ which contain it; one deduces that $\theta.\mathrm{F} = \mathrm{F}$ for every facet F of $\mathscr{I}$, whence $\theta = 1$. This makes it possible to use, for the study of $\operatorname{Aut}\mathscr{I}$, the results established in [39] on the automorphism group of a combinatorial building associated with a Tits system with finite Weyl group.

5.2. Axiomatic characterization of the family of the P_Ω. —

In n^os (5.2.1) to (5.2.27) we abandon the hypotheses and notation adopted in §§ 2 to 4, retaining only those of (1.5.2). We denote by G a group, by N a subgroup of G, by ν a surjective homomorphism from N onto $\mathbf{W}$, and by H the kernel of ν. We put $W = N/H$ and we denote by S the inverse image of $\mathbf{S}$ in W under the isomorphism of W onto $\mathbf{W}$ deduced from ν (an isomorphism by means of which one will sometimes identify W and $\mathbf{W}$). One is given, for every p. 97 affine root $\alpha \in \mathbf{\Sigma}$, a subgroup P_α of G. For every subset Ω of $\mathbf{A}$, one denotes by P_Ω the subgroup of G generated by H and the P_α for $\alpha \in \mathbf{\Sigma}$ and $\alpha \supset \Omega$. It is clear that

$$P_\Omega = P_{\mathrm{cl}(\Omega)}.$$

We shall impose on these data conditions ((5.2.1) (1) to (6)) which will imply that (G, P_C, N, S) is a double Tits system. Later (5.2.27) we shall show that, conversely, the subgroups P_α associated as in § 4 with a double Tits system satisfy these conditions, the notations P_Ω being concordant. This will make it possible to apply to the case of double Tits systems the results on the structure of the groups P_Ω which we shall prove in (5.2.6) et seq.

(5.2.1) We assume, up to (5.2.27), that the preceding data satisfy the following six conditions:

(A 1) *For every $n \in N$ and every $\alpha \in \mathbf{\Sigma}$, one has $nP_\alpha n^{-1} = P_{\nu(n).\alpha}$.*
(A 2) *Let $\alpha \in \mathbf{\Sigma}$. For every $\beta \in \mathbf{\Sigma}$ with $\beta \supsetneq \alpha$, one has $P_\beta \subsetneq P_\alpha$, and the intersection of the P_β for $\beta \supset \alpha$ is equal to H.*
(A 3) *Let $\alpha, \beta \in \mathbf{\Sigma}$ be such that $\alpha \cap \beta$ has nonempty interior, and let Ω be the intersection of the affine roots containing $\alpha \cap \beta$ and not containing α. Then $P_\alpha P_\Omega$ is a group.*
(A 4) *For every $\alpha \in \mathbf{\Sigma}$, the subset $P_\alpha P_{(\alpha^*)_+} \cup P_\alpha \nu^{-1}(r_\alpha) P_\alpha$ is a subgroup of G.*
(A 5) *For every $x \in \mathbf{A}$ and every vector chamber D, one has $P_{x+\mathrm{D}} \cap P_{x-\mathrm{D}} = H$.*
(A 6) *The group G is generated by the union of the P_α for $\alpha \in \mathbf{\Sigma}$.*

(5.2.2) It is clear that (A 1) implies

(A 1 bis) $$nP_\Omega n^{-1} = P_{\nu(n).\Omega} \qquad \text{for } n \in N \text{ and } \Omega \subset \mathbf{A}.$$

Using then the transitivity of ${}^v\mathbf{W}$ on the vector chambers, one sees at once that, granting (A 1), it suffices to assume (A 5) for *one* vector chamber D. Moreover, (A 2) shows that for $\Omega = \alpha$ the notation P_Ω is unambiguous.

On the other hand, let us place ourselves in the hypotheses of (A 3). Since every affine root containing $\alpha \cap \beta$ contains either α or Ω, one deduces at once from (A 2) and (A 3) that one has

(1) $$P_{\alpha\cap\beta} = P_\alpha P_\Omega.$$

Suppose moreover that $\beta \supsetneq \alpha^*$. Then one has $\Omega = \beta$ and

(2) $$P_{\alpha\cap\beta} = P_\alpha P_\beta.$$

In particular, one has

(3) $$P_{\alpha\cap(\alpha^*)_+} = P_\alpha P_{(\alpha^*)_+}.$$

(5.2.3) By (A 2) one has $H \subset P_\alpha$ for every $\alpha \in \mathbf{\Sigma}$. In what follows one denotes by U_α a system of representatives of P_α modulo H, i.e. a subset of P_α such that the map $(u, h) \mapsto uh$ from $U_\alpha \times H$ into P_α is bijective. One assumes the U_α chosen in such a way that $U_\alpha \supset U_\beta$ for $\alpha, \beta \in \mathbf{\Sigma}$ with $\alpha \subset \beta$. In the applications we have in view, the U_α will be subgroups of P_α. One also puts $U_{\mathbf{A}} = \{1\}$ and $\mathfrak{U}_a = \bigcup_{{}^v\alpha=a} U_\alpha$ for $a \in {}^v\mathbf{\Sigma}$. *p. 98*

(5.2.4) Let Ω be a nonempty subset of $\mathbf{A}$. One denotes by $\widetilde{\mathbf{\Sigma}}(\Omega)$ the set of affine roots containing Ω, and by $\mathbf{\Sigma}(\Omega)$ the set of minimal elements of $\widetilde{\mathbf{\Sigma}}(\Omega)$, ordered by inclusion. Since $\Omega \neq \emptyset$, every element of $\widetilde{\mathbf{\Sigma}}(\Omega)$ contains one and only one element of $\mathbf{\Sigma}(\Omega)$. The map $\alpha \mapsto {}^v\alpha$ restricted to $\mathbf{\Sigma}(\Omega)$ is evidently injective. Let ${}^v\mathbf{\Sigma}(\Omega)$ be its image; for $a \in {}^v\mathbf{\Sigma}(\Omega)$ one will write $a(\Omega)$ for the unique element of $\mathbf{\Sigma}(\Omega)$ satisfying ${}^va(\Omega) = a$. If $a \in {}^v\mathbf{\Sigma} - {}^v\mathbf{\Sigma}(\Omega)$, one puts $a(\Omega) = \mathbf{A}$. One has

$$\mathrm{cl}(\Omega) = \bigcap_{\alpha\in\widetilde{\mathbf{\Sigma}}(\Omega)} \alpha = \bigcap_{\alpha\in\mathbf{\Sigma}(\Omega)} \alpha = \bigcap_{a\in{}^v\mathbf{\Sigma}(\Omega)} a(\Omega) = \bigcap_{a\in{}^v\mathbf{\Sigma}} a(\Omega).$$

(5.2.5) Let Ω be a subset of $\mathbf{A}$ containing a sector of direction D. Then one has ${}^v\mathbf{\Sigma}(\Omega) \subset {}^v\mathbf{\Sigma}^+(\mathrm{D})$. Indeed, for every $\alpha \in \mathbf{\Sigma}$ there exists $x \in \mathbf{A}$ such that $\alpha = \{x + t \mid t \in {}^v\mathbf{A},\ {}^v\alpha(t) \geqslant 0\}$ (1.3.8). If ${}^v\alpha \notin {}^v\mathbf{\Sigma}^+(\mathrm{D})$ and $z \in \mathrm{D}$, one has ${}^v\alpha(z) < 0$. Let then $y \in \Omega$ and $\mu \in \mathbf{R}_+$ be such that $\mu^v\alpha(z) < {}^v\alpha(x - y)$. One has $y + \mu z \in \Omega + \mathrm{D} \subset \mathrm{cl}(\Omega)$ and ${}^v\alpha(y + \mu z - x) < 0$, whence $y + \mu z \notin \alpha$ and $\alpha \notin \widetilde{\mathbf{\Sigma}}(\Omega)$.

It follows (1.3.15) that *one may order ${}^v\mathbf{\Sigma}(\Omega)$ according to a nibbling order*.

Proposition **(5.2.6)**. — *Let Ω be a subset of $\mathbf{A}$ containing a sector and let $(a_1,\dots,a_n)$ be the elements of ${}^v\mathbf{\Sigma}(\Omega)$ arranged according to a nibbling order. The product map $\pi : (u_1,\dots,u_n,h) \mapsto u_1\dots u_n h$ is a bijection of $\prod_{1\leqslant j\leqslant n} U_{a_j(\Omega)} \times H$ onto P_Ω.*

We shall prove this proposition by induction on n, the assertion being obvious for $n = 0$ and $n = 1$. Suppose $n > 1$ and put $\Omega' = \bigcap_{2\leqslant j\leqslant n} a_j(\Omega)$, whence $\mathrm{cl}(\Omega) = a_1(\Omega) \cap \Omega'$. Since a_1 is extremal in $\{a_1,\dots,a_n\}$, the half-space $\{t \in {}^v\mathbf{A} \mid a_1(t) < 0\}$ does not contain the intersection of the half-spaces $\{t \in {}^v\mathbf{A} \mid a_j(t) < 0\}$ for $2 \leqslant j \leqslant n$, and one concludes immediately that no affine root equipollent to $a_1(\Omega)$ contains Ω', which implies that ${}^v\mathbf{\Sigma}(\Omega') = \{a_2,\dots,a_n\}$. Since $(a_2,\dots,a_n)$ is a nibbling order, one has by the induction hypothesis

$$P_{\Omega'} = U_{a_2(\Omega)} \dots U_{a_n(\Omega)} H.$$

Let us now show that $Q = U_{a_1(\Omega)} P_{\Omega'}$ is equal to P_Ω. For this it suffices to show that Q is stable under left multiplication by a generating system of P_Ω. Now it is obvious that $HQ \subset Q$ and that $U_{a_1(\Omega)} Q \subset Q$, since $HU_{a_1(\Omega)} \cup U^2_{a_1(\Omega)} \subset P_{a_1(\Omega)} = U_{a_1(\Omega)} H$ and since $H \subset P_{\Omega'}$. If $j \geqslant 2$, one has by (5.2.2) (1)

$$P_{a_1(\Omega)\cap a_j(\Omega)} = P_{a_1(\Omega)} P_{\Omega_j} = U_{a_1(\Omega)} P_{\Omega_j}$$

where Ω_j denotes the intersection of the affine roots containing $a_1(\Omega)\cap a_j(\Omega)$ and not containing $a_1(\Omega)$. But these affine roots contain Ω and do not contain $a_1(\Omega)$, hence they contain Ω'. One therefore has $\Omega_j \supset \Omega'$ and $P_{\Omega_j} \subset P_{\Omega'}$. Whence p. 99

$$\begin{aligned} U_{a_j(\Omega)} Q = U_{a_j(\Omega)} U_{a_1(\Omega)} P_{\Omega'} &\subset P_{a_1(\Omega)\cap a_j(\Omega)} P_{\Omega'} = U_{a_1(\Omega)} P_{\Omega_j} P_{\Omega'} \\ &\subset U_{a_1(\Omega)} P_{\Omega'} = Q. \end{aligned}$$

We have thus shown that π is *surjective*.

Let us now show that π is *injective*. Let $h, h' \in H$ and $u_j, u'_j \in U_{a_j(\Omega)}$ be such that $u_1 \dots u_n h = u'_1 \dots u'_n h'$. Since a_1 is extremal in $\{a_1,\dots,a_n\}$, there exists a vector chamber D such that $a_1 \in {}^v\mathbf{\Sigma}^+(\mathrm{D})$ and $a_j \in -{}^v\mathbf{\Sigma}^+(\mathrm{D})$ for $2 \leqslant j \leqslant n$. If x is a point of Ω, one then has $a_1(\Omega) \supset x - \mathrm{D}$ and $\Omega' \supset x + \mathrm{D}$, whence

$$u_1'^{-1} u_1 \in P_{a_1(\Omega)} \cap P_{\Omega'} \subset P_{x-\mathrm{D}} \cap P_{x+\mathrm{D}} = H$$

and $u_1 = u'_1$. The induction hypothesis then implies $u_j = u'_j$ for every j, whence the injectivity of π.

(5.2.7) One may express Proposition (5.2.6) in a slightly different form. Let Ω be a subset of $\mathbf{A}$ containing a sector of direction D. Let us arrange the elements of ${}^v\mathbf{\Sigma}^+(\mathrm{D})$ in a nibbling order $(a_1,\dots,a_m)$. Then *the product map $(u_1,\dots,u_m,h) \mapsto u_1\dots u_m h$ is a bijection of $\prod_{1\leqslant j\leqslant m} U_{a_j(\Omega)} \times H$ onto P_Ω*: this follows immediately from the fact that the order induced on ${}^v\mathbf{\Sigma}(\Omega)$ is nibbling (1.3.15), and from the definitions of $a_j(\Omega)$ (5.2.4) and of $U_\mathbf{A}$ (5.2.3).

(5.2.8) For every $\alpha \in \mathbf{\Sigma}$, the product map $(h,u) \mapsto hu$ is a bijection of $H \times U_\alpha^{-1}$ onto P_α. Consequently one deduces from (5.2.6) the following assertion, which generalizes (5.2.6): with the notation of (5.2.6) (or of (5.2.7)), the product map $(u_1,\dots,u_k,h,u_{k+1},\dots,u_n) \mapsto u_1\dots u_k h u_{k+1}\dots u_n$ is a bijection of $\prod_{1\leqslant j\leqslant k} U_{a_j(\Omega)} \times H \times \prod_{k+1\leqslant j\leqslant n} U^{-1}_{a_j(\Omega)}$ onto P_Ω, for any k with $0 \leqslant k \leqslant n$.

Proposition **(5.2.9)**. — *Let Ω be a nonempty subset of $\mathbf{A}$ and let D be a vector chamber. Let us arrange the elements of ${}^v\mathbf{\Sigma}(\Omega)\cap{}^v\mathbf{\Sigma}^+(\mathrm{D})$ (resp. ${}^v\mathbf{\Sigma}(\Omega)\cap{}^v\mathbf{\Sigma}^+(-\mathrm{D})$) in a nibbling order $(a_1,\dots,a_n)$ (resp. $(b_1,\dots,b_m)$). The product map $(u_1,\dots,u_n,h,v_1,\dots,v_m) \mapsto u_1\dots u_n h v_1\dots v_m$ from $\prod_{1\leqslant j\leqslant n} U_{a_j(\Omega)} \times H \times \prod_{1\leqslant k\leqslant m} U^{-1}_{b_k(\Omega)}$ into P_Ω is injective.*

One has $\mathrm{cl}(\Omega+\mathrm{D}) = \bigcap_{1\leqslant j\leqslant n} a_j(\Omega)$ and $\mathrm{cl}(\Omega-\mathrm{D}) = \bigcap_{1\leqslant k\leqslant m} b_k(\Omega)$, whence

$$U_{a_1(\Omega)}\dots U_{a_n(\Omega)}H \subset P_{\Omega+\mathrm{D}} \qquad \text{and} \qquad U^{-1}_{b_1(\Omega)}\dots U^{-1}_{b_m(\Omega)} \subset P_{\Omega-\mathrm{D}}.$$

If, with obvious notation,

$$u_1\dots u_n h v_1\dots v_m = u'_1\dots u'_n h' v'_1\dots v'_m,$$

then

$$g = (u'_1\dots u'_n)^{-1}u_1\dots u_n h = h'v'_1\dots v'_m(v_1\dots v_m)^{-1} \in P_{\Omega+\mathrm{D}}\cap P_{\Omega-\mathrm{D}}.$$

But, if $x\in\Omega$, one has $P_{\Omega+\mathrm{D}}\cap P_{\Omega-\mathrm{D}} \subset P_{x+\mathrm{D}}\cap P_{x-\mathrm{D}} = H$, whence $g\in H$. It then suffices to apply (5.2.6) and (5.2.8) to $\mathrm{cl}(\Omega+\mathrm{D})$ and to $\mathrm{cl}(\Omega-\mathrm{D})$ in order to draw $u_j = u'_j$ and $v_k = v'_k$ for $1\leqslant j\leqslant n$ and $1\leqslant k\leqslant m$.

Lemma **(5.2.10)**. — *Let Ω be a subset of* $\mathbf{A}$ *with nonempty interior. For every vector chamber* D, *put $Q_{\mathrm{D}} = P_{\Omega+\mathrm{D}}\cdot P_{\Omega-\mathrm{D}}$. The subset Q_{D} is independent of the choice of* D.

It suffices to prove that if r is the reflection with respect to a wall of a vector chamber D, one *p. 100*
has $Q_{\mathrm{D}} \subset Q_{r(\mathrm{D})}$. Now one may arrange the elements of ${}^v\boldsymbol{\Sigma}^+(\mathrm{D})$ in a nibbling order $(a_1,\dots,a_m)$ in such a way that r is the reflection with respect to the hyperplane $a_1 = 0$ (1.3.15). Then $(-a_1,\dots,-a_m)$ is a nibbling order on ${}^v\boldsymbol{\Sigma}^+(-\mathrm{D})$, and one has (5.2.8)

$$P_{\Omega+\mathrm{D}} = HU^{-1}_{a_1(\Omega)}\dots U^{-1}_{a_m(\Omega)} = U_{a_m(\Omega)}\dots U_{a_1(\Omega)}H$$

whence

$$(1) \qquad Q_{\mathrm{D}} = U_{a_m(\Omega)}\dots U_{a_1(\Omega)}HU^{-1}_{-a_1(\Omega)}\dots U^{-1}_{-a_m(\Omega)}.$$

Since $a_1(\Omega)\cap(-a_1(\Omega))$ contains the nonempty interior of Ω, one has ((5.2.2) (2))

$$(2) \qquad U_{a_1(\Omega)}HU^{-1}_{-a_1(\Omega)} \subset P_{a_1(\Omega)\cap -a_1(\Omega)} = P_{-a_1(\Omega)}P_{a_1(\Omega)} = U_{-a_1(\Omega)}HU^{-1}_{a_1(\Omega)}.$$

But $\mathrm{cl}(\Omega - r(\mathrm{D})) = (-a_1(\Omega))\cap\bigcap_{2\leqslant j\leqslant m} a_j(\Omega)$ and $\mathrm{cl}(\Omega + r(\mathrm{D})) = a_1(\Omega)\cap\bigcap_{2\leqslant j\leqslant m}(-a_j(\Omega))$. Hence (1) and (2) imply

$$Q_{\mathrm{D}} \subset P_{\Omega+r(\mathrm{D})}P_{\Omega-r(\mathrm{D})} = Q_{r(\mathrm{D})}.$$

Proposition **(5.2.11)**. — *Let Ω be a subset of* $\mathbf{A}$ *with nonempty interior, and let* D *be a vector chamber. One has $P_\Omega = P_{\Omega+\mathrm{D}}P_{\Omega-\mathrm{D}}$. If one arranges the elements of ${}^v\boldsymbol{\Sigma}(\Omega)\cap{}^v\boldsymbol{\Sigma}^+(\mathrm{D})$ (resp. ${}^v\boldsymbol{\Sigma}(\Omega)\cap{}^v\boldsymbol{\Sigma}^+(-\mathrm{D})$) in a nibbling order $(a_1,\dots,a_n)$ (resp. $(b_1,\dots,b_m)$), the product map*

$$(u_1,\dots,u_n,h,v_1,\dots,v_m) \mapsto u_1\dots u_n h v_1\dots v_m$$

is a bijection of $\prod_{1\leqslant j\leqslant n} U_{a_j(\Omega)}\times H\times\prod_{1\leqslant k\leqslant m} U^{-1}_{b_k(\Omega)}$ onto P_Ω.

For every root $a\in{}^v\boldsymbol{\Sigma}(\Omega)$ there exists a vector chamber D' contained in the half-space $a>0$. One then has $U_{a(\Omega)}P_{\Omega+\mathrm{D}'}\subset P_{\Omega+\mathrm{D}'}$. Since $P_{\Omega+\mathrm{D}}P_{\Omega-\mathrm{D}} = P_{\Omega+\mathrm{D}'}P_{\Omega-\mathrm{D}'}$ (Lemma (5.2.10)), one deduces that $P_{\Omega+\mathrm{D}}P_{\Omega-\mathrm{D}}$ is stable under left multiplication by H and by the $U_{a(\Omega)}$ for $a\in{}^v\boldsymbol{\Sigma}(\Omega)$, hence by P_Ω, whence the first assertion. The second then follows from (5.2.6) and (5.2.9).

Remark **(5.2.12)**. — As in (5.2.8), one may move the factor H within the product considered, provided one multiplies by -1 the exponent of those U_α which one moves from one side of H to the other.

Remark **(5.2.13)**. — *Let $a\in{}^v\boldsymbol{\Sigma}$. One may arrange the elements of ${}^v\boldsymbol{\Sigma}$ in an order $(a_1,\dots,a_m)$ such that $a=a_1$ and such that, with the notation of* (5.2.4), *the product map from $\prod_{1\leqslant j\leqslant k} U_{a_j(\Omega)}\times H\times\prod_{k<j\leqslant m} U^{-1}_{a_j(\Omega)}$ into P_Ω is injective (resp. bijective) for every subset Ω of* $\mathbf{A}$ *(resp. for every subset Ω of* $\mathbf{A}$ *with nonempty interior), and every k with $0\leqslant k\leqslant m$.*

It suffices indeed to choose the vector chamber D and the nibbling order on ${}^v\boldsymbol{\Sigma}^+(\mathrm{D})$ in such a way that a is its first element, then to choose a nibbling order on ${}^v\boldsymbol{\Sigma}^+(-\mathrm{D})$ and to apply (5.2.9) (resp. (5.2.11)).

Such an order on ${}^v\boldsymbol{\Sigma}$ will be called *admissible*.

Corollary **(5.2.14)**. — *Let $\alpha \in \boldsymbol{\Sigma}$ and let Ω' be a closed subset of* $\mathbf{A}$ *such that $\Omega = \alpha \cap \Omega'$ has nonempty interior and such that α and Ω' are transverse* (5.1.2). *One has $P_\Omega = P_\alpha \cdot P_{\Omega'}$.*

Let us arrange the elements of ${}^v\boldsymbol{\Sigma}$ in an admissible order $(a_1, \ldots, a_m)$ in such a way that $a_1 = {}^v\alpha$. One then has, by (5.2.13): *p. 101*

$$(1) \qquad P_\Omega = U_{a_1(\Omega)} U_{a_2(\Omega)} \ldots U_{a_m(\Omega)} H.$$

But the affine roots $a_j(\Omega)$ for $a_j \in {}^v\boldsymbol{\Sigma}(\Omega)$ and $j \geqslant 2$ do not contain $a_1(\Omega)$, hence contain Ω'. On the other hand, either $a_1(\Omega) \supset \Omega'$, or $a_1(\Omega) \supset \alpha$, whence $a_1(\Omega) = \alpha$. In both cases, (1) indeed implies $P_\Omega = P_\alpha \cdot P_{\Omega'}$.

Corollary **(5.2.15)**. — *Let $\alpha \in \boldsymbol{\Sigma}$ and let C be a chamber contained in α and having a panel contained in $\partial\alpha$. One has $P_{\mathrm{C}} = P_\alpha \cdot P_{\mathrm{C} \cup r_\alpha(\mathrm{C})}$.*

It suffices to apply (5.2.14) to $\Omega' = \mathrm{C} \cup r_\alpha(\mathrm{C})$.

Proposition **(5.2.16)**. — *Let Ω and Ω' be two closed subsets of* $\mathbf{A}$, *with nonempty interior and such that $\Omega \cap \Omega' \neq \emptyset$. Then one has*

$$P_\Omega \cap P_{\Omega'} = P_{\mathrm{cl}(\Omega \cup \Omega')}.$$

If moreover $P_\Omega \subset P_{\Omega'}$, one has $\Omega \supset \Omega'$.

It is clear that $P_{\mathrm{cl}(\Omega\cup\Omega')} \subset P_\Omega \cap P_{\Omega'}$. On the other hand, let $x \in \Omega \cap \Omega'$ and let $g \in P_\Omega \cap P_{\Omega'} \subset P_{\{x\}}$. Let us arrange the elements of ${}^v\boldsymbol{\Sigma}$ in an admissible order $(a_1, \ldots, a_m)$. By (5.2.13) one has

$$g = u_1 \ldots u_m h = v_1 \ldots v_m k$$

with $h, k \in H$, $u_j \in U_{a_j(\Omega)}$ and $v_j \in U_{a_j(\Omega')}$ for $1 \leqslant j \leqslant m$, whence $u_j, v_j \in U_{a_j(\{x\})}$. Applying (5.2.9) to $P_{\{x\}}$, one concludes that $u_j = v_j$ for every j, whence

$$u_j = v_j \in U_{a_j(\Omega)} \cap U_{a_j(\Omega')} = U_{a_j(\Omega) \cup a_j(\Omega')} \subset U_{a_j(\Omega\cup\Omega')}$$

and finally $g \in P_{\Omega\cup\Omega'}$, which proves the first assertion of the proposition.

If moreover $P_\Omega \subset P_{\Omega'}$, what precedes shows that $U_{a_j(\Omega)} \subset U_{a_j(\Omega')}$, whence (by (A 2)) $a_j(\Omega) \supset a_j(\Omega')$ for $1 \leqslant j \leqslant m$, and $\Omega \supset \Omega'$.

Corollary **(5.2.17)**. — *Let $\alpha \in \boldsymbol{\Sigma}$ and let C be a chamber contained in α and having a panel contained in $\partial\alpha$. One has $P_{\alpha^*} \cap P_{\mathrm{C}} = P_{(\alpha^*)_+}$.*

It suffices to apply (5.2.16) with $\Omega = \alpha^*$ and $\Omega' = \mathrm{C}$.

Lemma **(5.2.18)**. — *Let C and C' be two chambers. If $P_{\mathrm{C}} \supset P_{\mathrm{C}'}$, one has $\mathrm{C} = \mathrm{C}'$.*

Suppose $P_{\mathrm{C}} \supset P_{\mathrm{C}'}$ with $\mathrm{C} \neq \mathrm{C}'$. One can find $x \in \mathrm{C}$ and $y \in \mathrm{C}'$ such that $y - x$ belongs to a vector chamber D and such that the segment $[xy]$ meets no facet of codimension $\geqslant 2$. Let C_1 be the chamber adjacent to C meeting $[xy]$. One has

$$P_{\mathrm{C}_1} = P_{\mathrm{C}_1+\mathrm{D}} \cdot P_{\mathrm{C}_1-\mathrm{D}} \subset P_{x+\mathrm{D}} \cdot P_{y-\mathrm{D}} \subset P_{\mathrm{C}} \cdot P_{\mathrm{C}'} = P_{\mathrm{C}}.$$

Since $\overline{\mathrm{C}}_1 \cap \overline{\mathrm{C}} \neq \emptyset$, Proposition (5.2.16) then implies $\mathrm{C} \subset \overline{\mathrm{C}}_1$, which is absurd. One therefore indeed has $\mathrm{C} = \mathrm{C}'$.

Theorem **(5.2.19)**. — *Put $B = P_{\mathrm{C}}$. One has $B \cap N = H$ and the quadruple (G, B, N, S) is a saturated Tits system of affine type, the homomorphism ν defining by passage to the quotient an isomorphism of the Coxeter system (W, S) onto $(\mathbf{W}, \mathbf{S})$. Moreover, (G, B, N, S) is a double Tits system* (5.1.1) *and the notations P_Ω are coherent with those of* § 4*: if one identifies* $\mathbf{A}$ *with its canonical image in the building $\mathscr{I}$ of (G, B, N, S), the subgroup P_Ω is, for every subset Ω of* $\mathbf{A}$, *the fixer of Ω in G.* *p. 102*

The proof of this theorem will occupy n[os] (5.2.20) to (5.2.26).

(5.2.20) For every affine root $\alpha \in \boldsymbol{\Sigma}$, there exists $w \in \mathbf{W}$ such that $w.\mathbf{C} \subset \alpha$, whence $P_\alpha \subset P_{w.\mathbf{C}} = nBn^{-1}$ for $n \in \nu^{-1}(w)$. Condition (A 6) therefore implies that G *is generated by* $B \cup N$.

On the other hand, let $n \in B \cap N$. One then has $nBn^{-1} = B$, whence $P_{\nu(n).\mathbf{C}} = P_{\mathbf{C}}$ and $\nu(n).\mathbf{C} = \mathbf{C}$ by Lemma (5.2.18). One therefore has $\nu(n) = 1$ and $n \in H$. This shows that $B \cap N = H$ and that $B \cap N$ is normal in N, the quotient group being W. Consequently, axioms (T 1) and (T 2) of Tits systems are satisfied.

(5.2.21) Let us show that $H = \bigcap_{n\in N} nBn^{-1}$ (which will imply that the Tits system is *saturated* (1.2.12)). Indeed, let $g \in \bigcap_{n\in N} nBn^{-1}$. For every bounded closed subset Ω of $\mathbf{A}$ containing $\mathbf{C}$, one can find a sequence of chambers $(\mathrm{C}_i)_{0\leqslant i\leqslant n}$ with $\mathrm{C}_0 = \mathbf{C}$, such that, if one puts $\Omega_j = \bigcup_{0\leqslant i\leqslant j} \overline{\mathrm{C}}_i$, one has $\Omega_j \cap \overline{\mathrm{C}}_{j+1} \neq \emptyset$ for every j and $\Omega_n = \Omega$. Since $g \in P_{\mathrm{C}}$ for every chamber C, one sees by induction on j that $g \in P_{\Omega_j}$ for every j: it suffices to use (5.2.16). Hence $g \in P_\Omega$. Let us then arrange the elements of ${}^v\boldsymbol{\Sigma}$ in an admissible order $(a_1, \dots, a_m)$ and write $g = u_1 \dots u_m h$, with $h \in H$ and $u_i \in U_{a_i(\mathbf{C})}$. Since $g \in P_\Omega$, one has $u_i \in U_{a_i(\Omega)}$ for every bounded closed subset Ω of $\mathbf{A}$ containing $\mathbf{C}$, whence $u_i \in H$ by (A 2) and finally $g \in H$, which was to be proved.

(5.2.22) Let us now prove axiom (T 3). Let $r \in S$ and let α be the affine root containing $\mathbf{C}$ such that $r = r_\alpha$. By (5.2.15) one has $B = P_\alpha \cdot P_{\mathbf{C}\cup r(\mathbf{C})}$. Let $n \in N$; one therefore has

$$rBnB = rP_{\mathbf{C}\cup r(\mathbf{C})}P_\alpha nB = P_{\mathbf{C}\cup r(\mathbf{C})}rP_\alpha nB.$$

If $\nu(n)^{-1}(\alpha) \supset \mathbf{C}$, one has $n^{-1}P_\alpha n \subset B$, whence

$$rBnB \subset P_{\mathbf{C}\cup r(\mathbf{C})}rn(n^{-1}P_\alpha n)B \subset BrnB. \tag{1}$$

If $\nu(n)^{-1}(\alpha) \not\supset \mathbf{C}$, one has $\nu(rn)^{-1}(\alpha) \supset \mathbf{C}$ and $(rn)^{-1}P_\alpha rn \subset B$. On the other hand one has, by (A 4),

$$rP_\alpha r \subset (P_{(\alpha^*)_+}P_\alpha \cup P_\alpha rP_\alpha) \subset (B \cup BrP_\alpha)$$

whence

$$\begin{aligned} rBnB = P_{\mathbf{C}\cup r(\mathbf{C})}rP_\alpha rrnB &\subset (BrnB \cup BrP_\alpha rnB) \\ &= BrnB \cup Bn(rn)^{-1}P_\alpha rnB = BrnB \cup BnB \end{aligned} \tag{2}$$

and formulas (1) and (2) prove (T 3).

(5.2.23) Finally, Lemma (5.2.18) implies $rBr^{-1} \neq B$ for every $r \in S$, that is to say axiom (T 4). We have therefore indeed proved that (G, B, N, S) *is a Tits system of affine type*, and *saturated* by (5.2.21).

We may therefore construct the associated building $\mathscr{I}$, the canonical map $j : \mathbf{A} \longrightarrow \mathscr{I}$, and *p. 103*
consider the fixer in G of the image $j(\Omega)$ of a subset Ω of $\mathbf{A}$, a fixer which we shall provisionally denote by $\widetilde{P}_\Omega$. One knows that $\widetilde{P}_{\mathrm{C}} = P_{\mathrm{C}}$ for every chamber C (2.1.5). If $\alpha \in \boldsymbol{\Sigma}$ and $g \in P_\alpha$, one has $g \in P_{\mathrm{C}}$ for every chamber $\mathrm{C} \subset \alpha$, whence $g \in \widetilde{P}_\alpha$. Since P_Ω is by definition generated by H and the P_α for $\alpha \supset \Omega$, this implies $P_\Omega \subset \widetilde{P}_\Omega$ for every $\Omega \subset \mathbf{A}$.

(5.2.24) Consequently, by (5.2.11), one has

$$\widetilde{P}_{\mathrm{C}} = P_{\mathrm{C}} = P_{\mathrm{C+D}} \cdot P_{\mathrm{C-D}} \subset \widetilde{P}_{\mathrm{C+D}} \cdot \widetilde{P}_{\mathrm{C-D}} \subset \widetilde{P}_{\mathrm{C}}$$

for every chamber C and every vector chamber D, which shows that condition (iv) of Theorem (5.1.3) is satisfied. In other words, (G, B, N, S) *is indeed a double Tits system.*

(5.2.25) To complete the proof of Theorem (5.2.19) it only remains to prove that $\widetilde{P}_\Omega = P_\Omega$ for every subset Ω of $\mathbf{A}$.

Suppose first Ω closed and with nonempty interior. One may assume moreover that $\Omega \supset \mathbf{C}$. Let us arrange the elements of ${}^v\boldsymbol{\Sigma}$ in an admissible order $(a_1, \dots, a_m)$. If $g \in \widetilde{P}_\Omega \subset \widetilde{P}_{\mathbf{C}} = P_{\mathbf{C}}$, one may write $g = u_1 \dots u_m h$, with $u_j \in U_{a_j(\mathbf{C})}$ and $h \in H$. Let $j \in \{1, \dots, m\}$. There exists a chamber C $\subset \Omega$ with $a_j(\mathrm{C}) = a_j(\Omega)$. Let $\Gamma = (\mathbf{C}, \mathrm{C}_1, \ldots, \mathrm{C}_q = \mathrm{C})$ be a taut gallery between $\mathbf{C}$ and C. One has $\Gamma \subset \Omega$ and $g \in \widetilde{P}_{\mathrm{C}_i} = P_{\mathrm{C}_i}$ for every $i = 1, \ldots, q$. One may therefore write $g = u_1^i \ldots u_m^i h^i$ with $u_k^i \in U_{a_k(\mathrm{C}_i)}$ for $1 \leqslant k \leqslant m$ and $h^i \in H$. But $\overline{\mathrm{C}}_i \cap \overline{\mathrm{C}}_{i+1} \neq \emptyset$ and one sees, on placing oneself in $P_{\overline{\mathrm{C}}_i \cap \overline{\mathrm{C}}_{i+1}}$ and using (5.2.13), that $u_k^{i+1} = u_k^i$ for $1 \leqslant k \leqslant m$ and $0 \leqslant i < q$. It follows that u_k^i is independent of i and that $u_k \in U_{a_k(\mathrm{C})}$ for $1 \leqslant k \leqslant m$. In particular, one has $u_j \in U_{a_j(\mathrm{C})} = U_{a_j(\Omega)}$. Applying this for $j = 1, \ldots, m$, one concludes $g \in P_\Omega$, whence $\widetilde{P}_\Omega = P_\Omega$.

Let now Ω be an arbitrary nonempty closed subset of $\mathbf{A}$ (one has $\widetilde{P}_\emptyset = G = P_\emptyset$) and let us resume the notation L, F and Δ of (5.1.27) and (5.1.28); by (5.1.28) one has $\widetilde{P}_\Omega = \widetilde{P}_\Delta \widetilde{N}_\Omega \widetilde{P}_\Delta \widetilde{P}_{\widetilde{\Omega}}$ with $\widetilde{N}_\Omega = \widetilde{P}_\Omega \cap N$. Since Δ and $\widetilde{\Omega}$ have nonempty interior, one has $\widetilde{P}_\Delta = P_\Delta \subset P_\Omega$ and $\widetilde{P}_{\widetilde{\Omega}} = P_{\widetilde{\Omega}} \subset P_\Omega$, and it suffices to prove that $\widetilde{N}_\Omega \subset P_\Omega$. Now $\widetilde{N}_\Omega / H = \mathbf{W}_\mathrm{F}$ is generated by the reflections r_α associated with the affine roots α such that $\mathrm{F} \subset \partial\alpha$, or again $\Omega \subset \alpha \cap \alpha^*$. Our assertion therefore follows from the following lemma:

Lemma **(5.2.26)**. — *Let $\alpha \in \mathbf{\Sigma}$. The subset $\nu^{-1}(r_\alpha)$ of N is contained in the subgroup generated by $P_\alpha \cup P_{\alpha^*}$.*

If this assertion were false, one would have $P_{\alpha^*} \cap P_\alpha r_\alpha P_\alpha = \emptyset$ and consequently $P_{\alpha^*} \subset P_\alpha \cdot P_{(\alpha^*)_+}$ by (A 4) (or by condition (viii) of Theorem (5.1.3)). Using (5.2.2) (3), this would imply that P_{α^*} leaves fixed every chamber contained in $\alpha \cap (\alpha^*)_+$, which contradicts (5.1.11).

The proof of Theorem (5.2.19) is complete.

(5.2.27) We shall now prove a "converse" of Theorem (5.2.19). For this, let us resume the hypotheses and notation of 5.1, except that we provisionally denote by $\widetilde{P}_\Omega$ the fixer of Ω in G and that we reserve the notation P_Ω for the subgroup generated by the $\widetilde{P}_\alpha = P_\alpha$ for $\alpha \in \mathbf{\Sigma}$ and $\alpha \supset \Omega$. *p. 104*

Proposition. — *With the hypotheses and notation specified above, conditions* (A 1) *to* (A 6) *of* (5.2.1) *are satisfied as soon as* (G, B, N, S) *is a saturated double Tits system.*

The verification of (A 1) is immediate; (A 2) follows from (5.1.10) and from (2.2.5) (i); (A 4) follows from condition (viii) of (5.1.3); and (A 5) is verified immediately: if $g \in P_{x+\mathrm{D}} \cap P_{x-\mathrm{D}}$, then g leaves fixed all the points of the sector $x + \mathrm{D}$ and of the opposite sector $x - \mathrm{D}$, hence of their convex hull, which is nothing but $\mathbf{A}$, and $g \in \widetilde{P}_\mathbf{A} = H$.

To establish (A 3) and (A 6), we shall first prove that if Ω is a closed subset of $\mathbf{A}$ containing a sector, one has $\widetilde{P}_\Omega = P_\Omega$. For this we shall argue as in (5.2.6). Let $(a_1, \ldots, a_m)$ be the elements of ${}^v\mathbf{\Sigma}(\Omega)$ arranged in a nibbling order. Let us argue by induction on m, the cases $m = 0$ and $m = 1$ being trivial. Put $\Omega' = \bigcap_{2 \leqslant j \leqslant m} a_j(\Omega)$; we saw in (5.2.6) that ${}^v\mathbf{\Sigma}(\Omega') = \{a_2, \ldots, a_m\}$. By the induction hypothesis one therefore has $\widetilde{P}_{\Omega'} = P_{\Omega'}$. Moreover, $a_1(\Omega)$ and Ω' are transverse and, by (5.1.3) (iii), one has

$$\widetilde{P}_\Omega = \widetilde{P}_{a_1(\Omega)} \cdot \widetilde{P}_{\Omega'} = P_{a_1(\Omega)} \cdot P_{\Omega'} \subset P_\Omega$$

whence $\widetilde{P}_\Omega = P_\Omega$.

Let us then place ourselves in the hypotheses of (A 3). The closed subsets α and Ω are then transverse and one has, by (5.1.3) (iii), $P_{\alpha\cap\beta} \subset \widetilde{P}_{\alpha\cap\beta} = \widetilde{P}_{\alpha\cap\Omega} = \widetilde{P}_\alpha \cdot \widetilde{P}_\Omega = P_\alpha \cdot \widetilde{P}_\Omega$. But Ω contains a sector, hence $\widetilde{P}_\Omega = P_\Omega$, whence $P_{\alpha\cap\beta} \subset P_\alpha \cdot P_\Omega \subset P_{\alpha\cap\beta}$, which proves (A 3).

Finally, (5.1.3) (iv) shows that

$$B = \widetilde{P}_{\mathbf{C}+\mathrm{D}} \cdot \widetilde{P}_{\mathbf{C}-\mathrm{D}} = P_{\mathbf{C}+\mathrm{D}} \cdot P_{\mathbf{C}-\mathrm{D}} \subset P_\mathbf{C}$$

for every vector chamber D. Consequently the subgroup generated by the P_α contains B. One proves exactly as in (5.2.26) that it contains the subsets $\nu^{-1}(r)$ for $r \in S$, hence N. This establishes (A 6) and completes the proof of the proposition.

(5.2.28) It then follows from Theorem (5.2.19) that $\widetilde{P}_\Omega = P_\Omega$ for every subset Ω of $\mathbf{A}$. In other words, returning completely to the notation of 5.1, *the fixer P_Ω of an arbitrary subset Ω of $\mathbf{A}$ is, when (G, B, N, S) is a double Tits system, generated by the subgroups P_α for $\alpha \in \mathbf{\Sigma}$ and $\alpha \supset \Omega$, and the results which we established above on the structure of the subgroups P_Ω, and in particular those of* (5.2.6)*,* (5.2.9)*,* (5.2.11)*,* (5.2.13) *and* (5.2.16)*, are valid* (U_α being a subset of P_α satisfying the conditions of (5.2.3)).

(5.2.29) This implies that *the subgroup $\mathfrak{B}^0_{\mathrm{D}}$ (the union of the subgroups $P_{\mathfrak{C}}$ as $\mathfrak{C}$ ranges over the set of sectors of direction* D (4.1.5)*) is generated by the union of the subgroups P_α for ${}^v\alpha \in {}^v\mathbf{\Sigma}^+(\mathrm{D})$*. More precisely, if one arranges the elements of ${}^v\mathbf{\Sigma}^+(\mathrm{D})$ in

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(5.2.29, end) With the elements of ${}^v\Sigma^+(D)$ arranged in a nibbling order $(a_1, \ldots, a_m)$, the product map $(u_1, \ldots, u_m, h) \mapsto u_1 \cdots u_m h$ is a bijection of $\prod_{1 \le j \le m} \mathfrak{U}_{a_j} \times H$ onto $\mathfrak{B}^0_D$; here $\mathfrak{U}_a$ denotes the union of the U_α with ${}^v\alpha = a$. Likewise $(u_1, \ldots, u_m, t) \mapsto u_1 \cdots u_m t$ is a bijection of $\prod_{1 \le j \le m} \mathfrak{U}_{a_j} \times \nu^{-1}(\mathbf{V})$ onto $\mathfrak{B}_D$.

(5.2.30) We now study the case in which each P_α decomposes as a semidirect product of H by a normal subgroup U_α, these decompositions being suitably coherent. Precisely, until the end of § 5.2 we are given: a group G; a subgroup N of G; a surjective homomorphism ν of N onto $\mathbf{W}$ with kernel denoted H; and, for each $\alpha \in \mathbf{\Sigma}$, a subgroup U_α of G. These data are required to satisfy conditions (B 1)–(B 6) below, where for any subset $\Omega \subset \mathbf{A}$ we write U_Ω for the subgroup of G generated by the U_α with $\alpha \in \mathbf{\Sigma}$ and $\alpha \supset \Omega$ (condition (B 2) shows this notation is unambiguous when $\Omega = \alpha \in \mathbf{\Sigma}$).

(B 1) : For all $\alpha \in \mathbf{\Sigma}$ and all $n \in N$, $nU_\alpha n^{-1} = U_{\nu(n).\alpha}$.

(B 2) : Let $\alpha \in \mathbf{\Sigma}$. For every $\beta \in \mathbf{\Sigma}$ with $\beta \supsetneq \alpha$ one has $U_\beta \subsetneq U_\alpha$, and the intersection of the U_β for $\beta \in \mathbf{\Sigma}$, $\beta \supset \alpha$, is reduced to the identity.

(B′3) : Let $\alpha, \beta \in \mathbf{\Sigma}$ have non-parallel boundaries, and let Ω be the intersection of those $\gamma \in \mathbf{\Sigma}$ with $\gamma \supset \alpha \cap \beta$ and $\gamma \not\supset \alpha$. Then U_α normalizes U_Ω.

(B″3) : Let $\alpha, \beta \in \mathbf{\Sigma}$ be such that $\beta \supsetneq \alpha^*$. Then $U_\alpha H U_\beta$ is a subgroup of G.

(B 4) : Let $\alpha \in \mathbf{\Sigma}$. The subset $U_\alpha H U_{(\alpha^*)_+} \cup U_\alpha \nu^{-1}(r_\alpha) U_\alpha$ is a subgroup of G.

(B 5) : For every $x \in \mathbf{A}$ and every vector chamber D, $(H.U_{x+D}) \cap U_{x-D} = \{1\}$.

(B 6) : G is generated by the union of H and the U_α, $\alpha \in \mathbf{\Sigma}$.

Because of (B 1) it suffices to verify (B 5) for one vector chamber fixed in advance. Also (B 1) implies that H normalizes each U_α, hence each U_Ω. One sets $P_\Omega = H.U_\Omega$: this is the subgroup of G generated by H and the P_α for $\alpha \in \mathbf{\Sigma}$, $\alpha \supset \Omega$. By (B 5), P_Ω is the semidirect product of H by U_Ω as soon as Ω contains a sector.

Proposition (5.2.31). — *N, ν and the family of the P_α ($\alpha \in \mathbf{\Sigma}$) satisfy conditions (A 1)–(A 6) of (5.2.1).*

This is immediate. Moreover the subgroups U_α, $\alpha \in \mathbf{\Sigma}$, satisfy the conditions of (5.2.3), and $\mathfrak{U}_a$ is then a subgroup for every $a \in {}^v\Sigma$.

(5.2.32) The previous results therefore apply. In particular the quadruple (G, B, N, S) — with $B = P_{\mathfrak{C}}$ and S the inverse image of $\mathbf{S}$ in N/H — is a saturated double Tits system. Furthermore (5.2.6) and (5.2.11) can be made precise:

Proposition (5.2.32). — *Let Ω be a non-empty subset of $\mathbf{A}$.*

(i) *Suppose Ω contains a sector, and arrange the elements of ${}^v\Sigma(\Omega)$ in a nibbling order $(a_1, \ldots, a_m)$. Then $(u_1, \ldots, u_m) \mapsto u_1 \cdots u_m$ is a bijection of $\prod_{1 \le j \le m} U_{a_j(\Omega)}$ onto the subgroup U_Ω.*

(ii) *Suppose Ω has non-empty interior and let D be a vector chamber. Then $(u, h, v) \mapsto uhv$ is a bijection of $U_{\Omega+D} \times H \times U_{\Omega-D}$ onto P_Ω.*

The proof of (i) runs exactly as that of (5.2.6), using (B′3) in place of (A 3); (ii) follows from (i) together with (5.2.11).

(5.2.33) Proposition (5.1.28) may be sharpened similarly: in the notation of (5.1.28) one has $P_\Omega = U_\Delta N_\Omega U_\Delta U_{\widetilde{\Delta}}$.

(5.2.34) Let D be a vector chamber. If $\mathfrak{C}_1, \mathfrak{C}_2$ are two sectors of $\mathbf{A}$ of direction D, then $U_{\mathfrak{C}_1} \cup U_{\mathfrak{C}_2} \subset U_{\mathfrak{C}_1 \cap \mathfrak{C}_2}$, and the union of the subgroups $U_{\mathfrak{C}}$, as $\mathfrak{C}$ runs over the sectors of direction D, is a subgroup, denoted $\mathfrak{U}_D$. Arranging the elements of ${}^v\Sigma^+(D)$ in a nibbling order $(a_1, \ldots, a_m)$, it follows from (5.2.32) that $(u_1, \ldots, u_m) \mapsto u_1 \cdots u_m$ is a bijection of $\prod_{1 \le j \le m} \mathfrak{U}_{a_j}$ onto $\mathfrak{U}_D$. On the other hand (cf. (5.2.29)) $\mathfrak{B}_D^0$ is the semidirect product of H by the normal subgroup $\mathfrak{U}_D$, and $\mathfrak{B}_D$ is the semidirect product of $\nu^{-1}(\mathbf{V})$ by the normal subgroup $\mathfrak{U}_D$. Finally $G = \mathfrak{B}_D.N.\mathfrak{B}_D = \mathfrak{U}_D.N.\mathfrak{U}_D$, and one can even show that the evident map is a bijection of N onto $\mathfrak{U}_D \backslash G / \mathfrak{U}_D$.

(5.2.35) Let $\alpha \in \mathbf{\Sigma}$ and let Ω be a closed subset of $\mathbf{A}$. Assume α and Ω are transverse and that $\Omega' = \alpha \cap \Omega$ contains a chamber of $\mathbf{A}$. Then U_α normalizes U_Ω and $U_{\Omega'} = U_\alpha.U_\Omega$.

Proof sketch. Let $\beta \in \mathbf{\Sigma}$ with $\beta \supset \Omega$. For $\gamma \in \mathbf{\Sigma}$ with $\gamma \supset \alpha \cap \beta$ and $\gamma \not\supset \alpha$ one has $\gamma \supset \alpha \cap \Omega$ and $\gamma \not\supset \alpha$, hence $\gamma \supset \Omega$. So for $u \in U_\alpha$, condition (B′3) gives $uU_\beta u^{-1} \in U_\Omega$, which is the assertion.

Remark (5.2.36). — *Condition (B′3) is equivalent to:*

(B′3 bis) : *For $\alpha, \beta \in \mathbf{\Sigma}$ with non-parallel boundaries, the commutator group (U_α, U_β) is contained in the subgroup generated by the U_γ for $\gamma \in \mathbf{\Sigma}$ with $\gamma \supset \alpha \cap \beta$, $\gamma \not\supset \alpha$ and $\gamma \not\supset \beta$.*

Indeed (B′3 bis) trivially implies (B′3), and the converse follows from (5.2.6).[(2)]

Throughout § 6, V denotes a real vector space, V^* its dual, and Φ a root system in V^* with Weyl group ${}^vW$ (acting on V and on V^*). A ${}^vW$-invariant inner product on V, hence on V^*, is given. For $a \in \Phi$, r_a is the reflection attached to a, and $a^\vee$ is the *coroot* attached to a, i.e. the element of V with $r_a(v) = v - a(v)a^\vee$ for all $v \in V$; the $a^\vee$, $a \in \Phi$, form the root system dual to Φ. A chamber D_0 of Φ in V is fixed; Π is the corresponding basis of Φ ("system of simple roots"), and Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots for D_0 (cf. (1.3.10), (1.3.12)). We write Φ^{red} (resp. Φ^{nm}) for the set of non-divisible (resp. non-multipliable) roots, i.e. the set of $a \in \Phi$ with $a/2 \notin \Phi$ (resp. $2a \notin \Phi$), and we put $\Phi^{+\text{red}} = \Phi^{\text{red}} \cap \Phi^+$, $\Phi^{-\text{red}} = \Phi^{\text{red}} \cap \Phi^-$.

In a group X, the normalizer of a subset $Y \subset X$ is written $\mathcal{N}_X(Y)$, or simply $\mathcal{N}(Y)$ when no confusion is possible.

6. Valued root data

(2) In [12] and [16] a triple $(N, \nu, (U_\alpha)_{\alpha \in \mathbf{\Sigma}})$ satisfying (B 1), (B 2), (B′3 bis), (B″3), (B 4), (B 6) together with a condition (DR 6) analogous to but distinct from (B 5) was called an *affine root datum of type* $\mathbf{\Sigma}$. In fact that condition (DR 6), although in general satisfied in the applications we have in view, does not seem to be a consequence of the conditions of (5.2.30); and moreover it is perhaps insufficient to establish axiom (T 4) of Tits systems, contrary to what we asserted in [12] and [16].

6.1. Root data. —

Definition (6.1.1). — *A* root datum of type Φ *in a group G is a system* $(T, (U_a, M_a)_{a\in\Phi})$ *with the following properties.*

(DR 1) : T is a subgroup of G and, for every $a \in \Phi$, U_a is a subgroup of G not reduced to the identity element.
(DR 2) : For $a, b \in \Phi$, the commutator group (U_a, U_b) is contained in the group generated by the U_{pa+qb} for p, q strictly positive integers with $pa + qb \in \Phi$.
(DR 3) : If a and $2a$ belong to Φ, then $U_{2a} \subsetneq U_a$.
(DR 4) : For $a \in \Phi$, M_a is a right coset of T and $U_{-a} - \{1\} \subset U_a M_a U_a$.
(DR 5) : For $a, b \in \Phi$ and $n \in M_a$, $nU_b n^{-1} = U_{r_a(b)}$.
(DR 6) : If U^+ (resp. U^-) denotes the group generated by the U_a for $a \in \Phi^+$ (resp. Φ^-), then $TU^+ \cap U^- = \{1\}$.

The root datum is called *generating* if T and the U_a generate G.

(6.1.2) Some immediate consequences of these axioms. For $a \in \Phi$ we put $U_a^* = U_a - \{1\}$.

(1) For $a \in \Phi$ one has $U_a \neq U_{-a}$ and $U_a M_a U_a \cap \mathcal{N}(U_a) = \emptyset$; in particular $U_{-a} \cap \mathcal{N}(U_a) = \{1\}$.

Proof sketch. The first relation follows from (DR 1) and (DR 6). For the second: if $g \in U_a m U_a \cap \mathcal{N}(U_a)$ with $m \in M_a$, then $gU_a g^{-1} = U_a$, whence by (DR 5) $U_a = mU_a m^{-1} = U_{-a}$.

(2) For $a \in \Phi$ and $u \in U_{-a}^*$ there is one and only one element of M_a, written $m(u)$, with $u \in U_a.m(u).U_a$. More precisely there is exactly one triple $(u', m, u'') \in U_a \times G \times U_a$ with

$$u = u'mu'', \qquad mU_a m^{-1} = U_{-a}, \qquad mU_{-a}m^{-1} = U_a,$$

and one then has $m \in M_a$ and $u' \neq 1$.

Proof sketch. Given (DR 4) and (DR 5) it suffices to show that two triples with the stated properties coincide. Setting $u_1'^{-1}u' = u_2' \in U_a$ and $mu_1''u''^{-1}m^{-1} = u_2'' \in U_{-a}$, one finds $u_2'u_2''^{-1} = m_1 m^{-1} \in \mathcal{N}(U_a) \cap \mathcal{N}(U_{-a})$, so $u_2' \in \mathcal{N}(U_{-a})$ and $u_2'' \in \mathcal{N}(U_a)$; by (1) both equal 1. If u' were 1 one would get $m = mu''^{-1}m^{-1}.u \in U_{-a}$, hence $U_a = mU_{-a}m^{-1} = U_{-a}$, contradicting (1).

For $a \in \Phi$ we put $M_a^0 = m(U_a^*) = \{m(u) \mid u \in U_a^*\}$.

(3) For $a \in \Phi$, T normalizes U_a and M_a.

Proof sketch. Let $t \in T$ and take u, u', m, u'' as in (2). Since $m \in M_a$, also $tm \in M_a$ by (DR 4), and (DR 5) shows that $t = (tm)m^{-1}$ normalizes U_a; similarly for U_{-a}. Applying (2) to ${}^t u = {}^t u' \cdot {}^t m \cdot {}^t u''$ gives ${}^t m \in M_a$; since M_a is a right coset of T, t normalizes M_a.

(4) One has $M_a = M_a^{-1} = M_{-a}$, and $M_a = M_{a/2}$ if $a/2 \in \Phi$.

Proof sketch. With u, u', m, u'' as in (2): if $a/2 \in \Phi$ then (2) and (DR 5) give $m \in M_{a/2}$, hence $M_a = T.m = M_{a/2}$. In all cases

$$u^{-1} = u''^{-1}m^{-1}u'^{-1}, \qquad u'^{-1} = (mu''m^{-1}).m.u^{-1},$$

so by (2) $m^{-1} \in M_a$ and $m \in M_{-a}$; with (3) and (DR 4) this gives $M_a^{-1} = m^{-1}T \subset M_a T = M_a$ and $M_{-a} = T.m = M_a$.

(5) $T \cup M_a$ is a subgroup of G admitting T as a normal subgroup of index 2. (Immediate from (3) and (4).)

(6) For $a \in \Phi$, $U_a^*.M_a.U_a = U_{-a}^*.T.U_a$. This follows from (DR 4), (2) and (5) via the chain

$$U_a^* M_a U_a \subset U_{-a}^* M_a U_{-a} M_a U_a \subset U_{-a}^* T U_a \subset U_a^* M_a U_a T U_a = U_a^* M_a U_a.$$

(7) If L_a denotes the group generated by U_a, U_{-a} and T, then

$$L_a = TU_a \cup U_a M_a U_a = M_a U_a \cup U_{-a} T U_a \neq U_{-a} T U_a.$$

Proof sketch. The second equality is (6). The first holds because the second and third members are visibly contained in L_a and invariant under right multiplication by T (by (3)), the second being invariant under right multiplication by U_a and the third under right multiplication by U_{-a}. Finally $L_a = U_{-a}TU_a$ is impossible: otherwise there would exist $u \in U_a$, $u' \in U_{-a}$, $t \in T$ with $m = u'tu \in M_a$, whence $U_{-a} = mU_am^{-1} = u'U_au'^{-1}$ and $U_{-a} = U_a$, contradicting (1).

(8) $\mathcal{N}(U_a) \cap L_a = TU_a$ and $\mathcal{N}(U_a) \cap \mathcal{N}(U_{-a}) \cap L_a = T$. (The first from (7) and (1); the second then follows using (DR 6).)

From (8), (DR 4) and (DR 5) one gets at once

$$M_a = \{x \in L_a \mid xU_ax^{-1} = U_{-a} \text{ and } xU_{-a}x^{-1} = U_a\}. \tag{21}$$

In particular M_a is entirely determined by the data of T, U_a and U_{-a}, which allows us to speak, by abuse of language, of *the root datum* $(T, (U_a)_{a\in\Phi})$.

(10) Let N be the group generated by T and the union of the M_a, $a \in \Phi$. If $\Phi \neq \emptyset$ this is also the group generated by the M_a. By (DR 5) there is a unique epimorphism ${}^v\nu : N \longrightarrow {}^vW$ such that for all $a \in \Phi$ and all $n \in N$ one has $nU_an^{-1} = U_b$ with $b = {}^v\nu(n)(a)$. Moreover ${}^v\nu(M_a) = \{r_a\}$ for every $a \in \Phi$.

(11) Since N normalizes T by (8), transitivity of ${}^vW$ on the chambers of Φ shows that condition (DR 6) remains satisfied if the chamber used to define Φ^+ and Φ^- is replaced by another chamber.

(12) Let T^0 be the intersection of T with the group generated by the union of the M_a^0, $a \in \Phi$ (cf. (2)); it is obviously a normal subgroup of T, and even of N. Then $(T^0, (U_a, T^0M_a^0)_{a\in\Phi})$ is a generating root datum of the group G' generated by the union of the U_a, $a \in \Phi$. More generally, if X is a subgroup of T normalized by M_a^0, the pair $(XT^0, (U_a, XT^0M_a^0)_{a\in\Phi})$ is a generating root datum of the group generated by X and G'. These assertions are evident.

Examples (6.1.3). — *Let K be a field.*

a) Let $\Phi = \{a, -a\}$ and $G = SL_2(K)$. Let T be the subgroup of diagonal matrices, U_a (resp. U_{-a}) the subgroup of "unipotent" upper (resp. lower) triangular matrices, $m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $M_a = M_{-a} = Tm$. One obtains in this way a generating root datum in G. Verification of the axioms is immediate; for (DR 4) one uses the relation

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \qquad (u \in K^*), \tag{22}$$

from which it also follows, in the notation of (6.1.2)(2), that

$$m\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}.$$

b) Assume from now on that K is commutative; let $\mathscr{G}$ be a connected simple algebraic group defined and split over K, and $\mathscr{T}$ a maximal torus of $\mathscr{G}$ defined and split over K. Let Φ be the root system of $\mathscr{G}$ relative to $\mathscr{T}$. It is known that for each $a \in \Phi$ there is a unique algebraic subgroup $\mathscr{U}_a$ of $\mathscr{G}$ admitting an isomorphism x_a of the additive group $\mathscr{A}dd$ onto $\mathscr{U}_a$ satisfying $t.x_a(u).t^{-1} = x_a(a(t).u)$ for $u \in \mathscr{A}dd$, $t \in \mathscr{T}$; choosing such an isomorphism amounts to choosing a generator X_a of the Lie algebra of $\mathscr{U}_a$. It is known ([17], [19]) that the x_a (or the X_a) may be chosen so that:

(2) for every $a \in \Phi$ there is a unique epimorphism ζ_a defined over K from SL_2 onto the algebraic subgroup generated by $\mathscr{U}_a$ and $\mathscr{U}_{-a}$, with $x_a(u) = \zeta_a\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and $x_{-a}(u) = \zeta_a\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ for all $u \in \mathscr{A}dd$;

(3) for $a, b \in \Phi$ with $b \neq -a$, and $u, v \in \mathscr{A}dd$,

$$(x_a(u), x_b(v)) = \prod_{pa+qb\in\Phi,\ p,q\in\mathbf{N}^*} x_{pa+qb}\big(C_{a,b;p,q}\, u^p v^q\big),$$

where the coefficients $C_{a,b;p,q}$ are *integers* (the roots $pa+qb$ being arranged in a prescribed order).

This choice, which we assume made, corresponds to the choice of a "Chevalley basis" of the Lie algebra of $\mathscr{G}$ in the sense of [8], or again of a "pinning" of $\mathscr{G}$ in the sense of [19].

Now let $G = \mathscr{G}(K)$ be the set of K-rational points of $\mathscr{G}$, and likewise $T = \mathscr{T}(K)$, $U_a = \mathscr{U}_a(K)$ and $M_a = T.\zeta_a\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ ($a \in \Phi$). Then $(T, (U_a, M_a))$ is a generating root datum of type Φ in G. Conditions (DR 1) and (DR 3) are obviously satisfied, (DR 2) follows from (3), and (DR 4) follows from example a). Finally (DR 5) and (DR 6) are well known (*loc. cit.*). Note that the group N is then equal to $\mathcal{N}(\mathscr{T})(K)$, where $\mathcal{N}(\mathscr{T})$ denotes the normalizer of $\mathscr{T}$ in $\mathscr{G}$.

c) More generally, let $\mathscr{G}$ be a connected semisimple algebraic group defined over K, $\mathscr{S}$ a maximal K-split torus of $\mathscr{G}$ defined over K, and $\mathscr{S}'$ a maximal torus of $\mathscr{G}$ defined over K and containing $\mathscr{S}$; let Φ (resp. Φ') be the root system of $\mathscr{G}$ relative to $\mathscr{S}$ (resp. $\mathscr{S}'$). For $a \in \Phi$, let $\mathscr{U}_a$ be the algebraic subgroup of $\mathscr{G}$ generated by the subgroups $\mathscr{U}_{a'}$ for those $a' \in \Phi'$ whose restriction to $\mathscr{S}$ equals a or $2a$ (here $\mathscr{U}_{a'}$ is the subgroup defined as in b) above, replacing K by an algebraic closure of K, or by an extension of K splitting $\mathscr{S}'$). Let $\mathscr{Z}(\mathscr{S})$ (resp. $\mathcal{N}(\mathscr{S})$) be the centralizer (resp. normalizer) of $\mathscr{S}$ in $\mathscr{G}$. Put $G = \mathscr{G}(K)$, $T = \mathscr{Z}(\mathscr{S})(K)$ and $U_a = \mathscr{U}_a(K)$ (for $a \in \Phi$). The results of [3] then show that $(T, (U_a))$ is a generating root datum of type Φ in G, for which the group N equals $\mathcal{N}(\mathscr{S})(K)$.

(6.1.4) In all the sequel of § 6, $(T, (U_a, M_a))$ denotes a generating root datum of type Φ in a group G, and the symbols L_a, U^+, U^-, N, ${}^v\nu$ have the meaning defined in (6.1.1) or (6.1.2). To simplify notation we set $U_{2a} = \{1\}$ when $a \in \Phi$ and $2a \notin \Phi$.

(6.1.5) Let $\Phi = \bigcup_{i \in I} \Phi_i$ be the decomposition of Φ into irreducible components. For $i \in I$ let G_i^0 be the subgroup of G generated by the U_a, $a \in \Phi_i$. It is immediate that $(T \cap G_i^0, (U_a, M_a \cap G_i^0)_{a \in \Phi_i})$ is a generating root datum of type Φ_i in G_i^0. The subgroup T normalizes each G_i^0 and, by (DR 2), every element of G_j^0 for $j \neq i$ centralizes G_i^0. Since $(T, (U_a))$ is generating, one sees that each G_i^0 is a normal subgroup of G and that $G_i^0 \cap G_j^0$, for $i \neq j$, is contained in the center of the normal subgroup G^0 of G generated by the U_a (or by the G_i^0) — a center which is moreover contained in T, as we shall see later. Note that $G = T.G^0$.

Proposition (6.1.6). — *The group U^+ is nilpotent. Moreover, let Ψ be a subset of Φ^+ and $\Psi^{\mathrm{red}} = \{a \in \Psi \mid a/2 \notin \Psi\}$. For each $a \in \Psi$ let Y_a be a subgroup of U_a, and put $X_a = Y_a$ or $Y_a.Y_{2a}$ according as $2a \notin \Psi$ or $2a \in \Psi$. Let X be the group generated by the Y_a, and assume they have the following property:*

(i) for $a, b \in \Psi$, (Y_a, Y_b) is contained in the group generated by the Y_{ma+nb} for $m, n \in \mathbf{N}^*$ with $ma + nb \in \Psi$.

Then the product map $\prod_{a \in \Psi^{\mathrm{red}}} X_a \longrightarrow X$ is bijective, whatever the order in which the factors of the product are arranged.

This is an immediate consequence of (6.1.2)(11) and of the following lemma.

Lemma (6.1.7). — *Let X be a group, and Ψ, Ψ^{red} as in (6.1.6). For each $a \in \Psi$ let Y_a be a subgroup of X and put $X_a = Y_a$ or $Y_a.Y_{2a}$ according as $2a \notin \Psi$ or $2a \in \Psi$. Assume the Y_a generate X and possess property (i) of (6.1.6) as well as the following:*

(ii) for $v \in V$, the intersection of the group generated by the Y_a ($a \in \Psi$, $a(v) \leq 0$) with the group generated by the Y_a ($a \in \Psi$, $a(v) > 0$) is reduced to the identity element.

Then X is nilpotent and the product map

(23)
$$\prod_{a \in \Psi^{\mathrm{red}}} X_a \longrightarrow X$$

is bijective, whatever the order in which the factors are arranged.

Proof sketch. Induction on $\#\Psi$. A maximal element a_0 of Ψ for the order attached to D_0 satisfies $ma_0 + nb \notin \Psi$, so Y_{a_0} is central by (i); pass to $X' = X/Y_{a_0}$ and apply the induction hypothesis. Surjectivity of (1) follows since Y_{a_0} is central. Injectivity: if $\prod x_a = \prod y_a$, the image in X' gives $x_a X_{a_0} = y_a X_{a_0}$; for a not proportional to a_0, hypothesis (ii) applied at a point v with $a(v) \leq 0 < a_0(v)$ forces $x_a = y_a$, and the remaining factor then agrees as well. *(Original: p. 111–112.)*

Proposition (6.1.8). — *Suppose Φ has rank 2. Let $a_1, \ldots, a_{2k}$ be the elements of Φ^{red} arranged in "circular order", i.e. in such a way that*

$$\Phi^{\text{red}} \cap (\mathbf{Q}_+ a_{i-1} + \mathbf{Q}_+ a_{i+1}) = \{a_{i-1}, a_i, a_{i+1}\} \qquad \textit{for } 1 < i < 2k.$$

The roots $a = a_1$ and $b = a_k$ then form a basis of Φ, and the corresponding positive elements of Φ^{red} are the a_i for $1 \leq i \leq k$. Define a_i for every $i \in \mathbf{Z}$ by the condition $a_{i+2k} = a_i$ for all $i \in \mathbf{Z}$. Let further $u \in U_a$ and $u' \in U_b$ with $u, u' \neq 1$; put $m = m(u)$, $m' = m(u')^{-1}$ and $n = mm'$. There exists one and only one sequence $(u_i)_{i \in \mathbf{Z}}$ with $u_i \in U_{a_i}$, $u_1 = u^{-1}$ and $u_k = u'$, such that the two following conditions hold:

$$(u, u') = u_2 \cdots u_{k-1}, \tag{24}$$

$$u_i \in U_{-a_i} . M_{a_i} . u_{k+i}^{-1} \qquad \textit{for all } i \in \mathbf{Z}. \tag{25}$$

Put $m_i = m(u_i)^{-1}$, so that $m_1 = m$ and $m_k = m'$. Then the following relations hold for all $i \in \mathbf{Z}$:

$$m_{k+i} = m_i; \tag{26}$$

$$(u_i^{-1}, u_{k+i-1}) = u_{i+1} \cdots u_{k+i-2}; \tag{27}$$

$$u_{i+1} = m_i u_{k+i-1} m_i^{-1}; \tag{28}$$

$$u_2 = m u' m^{-1}; \tag{29}$$

$$m_{i+1} m_i = n; \tag{30}$$

$$m_{2i} = n^i m' n^{-i} \quad \textit{and} \quad m_{2i+1} = n^i m n^{-i}; \tag{31}$$

$$mm'mm' \cdots = m'mm'm \cdots , \tag{32}$$

each of the two members of the last relation having k factors.

Proof sketch. The heart of the argument is the induction on $j \geq 0$ over the statement (P_j): *there is a unique finite sequence $(u_i)_{1 \leq i \leq k+j}$ with $u_i \in U_{a_i}$, $u_1 = u^{-1}$, $u_k = u'$, with $u_i \neq 1$ for $1 \leq i \leq j+1$ and $k \leq i \leq k+j$, satisfying* (4) *for $1 \leq i \leq j+1$ and* (2)*,* (5) *for $1 \leq i \leq j$.* (P_0) comes from (DR 2) and (6.1.6). The inductive step writes $u_j = v_{k+j} m_j^{-1} u_{k+j}^{-1}$ and sets $u'_{j+1} = m_j u_{k+j-1} m_j^{-1}$; commuting $U_{a_{k+j}}$ past $U_{a_{k+j-1}}$ turns the commutator (u_j^{-1}, u_{k+j-1}) into $u'_{j+1} \cdot (u'^{-1}_{j+1}, u_{k+j}) \cdot u_{k+j-1}^{-1}$, and (DR 2) with (6.1.6) forces $u_{j+1} = u'_{j+1} \neq 1$, yielding (5) for $i = j$ and (4) for $i = j+1$. The symmetric argument (swap a, b and u, u') produces the sequence for $i \leq 0$, and the two halves match up. Relations (7)–(9) follow from $m_{i+1} = m_i m_{i-1} m_i^{-1} = n m_{i-1} n^{-1}$, using $m_1 = m$, $m_0 = m_k = m'$ and taking i with $2i = k$ or $2i+1 = k$. *(Original: pp. 113–114.)*

Remarks (6.1.9). —

a) It is easy to see that the proof of (6.1.8) remains valid if one takes a weaker definition of root data, for example that of [36], which includes the case of the Ree groups of type ${}^2F_4$.

b) Let $a \in \Phi$; for $x \in U_a - \{1\}$ write $f(x)$ for the unique element of U_a with $x M_a f(x) \cap U_{-a} \neq \emptyset$. In many cases the map f is the identity, for example for the canonical root datum of $SL_2(K)$ ((6.1.3)(1)), and more generally for the root datum of the group of rational points of a connected semisimple algebraic group described in (6.1.3) c), when a is a non-multipliable root ([3], § 7).

In all cases, from the relation $u_i = f(u_{k+i})^{-1}m_i^{-1}u_{k+i}^{-1}$ one draws

$$f(u_{k+i})^{-1} = u_i.m_i.m_i^{-1}u_{k+i}m_i, \qquad \text{whence} \qquad f(u_i) = m_i^{-1}u_{k+i}m_i.$$

From (5) and (7) one then deduces

$$u_{i+2} = n.(m_i^{-1}u_{k+i}m_i).n^{-1} = n.f(u_i).n^{-1},$$

whence

$$u_{2i} = n^{i-1}mf^{i-1}(u')m^{-1}n^{1-i} \qquad \text{and} \qquad u_{2i+1} = n^i f^i(u^{-1})n^{-i}.$$

If in particular f is the identity for each of the a_i, one has

$$u_{2i+1} = n^i u^{-1}n^{-i} \qquad \text{and} \qquad u_{2i} = n^{i-1}mu'm^{-1}n^{1-i}.$$

c) Relation (9) is not the only identity between m and m' whose image in ${}^vW$ gives the well-known relation $({}^v\nu(mm'))^k = 1$. For example, if Φ is of type C_2 and a_1 is a long root, then a_1 and a_3 are strongly orthogonal and m_1, m_3 commute. One deduces that

$$mm'mm'^{-1} = m'mm'^{-1}m.$$

Likewise, if Φ is of type G_2 and a_1 is long, then a_1 and a_4 are strongly orthogonal, and one deduces that

$$mm'mm'm^{-1}m'^{-1} = m'mm'mm^{-1}m'^{-1}m.$$

Corollary (6.1.10). — *Let Ψ be a root subsystem of Φ and Δ a basis of Ψ. For $a \in \Delta$ let N_a be a non-empty subset of M_a^0, and let N_1 be the group generated by the N_a, $a \in \Delta$. Let X be a subgroup of T normalized by N_1 and containing the N_a^2 for $a \in \Delta$. Then the map λ sending $n \in N_1X$ to the restriction of ${}^v\nu(n)$ to the subspace generated by Ψ is an epimorphism of N_1X onto the Weyl group ${}^vW_1$ of Ψ, with kernel X.*

Proof sketch. It is immediate that $\lambda^{-1}(1) \supset X$ and that $\lambda(N_1X) = {}^vW_1$ (since ${}^v\nu(N_a) = \{r_a\}$ for $a \in \Delta$). For every $a \in \Delta$ the image of N_a in the quotient group N_1X/X reduces to a single element $\overline{m}_a$ of order 2, and the $\overline{m}_a$ generate N_1X/X. Let $a, b \in \Delta$ with $a \neq b$; applying (6.1.8)(9) to the root datum $(T, (U_c))$, where c runs over the root system formed by the $pa + qb$ with $p, q \in \mathbf{Z}$ and $pa + qb \in \Phi$, one sees, taking into account the relations $\overline{m}_a^2 = \overline{m}_b^2 = 1$, that $(\overline{m}_a\overline{m}_b)^k = 1$ with k the order of r_ar_b. Consequently the $\overline{m}_a$, $a \in \Delta$, satisfy the defining relations of the Weyl group ${}^vW_1$, and the canonical map $N_1X/X \to {}^vW_1$ is an isomorphism.

Corollary (6.1.11). — (i) *If $\Phi \neq \emptyset$, then N is generated by the union of the M_a for $a \in \Pi$.* (ii) *One has ${}^v\nu^{-1}(1) = T = N \cap TU^+$.*

Proof sketch. Let N' be the group generated by the M_a, $a \in \Pi$; then ${}^v\nu(N') = {}^vW$. For any root $b \in \Phi$ there is $w \in {}^vW$ such that $w(b)$ is proportional to a simple root $a \in \Pi$ ([5], ch. VI, § 1, prop. 15); if $n \in {}^v\nu^{-1}(w) \cap N'$, then by (6.1.2)(9) and (10)

$$M_b = n^{-1}M_an \subset N',$$

whence (i). The relation ${}^v\nu^{-1}(1) = T$ follows from (6.1.10). Finally, if $n \in N \cap TU^+$, then $w = {}^v\nu(n)$ preserves Φ^+ (by (DR 1), (DR 6) and (6.1.2)(10)), hence is the identity element, and $n \in T$.

Proposition (6.1.12). — *Put $R = \{M_a \mid a \in \Pi\} \subset N/T$. Then (G, TU^+, N, R) is a saturated Tits system with Weyl group N/T isomorphic to ${}^vW$.*

Proof sketch. By (6.1.11) it suffices to prove the following relations, for $n \in N$, $a \in \Pi$ and $m \in M_a$:

$$mTU^+m^{-1} \neq TU^+, \tag{33}$$

$$TU^+n.\{1, m\}.TU^+m = TU^+n.\{1, m\}.TU^+, \tag{34}$$

$$\bigcap_{x \in N} xTU^+x^{-1} \subset T. \tag{35}$$

The first follows immediately from (DR 1), (DR 6) and the fact that $U_{-a} \subset mTU^+m^{-1}$. The second being invariant when n is replaced by nm, we need only establish it in the case where $a' = {}^v\nu(n)(a) \in \Phi^+$. Let U' be the group generated by the U_b for $b \in \Phi^+$ and $b \notin \mathbf{R}a$. By (6.1.6) one has $U^+ = U_aU'$, and with the notation of (6.1.2)(7),

$$TU^+n.\{1,m\}.TU^+ = TU^+n.\{1,m\}.U_aTU' \subset TU^+nL_aTU'.$$

But by (6.1.2)(7),

$$nL_a \subset nTU^+ \cup nU_amU_aT = nTU^+ \cup U_{a'}nmU_aT \subset TU^+.\{n,nm\}.TU^+,$$

whence (2). Finally (3) is a consequence of (DR 6).

Corollary (6.1.13). — *Keep the notation of (6.1.12). Then*

$$\mathcal{N}(U^+) = TU^+, \tag{36}$$

$$T = \bigcap_{a\in\Phi} \mathcal{N}(U_a) = \mathcal{N}(U^+) \cap \mathcal{N}(U^-). \tag{37}$$

Proof sketch. The normalizer of U^+ contains TU^+, so it has the form $TU^+N'TU^+$ with $N' \subset N$ (1.2.9); but if an element n of N normalizes U^+, its image under ${}^v\nu$ preserves Φ^+ by (DR 1) and (DR 6), hence is the identity element, so that $n \in {}^v\nu^{-1}(1) = T$. Thus $N' \subset T$ and $\mathcal{N}(U^+) = TU^+$. Relation (2) now follows from the inclusions (for the first, cf. (6.1.2)(3))

$$T \subset \bigcap_{a\in\Phi} \mathcal{N}(U_a) \subset \mathcal{N}(U^+) \cap \mathcal{N}(U^-) = TU^+ \cap TU^- = T.$$

Remark (6.1.14). — *The preceding corollary shows that a* generating *root datum is entirely determined by the system of the U_a.*

(6.1.15) Let us note some further consequences of (6.1.12).

a) *The injection of N into G defines a bijection of N (resp. ${}^vW$) onto the set of double cosets $U^+\backslash G/U^+$ (resp. $TU^+\backslash G/TU^+$).* Indeed, by (1.2.7) it suffices to prove that if $n \in N$ and $t \in T$ are such that $tn \in U^+nU^+$, then $t = 1$. Now by (6.1.6) one has $nU^+n^{-1} \subset U^+.U^-$, and our assertion follows from (DR 6).

b) Let $w \in {}^vW$; the subgroup $nU^+n^{-1} \cap U^-$ is independent of $n \in {}^v\nu^{-1}(w)$: it follows from (6.1.6) and (DR 6) that it is the subgroup generated by the U_a for $a \in \Phi^- \cap w(\Phi^+)$. Denote it U_w^+. One then checks easily that, for every $n \in {}^v\nu^{-1}(w)$, the map $(u,t,u') \mapsto utnu'$ (resp. $(u,u') \mapsto unu'$) is a bijection of $U^+ \times T \times U_w^+$ (resp. $U^+ \times U_w^+$) onto U^+TnU^+ (resp. U^+nU^+).

c) There exists $n \in N$ with $nU^+n^{-1} = U^-$ and $nU^-n^{-1} = U^+$: it suffices to choose n so that ${}^v\nu(n)$ is the element of ${}^vW$ carrying the chamber D to $-D$. Consequently, *the injection of N into G also defines a bijection of N (resp. ${}^vW$) onto $U^+\backslash G/U^-$ (resp. $TU^+\backslash G/TU^-$).*

d) Let X be a subset of Π, let ${}^vW_X$ be the subgroup of ${}^vW$ generated by the r_a for $a \in X$, and put $G_X = U^+T\,{}^vW_XU^+$. Then G_X is a parabolic subgroup of G, and the map $X \mapsto G_X$ is a bijection of $\mathscr{P}(\Pi)$ onto the set of subgroups of G containing TU^+: this is only a restatement of (1.2.9).

6.2. Valuations of a root datum. —

The notation of (6.1.4) is retained.

Definition (6.2.1). — *A* valuation *of the root datum $(T, (U_a, M_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$, where φ_a is a map of U_a into $\mathbf{R} \cup \{\infty\}$, having the following properties.*

(V 0) : for every $a \in \Phi$, the image of φ_a contains at least three elements;
(V 1) : for $a \in \Phi$ and $k \in \mathbf{R} \cup \{\infty\}$, the set $U_{a,k} = \varphi_a^{-1}([k,\infty])$ is a subgroup of U_a, and $U_{a,\infty} = \{1\}$;

(V 2) : for every $a \in \Phi$ and every $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on $U^*_{-a} = U_{-a} - \{1\}$;

(V 3) : let $a, b \in \Phi$ and $k, \ell \in \mathbf{R}$; if $b \notin -\mathbf{R}_+ a$, the commutator group $(U_{a,k}, U_{b,\ell})$ is contained in the group generated by the $U_{pa+qb,\ pk+q\ell}$ for $p, q \in \mathbf{N}^*$ with $pa + qb \in \Phi$;

(V 4) : if a and $2a$ belong to Φ, then φ_{2a} is the restriction of $2\varphi_a$ to U_{2a};

(V 5) : let $a \in \Phi$, $u \in U_a$ and $u', u'' \in U_{-a}$; if $u'uu'' \in M_a$, then $\varphi_{-a}(u') = -\varphi_a(u)$.

(6.2.2) Let $\varphi = (\varphi_a)$ be a valuation of $(T, (U_a))$. By convention we set $U_{2a,k} = \{1\}$ when $a \in \Phi$ and $2a \notin \Phi$ ($k \in \mathbf{R} \cup \{\infty\}$). For $a \in \Phi$ and $u \in U_a$,

$$\varphi_a(u) = \sup\{k \in \mathbf{R} \cup \{\infty\} \mid u \in U_{a,k}\}. \tag{38}$$

By (V 1),

$$\varphi_a(u^{-1}) = \varphi_a(u). \tag{39}$$

It follows that, when (V 1) holds, axiom (V 5) is equivalent to

(V 5 bis) : let $a \in \Phi$, $u \in U_a$ and $u', u'' \in U_{-a}$; if $u'uu'' \in M_a$, then $\varphi_{-a}(u'') = -\varphi_a(u)$.

For $a \in \Phi$ and $k \in \mathbf{R}$ one puts

$$M_{a,k} = M_a \cap U_{-a}\varphi_a^{-1}(k)U_{-a}.$$

By (6.1.2)(2), $M_{a,k}$ is non-empty if and only if $k \in \varphi_a(U_a^*)$, and

$$\varphi_a^{-1}(k) \subset U_{-a,-k} M_{a,k} U_{-a,-k} \qquad (a \in \Phi,\ k \in \mathbf{R}). \tag{40}$$

When $2a \in \Phi$ one has $M_{2a,2k} \subset M_{a,k}$.

It follows from (V 1) and (V 3) that the $U_{a,k}$ form a basis of the filter of neighborhoods of the identity for a topological group structure on U_a.

For $a \in \Phi$ one puts

$$\Gamma_a = \varphi_a(U_a^*), \qquad \Gamma'_a = \{\varphi_a(u) \mid u \in U_a^*,\ \varphi_a(u) = \sup \varphi_a(uU_{2a})\}.$$

By (V 5) and (6.1.2)(2) one has $\Gamma_{-a} = -\Gamma_a$. If $2a \notin \Phi$ then $\Gamma'_a = \Gamma_a$. By (V 1) and (V 4), $\Gamma_a = \Gamma'_a \cup (1/2)\Gamma_{2a}$. Note that Γ'_a may be empty: for this it suffices that U_{2a} be dense in U_a (for an example of this situation, cf. (6.2.3) d)); this condition is also necessary when Γ_a is discrete in $\mathbf{R}$.

Examples (6.2.3). — *Let K be a field equipped with a non-improper real valuation ω.*

a) Let $G = SL_2(K)$ and take up again the root datum in G defined in (6.1.3) a). For $u \in K$ put

$$\varphi_a\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right) = \varphi_{-a}\left(\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\right) = \omega(u).$$

Conditions (V 0), (V 1), (V 3) and (V 4) are immediately verified. If $m = \begin{pmatrix}0 & x\\ y & 0\end{pmatrix} \in M_a$, then $m\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}m^{-1} = \begin{pmatrix}1 & xuy^{-1}\\ 0 & 1\end{pmatrix}$, whence (V 2). Finally (V 5) follows at once from formula (1) of (6.1.3) a). We have thus indeed defined a valuation of the root datum $(T, (U_a))$. Note that

$$M_{a,k} = \left\{\begin{pmatrix}0 & u\\ -u^{-1} & 0\end{pmatrix} \,\middle|\, \omega(u) = k\right\}. \tag{41}$$

b) Take up the notation of (6.1.3) b). For $a \in \Phi$ and $u = x_a(t) \in U_a$ ($t \in K$), put $\varphi_a(u) = \omega(t)$. The family $\varphi = (\varphi_a)$ is a valuation of the root datum $(T, (U_a))$. Here again (V 0), (V 1) and (V 4) are evident. Condition (V 3) follows from assertion (3) of (6.1.3) b). Moreover the family $(\varphi_a \circ \zeta_a, \varphi_{-a} \circ \zeta_{-a})$ is none other than the valuation of the root datum of $SL_2(K)$ described in example a); one deduces at once (V 5), and also (V 2) when $m = n_a = \zeta_a\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Since

$M_a = Tn_a$, to finish the proof it suffices to show that $u \mapsto \varphi_a(tut^{-1}) - \varphi_a(u)$ is constant on U_a^* for every $t \in T$, which is evident since $tx_a(s)t^{-1} = x_a(a(t)s)$ for $s \in K, t \in T$.

c) One of the goals of this work is to show that if K is a local field and $\mathscr{G}$ a connected semisimple algebraic group defined over K, then the root datum $(T, (U_a))$ described in (6.1.3) c) can be valued: this is what we shall see in a later chapter (see also § 10).

d) Suppose the valued field K commutative, imperfect of characteristic 2, and let $L \subsetneq K$ be a K^2-vector subspace of K. Put $G = SL_2(K)$, $\Phi = \{\pm a, \pm 2a\}$, let $U_{\pm a}$ and $\varphi_{\pm a}$ be defined as in (6.1.3) a) and (6.2.3) a), put

$$U_{2a} = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \middle| \, u \in L \right\}, \qquad U_{-2a} = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \middle| \, u \in L \right\},$$

and $\varphi_{\pm 2a} = 2\varphi_{\pm a} \,|\, U_{\pm 2a}$. Then $(T, (U_b)_{b\in\Phi})$ is a root datum of type Φ (hence of type BC_1) in G, and $\varphi = (\varphi_b)$ is a valuation of it. Moreover U_{2a} is dense in U_a (and $\Gamma'_a = \emptyset$) as soon as L is dense in K. This is so, for example, when $K = k((t))$ is a field of formal series over a field k of infinite degree over k^2, if one takes for L the group of series $\sum_i a_i t^i$ such that the k^2-vector subspace of k generated by the a_i is finite-dimensional.

e) Take up the notation of (5.2.30) and suppose that the root datum $(T, (U_a)_{a\in\Phi})$ and the system $(N, \nu, (U_\alpha)_{\alpha\in\mathbf{\Sigma}})$ are *compatible*, in the sense that the following conditions hold:

(i) $V = {}^v\mathbf{A}$, $V^* = {}^v\mathbf{A}^*$ and ${}^v\mathbf{W} = {}^vW$, which amounts to saying that there is a surjective map $a \mapsto \bar{a}$ of Φ onto ${}^v\Sigma$ with $\bar{a} \in \mathbf{R}_+ a$ for every $a \in \Phi$;

(ii) the subgroup N coincides with the one also denoted N in (6.1.2);

(iii) for every $a \in \Phi$, the subgroup U_a is the union of the U_α for $\alpha \in \mathbf{\Sigma}$ with ${}^v\alpha = a$, i.e. is the subgroup denoted $\mathfrak{U}_a$ in § 5.2.

Comparing (6.1.2)(10) with condition (B 1) of (5.2.30), one sees at once that the homomorphism ${}^v\nu : N \longrightarrow {}^v\mathbf{W} = {}^vW$ defined in (6.1.2) coincides with the one also denoted ${}^v\nu$ in § 5.2.

Identifying now $\mathbf{A}$ and V by the choice of an origin o in $\mathbf{A}$, denote for $a \in V^*$ and $k \in \mathbf{R}$ by $\alpha_{a,k}$ the closed half-space $\{x \in \mathbf{A} \mid a(x) + k \geq 0\}$. For $a \in \Phi^{\mathrm{red}}$ and $u \in U_a$ put

$$\varphi_a(u) = \sup\{k \in \mathbf{R} \mid u \in U_{\alpha_{a,k}}\},$$

and for $a \in \Phi - \Phi^{\mathrm{red}}$ and $u \in U_{2a}$ put $\varphi_{2a}(u) = 2\varphi_a(u)$. Then the family $\varphi = (\varphi_a)_{a\in\Phi}$ so defined is a *quasi-valuation* of the root datum $(T, (U_a))$, in the sense that it satisfies (V 0), (V 1), (V 2), (V 4), (V 5) and the condition (V 3 bis), weaker than (V 3), obtained by replacing in (V 3) the hypothesis "$b \notin -\mathbf{R}_+ a$" by "$b \notin \mathbf{R}a$". Let us note moreover that part of the results we shall establish on valuations of a root datum remain valid for quasi-valuations.

Proof sketch. (V 0), (V 1), (V 4) are immediate. For (V 2), reduce by (V 4) and (6.1.2)(4) to $a \in \Phi^{\mathrm{red}}$; then $\nu(m)$ is the orthogonal reflection in a hyperplane $a(x) + \ell = 0$, and (B 1) gives $mU_{a,k}m^{-1} = U_{-a,k-2\ell}$, whence $\varphi_{-a}(u) = \varphi_a(mum^{-1}) - 2\ell$.

For (V 3 bis) one identifies the affine roots containing $\alpha_{a,k} \cap \alpha_{b,\ell}$ as the $\alpha_{pa+qb,h}$ with $h \geq pk + q\ell$, applies (5.2.32) to get U_Ω as an ordered product of the $U_{s,h(s)}$ and (6.1.6) to get the analogous product decomposition of the group X generated by suitable X_s, so that $X \cap U_\Omega = \prod_{pa+qb\in\Phi,\, p,q\in\mathbf{N}^*} U_{pa+qb,\, pk+q\ell}$; the commutator (x, y) lies in U_Ω by (B′3 bis) and in X by (DR 2).

For (V 5), again with $a \in \Phi^{\mathrm{red}}$: put $k = \varphi_a(u)$, $\alpha = \alpha_{a,k}$, use (B 4) to write $u = u'_1 r u''_1$ with $u'_1, u''_1 \in U_{-a,-k}$, so $u' = u'_1$ by (6.1.2)(2) and $\varphi_{-a}(u') \geq -k$; strict inequality would give $\alpha_+ \cap \alpha^* = (\alpha^*)_+ \cap \alpha$, which is absurd. *(Original: pp. 119–120.)*

A partial converse will be seen later (6.5).

(6.2.4) In all the sequel of § 6, $\varphi = (\varphi_a)$ denotes a valuation of the root datum $(T, (U_a))$. When ambiguity about φ is to be feared, we write ${}^\varphi U_{a,k}$, ${}^\varphi M_{a,k}$, etc. for the sets denoted simply $U_{a,k}$, $M_{a,k}$, etc. above; an analogous notation is used without further warning for the other symbols attached to φ defined below.

(6.2.5) Let $\lambda : \Phi \to \mathbf{R}_+^*$ be a function constant on the irreducible components of Φ, and let $v \in V$. It is easy to see that the family $\psi = (\psi_a)$ defined by $\psi_a(u) = \lambda(a)\varphi_a(u) + a(v)$ (for $a \in \Phi$, $u \in U_a$) is a valuation of $(T, (U_a))$, which we denote $\lambda\varphi + v$. The valuations φ and $\psi = \lambda\varphi + v$ are said to be *equivalent*, and *equipollent* if $\lambda = 1$. The map $(\varphi, v) \mapsto \varphi + v$ is an action of the additive group of V on the set of valuations, and an equipollence class of valuations is an orbit of V for this action.

On the other hand let $n \in N$ and $w = {}^v\nu(n)$. For $a \in \Phi$ and $u \in U_a$ one has $n^{-1}un \in U_{w^{-1}(a)}$ by (6.1.2)(10). Put this time

$$\psi_a(u) = \varphi_{w^{-1}(a)}(n^{-1}un).$$

Here again it is immediate — the inner automorphism defined by n being an automorphism of the root datum $(T, (U_a))$ — that the family $\psi = (\psi_a)$ so defined is a valuation of $(T, (U_a))$, which we denote $n.\varphi$. One thus obtains an action of the group N on the set of valuations of $(T, (U_a))$, and one checks at once that for $n \in N$, $v \in V$ and λ as above,

$$n.(\lambda\varphi + v) = \lambda(n.\varphi) + {}^v\nu(n)(v). \tag{42}$$

(6.2.6) Let A be the set of valuations equipollent to φ: it is an affine space under V, which we equip with the Euclidean distance corresponding to the given inner product on V. For $a \in V^*$ and $k \in \mathbf{R}$ we write, as above, $\alpha_{a,k}$ (or ${}^\varphi\alpha_{a,k}$ if confusion is to be feared) for the closed half-space of A formed by the $x \in A$ with $a(x - \varphi) + k \geq 0$. The boundary of a closed half-space α is denoted $\partial\alpha$. For $\alpha = \alpha_{a,k}$ with $a \in \Phi^{\mathrm{red}}$ one puts

$$U_\alpha = U_{a,k} \qquad \text{and} \qquad U_{\alpha_+} = \bigcup_{h>k} U_{a,h}.$$

The *affine roots* of A are the half-spaces $\alpha_{a,k}$ for $a \in \Phi$ and $k \in \Gamma'_a$, and the *walls* of A are the boundaries of the affine roots. The set of affine roots is denoted Σ. Finally, $\mathscr{E}$ denotes the set of pairs $(\alpha, a) \in \Sigma \times \Phi$ such that $\alpha = \alpha_{a,k}$ with $k \in \Gamma'_a$. It is clear that Σ, $\mathscr{E}$ and the map $\alpha \mapsto U_\alpha$ remain unchanged if φ is replaced by an equipollent valuation.

Proposition (6.2.7). — *For $i = 1, \ldots, p$ let $a_i \in \Phi$, $k_i \in \Gamma_{a_i}$ and $m_i \in M_{a_i,k_i}$. Put $n = m_1 \cdots m_p$ and $r_i = r_{a_i}$. Then*

$$n.\varphi = \varphi - v \qquad \textit{with} \qquad v = \sum_{1 \leq i \leq p} k_i (r_1 \cdots r_{i-1})(a_i^\vee) \in V.$$

Proof sketch. Induction on p reduces, via (6.2.5)(1), to $p = 1$ with $a = a_1 \in \Phi^{\mathrm{red}}$; write $m = u'uu''$, so $\varphi_a(u) = -\varphi_{-a}(u') = -\varphi_{-a}(u'') = k$. For b not proportional to a, the group $U' = \prod_{c \in \Psi^{\mathrm{red}}} U_{c,h(c)}$ (with Ψ the set of $pa + qb \in \Phi$, $p \in \mathbf{Z}$, $q \in \mathbf{N}^*$, and $h(pa + qb) = pk + q\ell$) is normalized by m by (V 3), and intersecting with U_b, $U_{r(b)}$ gives $m^{-1}U_{b,\ell}m = U_{r(b),\,\ell - kb(a^\vee)}$, i.e. (1). The cases $b = -a$ and $b = a$ are handled from $u'' = u^{-1}m(m^{-1}u'^{-1}m)$ and $u' = (mu''^{-1}m^{-1})mu^{-1}$ together with (V 2) and $a(a^\vee) = 2$, giving (2) and (3); with (V 4) these say $m.\varphi = \varphi - ka^\vee$. *(Original: p. 121.)*

Corollary (6.2.8). — *Let φ and ψ be two valuations of the root datum $(T, (U_a))$. If $\varphi_a = \psi_a$ for every $a \in \Pi$, then $\varphi = \psi$.*

Proof sketch. One has ${}^\varphi M_{a,k} = {}^\psi M_{a,k}$ for every $a \in \Pi$ and every $k \in \mathbf{R}$. Let $b \in \Phi^{\mathrm{red}}$. There is $w \in {}^vW$ with $c = w^{-1}(b) \in \Pi$, and a sequence $a_1, \ldots, a_p$ of elements of Π with $w = r_{a_1} \cdots r_{a_p}$. Choosing, for $i = 1, \ldots, p$, an element $k_i \in \Gamma_{a_i}$ and an element $m_i \in {}^\varphi M_{a_i,k_i} = {}^\psi M_{a_i,k_i}$, and putting $m = m_1 \cdots m_p$, the preceding proposition shows that

$$\varphi_b(u) - \varphi_c(mum^{-1}) = \psi_b(u) - \psi_c(mum^{-1}) \qquad (u \in U_b),$$

which proves the assertion.

Corollary (6.2.9). — *Let $a, b \in \Phi$ with $b \notin \mathbf{R}a$ and $b - a \notin \Phi$, let $\Psi = \Phi \cap (\mathbf{Z}a + \mathbf{Z}b)$ be the root subsystem they generate, let $a = a_1, a_2, \ldots, a_s = b, a_{s+1}, \ldots, a_{2s}$ be the elements of Ψ^{red}*

arranged in "circular order" (cf. (6.1.8)), let $u \in U_a$, $u' \in U_b$ and $(u, u') = u_2 \cdots u_{s-1}$ with $u_i \in U_{a_i}$. Then, if $i, p, q \in \mathbf{N}$ are such that $2 \le i \le s-1$ and $a_i = pa + qb$, one has

$$\varphi_{a_i}(u_i) = p\,\varphi_a(u) + q\,\varphi_b(u').$$

Proof sketch. One may obviously assume $u \neq 1 \neq u'$. By (6.1.8) one has $u_2 = m(u).u'.m(u)^{-1}$, and applying (6.2.7) with $n = m(u)$ gives

$$\varphi_{a_2}(u_2) = \varphi_b(u') - b(a^\vee)\varphi_a(u). \tag{43}$$

Since $a_2 = r_a(b) = b - b(a^\vee).a$, this establishes the assertion for $i = 2$.

Assume now $i > 2$ and argue by induction on i. Let $u'' \in U^*_{-a}$ be defined by $u \in U_a M_a u''$. By axiom (V 5 bis) of valuations (6.2.2),

$$\varphi_{-a}(u'') = -\varphi_a(u). \tag{44}$$

On the other hand it follows from (6.1.8) that $(u_2^{-1}, u'') = u_3 \cdots u_{s-1}u'$. But $a_i = pa + qb = qa_2 - (p + q\,b(a^\vee)).(-a)$. Hence, by the induction hypothesis and taking (1) and (2) into account,

$$\varphi_{a_i}(u_i) = q\,\varphi_{a_2}(u_2) - (p + q\,b(a^\vee))\,\varphi_{-a}(u'') = p\,\varphi_a(u) + q\,\varphi_b(u').$$

Proposition (6.2.10). — *Let A be the affine space of valuations equipollent to φ.*

(i) *A is stable under the action of N. For $n \in N$, the map $\nu(n) : \psi \mapsto n.\psi$ of A into itself is an automorphism of the Euclidean space A, whose canonical image in* Aut V *equals ${}^v\nu(n)$.*

(ii) *For every $a \in \Phi$ and every $k \in \Gamma_a$, the image under ν of an element of $M_{a,k}$ is the orthogonal reflection $r_{a,k}$ in the hyperplane $\partial\alpha_{a,k} = \{x \in A \mid a(x - \varphi) + k = 0\}$.*

(iii) *For every $n \in N$ and every $\alpha \in \Sigma$ one has $\nu(n)(\alpha) \in \Sigma$ and $nU_\alpha n^{-1} = U_{\nu(n)(\alpha)}$.*

Proof sketch. (i) By (6.2.5)(1) and (6.2.7), $n.A = A$ for n in the subgroup N' generated by the $M_{a,k}$; as $N = N'T$ it remains to treat $t \in T$, where (V 2) gives a constant $c(t, a)$ with $(t.\psi)_a = \psi_a + c(t, a)$, and the unique $v \in V$ with $a(v) = c(t, a)$ for $a \in \Pi$ yields $t.\psi = \psi + v$ by (6.2.8). (ii) For $m \in M_{a,k}$ one has ${}^v\nu(m) = r_a$ and, by (6.2.7), $m.(\varphi - \frac{1}{2}ka^\vee) = \varphi - \frac{1}{2}ka^\vee$. (iii) Unwinding $u \in nU_\alpha n^{-1}$ gives $u \in U_\beta$ with $\beta = \{\psi \in A \mid a(n^{-1}.\psi - \varphi) + k \ge 0\} = \nu(n)(\alpha)$. *(Original: pp. 122–123.)*

(6.2.11) One puts $H = \nu^{-1}(1)$ and $\widehat{W} = \nu(N)$, and denotes by W the subgroup of $\widehat{W}$ generated by the reflections $r_{a,k}$ for $a \in \Phi$ and $k \in \Gamma_a$. Since N permutes the $M_{a,k}$, this is a normal subgroup of $\widehat{W}$.

Put $N' = \nu^{-1}(W)$, $T' = T \cap N'$, and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \emptyset$ for $a \in \Phi$, one sees easily that $(T', (U_a))$ is a generating root datum of type Φ in G', for which the group N of (6.1.2)(10) is none other than N'.

Remarks (6.2.12). —

a) Let $\Phi = \bigcup_i \Phi_i$ be the decomposition of Φ into irreducible components. It is clear that $\varphi_i = (\varphi_a)_{a \in \Phi_i}$ is a valuation of the root datum $(T \cap G_i^0, (U_a)_{a \in \Phi_i})$ of the group G_i^0 defined in (6.1.5). On the other hand, to the decomposition of Φ there corresponds a canonical decomposition of A into a product of affine spaces A_i under the dual vector space of the subspace generated by Φ_i (cf. (1.3.9)). It is clear that the action of N is compatible with this product decomposition of A, and that W decomposes as the direct product of the groups W_i defined from G_i^0 and φ_i as W was defined from G and φ. This often allows one to reduce to the case where Φ is irreducible.

b) Let $a \in \Phi$ and $u \in U_a$, $u \neq 1$. It follows from (V 5) and (6.2.10)(ii) that *the valuation $\varphi_a(u)$ is the unique real number k such that $\nu(m(u))$ is the orthogonal reflection in the hyperplane $\partial\alpha_{a,k}$*. One sees therefore that φ is completely determined by the datum of the homomorphism $\nu : N \to \operatorname{Aut} A$.

More precisely, let A_0 be an affine space under V, let ν_0 be a homomorphism of N into $\operatorname{Aut} A_0$ whose composite with the canonical homomorphism $\operatorname{Aut} A_0 \to \operatorname{Aut} V$ equals ${}^v\nu$, and let

φ_0 be a point of A_0. For a and u as above, $\nu_0(m(u))$ is an orthogonal reflection in a hyperplane of A_0 of the form $\{x \in A_0 \mid a(x - \varphi_0) + k = 0\}$ (with $k \in \mathbf{R}$), since ${}^v\nu_0(m(u)) = r_a$. Put then $\psi_a(u) = k$ and $\psi_a(1) = \infty$. It is clear that if there exist a valuation φ of $(T, (U_a))$ and an isomorphism j of ${}^\varphi A$ onto A_0 with $j(\varphi) = \varphi_0$ and $\nu_0(n) = j\nu(n)j^{-1}$ for all $n \in N$, then $\varphi_a = \psi_a$ for all $a \in \Phi$.

It is moreover immediate that the family $\psi = (\psi_a)$ always satisfies conditions (V 2), (V 4) and (V 5); condition (V 0) is satisfied as soon as $\nu_0(T)$ generates V. To be sure that ψ is indeed a valuation, it remains only to verify conditions (V 1) and (V 3). It is possible that (V 1) suffices to entail that ψ is a "quasi-valuation" of $(T, (U_a))$, i.e. that for a family ψ of the type considered here, (V 1) entails condition (V 3 bis) of (6.2.3) e).

Application. Let K be a field and $G = SL_2(K)$; take up the notation of (6.1.3) a) and suppose the root datum $(T, (U_a, U_{-a}))$ equipped with a valuation φ. Replacing φ by an equipollent valuation, one may assume $\varphi_a\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) = 0$. For $x \in K^*$ let $\omega(x) \in \mathbf{R}$ be such that $\nu\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right) = \omega(x)a^\vee$. One defines in this way a homomorphism $\omega : K^* \to \mathbf{R}$. Extend ω to all of K by setting $\omega(0) = \infty$. Then *ω is a non-improper valuation of K*. Indeed, by (6.1.3)(1) and (6.2.10)(ii), $\nu\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ is the reflection about φ; consequently, for $u \in K^*$,

$$\nu\left(\begin{pmatrix}0 & u\\ -u^{-1} & 0\end{pmatrix}\right) = \nu\left(\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix}\right) \circ \nu\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)$$

is the reflection about $\varphi + (1/2)\omega(u)a^\vee$. Using once more (6.1.3)(1) and (6.2.10)(ii), one sees that $\varphi_a\left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right) = \omega(u)$ for all $u \in K$. Condition (V 1) then shows that $\omega(u + v) \geq \inf(\omega(u), \omega(v))$ for $u, v \in K$, hence that ω is a valuation of K, and (V 0) shows that ω is not improper.

Taking (6.2.3) a) into account, one sees therefore that *the equipollence classes of valuations of the "canonical" root datum of $SL_2(K)$ correspond bijectively to the non-improper valuations of K*. This result will be generalized in 10.2.

c) Suppose Φ irreducible and let ψ be another valuation of the root datum $(T, (U_a))$. *If there exists $a \in \Phi$ with $\varphi_a = \psi_a$, then the valuations φ and ψ are equipollent.*

Proof sketch. Reduce to $a \in \Pi \cup 2\Pi$ with $u_b \in U_b$ satisfying $\varphi_b(u_b) = \psi_b(u_b) = 0$ for each $b \in \Pi$. By b) it is enough, after normalizing $\varphi = \psi = 0$, to show ${}^\varphi\nu = {}^\psi\nu$. On the group N_0 generated by the $m(u_b)$ both give the reflection $r_{b,0}$, and $N = N_0.T$; for $t \in T$ one gets $c({}^\varphi\nu(t)) = c({}^\psi\nu(t))$ for every $c \in {}^v\nu(N_0)(a)$, and irreducibility of Φ makes the ${}^vW$-orbit of a span V^*. *(Original: p. 124.)*

d) If $a, b \in \Phi$, there exists a map $g : U_a \to U_b$ such that, for every valuation φ of the root datum $(T, (U_c)_{c\in\Phi})$, one has p. 125

$$\varphi_b(g(u)) = -b(a^\vee) \cdot \varphi_a(u) + \text{const.} \qquad (u \in U_a)$$

(the constant depending of course on φ).

Indeed, it suffices to choose arbitrarily an element $u_0 \in U^*_{r_a(b)}$ and to put

$$g(u) = \begin{cases} 1 & \text{if } u = 1,\\ m(u) \cdot u_0 \cdot m(u)^{-1} & \text{if } u \neq 1.\end{cases}$$

(6.2.13) We shall now study the sets Γ_a and Γ'_a defined in (6.2.2). If $0 \in \Gamma_a$, we denote by $\widetilde{\Gamma}_a$ the subgroup of $\mathbf{R}$ generated by Γ_a.

Definition. — The valuation φ is said to be special *if, for every root $a \in \Phi^{\mathrm{red}}$, one has $0 \in \Gamma_a$.*

Proposition (6.2.14). — In order that φ be special it is necessary and sufficient that 0 belong to Γ_a for every $a \in \Pi$. One then has $\Gamma'_{w(b)} = \Gamma'_b$ for every $w \in {}^vW$ and every $b \in \Phi$.

Indeed, (6.2.7) implies that, for $a, b \in \Phi$, one has

$$\Gamma'_{r_a(b)} \supset \Gamma'_b - b(a^\vee)\,\Gamma_a. \tag{45}$$

In particular, $\Gamma'_{r_a(b)} \supset \Gamma'_b$ as soon as $0 \in \Gamma_a$. Since the r_a for $a \in \Pi$ generate ${}^vW$, and since every root is the transform of a root belonging to $\Pi \cup 2\Pi$ by an element of ${}^vW$, this yields the proposition.

Corollary (6.2.15). — There exists a special valuation equipollent to φ, and even a valuation ψ equipollent to φ such that $0 \in \Gamma_a = \Gamma'_a$ for every root a belonging to the set Φ^{nm} of non-multipliable roots.

It suffices to take $\psi = \varphi + v$, with $-a(v) \in \Gamma_a$ for every $a \in (\Pi \cup 2\Pi) \cap \Phi^{\mathrm{nm}}$.

Corollary (6.2.16). — Let $b \in \Phi$. One has $\Gamma'_{-b} = -\Gamma'_b$. If $0 \in \Gamma_b$, one has $\Gamma'_b = \Gamma'_{-b} = \Gamma'_b + 2\widetilde{\Gamma}_b$. If moreover Γ'_b is a discrete subset of $\mathbf{R}$ and if $0 \in \Gamma'_b$, then Γ'_b is a subgroup of $\mathbf{R}$ isomorphic to $\mathbf{Z}$.

The first two assertions follow from (6.2.14) and from (6.2.14) (1), in which one replaces a by b. If moreover Γ'_b is discrete and contains 0, then $2\widetilde{\Gamma}_b$ is contained in Γ'_b, hence is a discrete group, isomorphic to $\mathbf{Z}$ in view of (V 0). The relations $2\widetilde{\Gamma}_b \subset \Gamma'_b = \Gamma'_b + 2\widetilde{\Gamma}_b \subset \widetilde{\Gamma}_b$ then imply $\Gamma'_b = \widetilde{\Gamma}_b$ or $\Gamma'_b = 2\widetilde{\Gamma}_b$.

Example (6.2.17). — Let $\widetilde{\Gamma}$ be a subgroup of $\mathbf{R}$ such that $\widetilde{X} = \widetilde{\Gamma}/2\widetilde{\Gamma}$ is a vector space of dimension > 1 over the field $\mathbf{Z}/2\mathbf{Z}$. Let X be a generating system of $\widetilde{X}$, containing 0 and different from the whole of $\widetilde{X}$, and let Γ be the inverse image of X in $\widetilde{\Gamma}$. Then Γ is not a group, but contains 0 and is stable under addition of an element of $2\widetilde{\Gamma}$, and $\widetilde{\Gamma}$ is the group generated by Γ; it is moreover easy to see that we have just described all the subsets of $\mathbf{R}$ possessing these properties.

On the other hand, let k be a field of characteristic 2 and let K_0 be the group algebra of the group $\widetilde{\Gamma}$ with coefficients in k. Denote by $(X^\gamma)_{\gamma \in \widetilde{\Gamma}}$ the canonical basis of K_0. One obtains a valuation ω of K_0 with values in $\widetilde{\Gamma}$ by putting p. 126

$$\omega\Big(\sum_\gamma a_\gamma X^\gamma\Big) = \inf\{\gamma \mid a_\gamma \neq 0\}. \tag{46}$$

The completion K of K_0 for this valuation is a field, sometimes called the "field of formal series over k with exponents in $\widetilde{\Gamma}$". The elements of K are the series $\sum_\gamma a_\gamma X^\gamma$ such that, for every $\lambda \in \mathbf{R}$, the set of $\gamma \leq \lambda$ with $a_\gamma \neq 0$ is finite; the valuation ω extends to K and (1) remains valid.

Now let L be the subset of K consisting of the series $\sum_\gamma a_\gamma X^\gamma$ such that $a_\gamma = 0$ for $\gamma \notin \Gamma$. One sees at once that L is a K^2-vector subspace of K, containing K^2, but is not a subfield of K, although $x \in L$, $x \neq 0$, implies $x^{-1} = (x^{-2})x \in L$.

Finally let G be the subset of $\mathrm{SL}_2(K)$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the products ab, ac, bd and cd belong to L. One verifies easily that G is a subgroup of $\mathrm{SL}_2(K)$. Taking up again the notation of (6.2.3) a): it is immediate that $(T \cap G, (U_a \cap G))$ is a root datum in G and that $\psi = (\varphi_a \mid U_a \cap G)$ is a valuation of it, for which one has $\Gamma_a = \Gamma'_a = \Gamma$.

We thus see that (6.2.16) is "the best possible result" in the general case.

Proposition (6.2.18). — Assume φ special. Let a and b be two non-orthogonal roots.

(i) If a and b have the same length and $a \neq \pm b$, one has $\Gamma_a = \Gamma'_a = \Gamma_b$ and Γ_a is a group.

(ii) If $(b \mid b) = 2(a \mid a)$ and if $0 \in \Gamma'_a$, one has $\Gamma_b + 2\widetilde{\Gamma}_a \subset \Gamma_b = \Gamma'_b$ and $\Gamma'_a + \widetilde{\Gamma}_b \subset \Gamma'_a$.

(iii) If $(b \mid b) = 3(a \mid a)$, one has $\Gamma_b + 3\Gamma_a \subset \Gamma_b = \Gamma'_b \subset \Gamma_a = \Gamma'_a$ and Γ_a and Γ_b are groups.

Under the hypothesis of (i) one has $2a \notin \Phi$, hence $\Gamma_a = \Gamma'_a$ and $a(b^\vee) = b(a^\vee) = \pm 1$. Replacing b by $-b$ if necessary, we may assume that $a(b^\vee) = -1$. Then $a+b$ is a root of the same length as a and b and one has $\Gamma_{a+b} = \Gamma_a = \Gamma_b$ (6.2.14). On the other hand (6.2.14) (1) implies that $\Gamma_{a+b} \supset \Gamma_a + \Gamma_b$, whence (i).

Under the hypothesis of (ii) one has $2b \notin \Phi$, hence $\Gamma_b = \Gamma'_b$; moreover, replacing b by $-b$ if necessary, we may assume that $b(a^\vee) = 2a(b^\vee) = -2$, and (ii) then follows from (6.2.14) (1) applied to the pairs (a, b) and (b, a).

Under the hypothesis of (iii), the root system generated by a and b is of type G_2 and there exists a root c of the same length as a (resp. b) and not orthogonal to a (resp. b). Hence $\Gamma_a = \Gamma'_a$ and $\Gamma_b = \Gamma'_b$ are groups by (i), and the rest of (iii) is proved as (ii).

Proposition (6.2.19). — If φ is special, the group W (resp. $\widehat{W}$) is the semi-direct product of the stabilizer of φ by the subgroup $W \cap V$ (resp. $\widehat{W} \cap V$) of the translations of W (resp. $\widehat{W}$). Moreover, $W \cap V$ is the group generated by the translations $ka^\vee$, with $a \in \Phi^{\mathrm{red}}$ and $k \in \widetilde{\Gamma}_a$.

If φ is special, one has $M_{a,0} \neq \emptyset$ for every $a \in \Phi^{\mathrm{red}}$, whence $r_{a,0} \in W$. The image in ${}^vW$ of the stabilizer of φ is therefore the whole of ${}^vW$, whence the first assertion. The second follows from (6.2.14) and from the fact that ${}^vW$ permutes the coroots $a^\vee$.

Proposition (6.2.20). — Assume φ special. One has

$$W \cap V = \Big\{ \sum_a \gamma_a a^\vee \;\Big|\; a \in \Pi,\ \gamma_a \in \widetilde{\Gamma}_a \Big\}$$

$$\widehat{W} \cap V \subset \Big\{ \sum_a \gamma_a \varpi_a \;\Big|\; a \in \Pi,\ \gamma_a \in \widetilde{\Gamma}_a \Big\}$$

where (ϖ_a) denotes the basis of V dual to the basis Π of V^* (fundamental weights of $(\Phi^{\mathrm{red}})^\vee$).

For every root $b \in \Phi^{\mathrm{red}}$ one has $b^\vee = \sum_{a \in \Pi} n_{b,a}\, a^\vee$ with $n_{b,a} \in \mathbf{Z}$. Moreover, it is well known that $n_{b,a}$ is even (resp. a multiple of 3) as soon as $(a \,|\, a) = 2(b \,|\, b)$ (resp. $(a \,|\, a) = 3(b \,|\, b)$). The first assertion then follows from (6.2.14), (6.2.18) and (6.2.19).

Now let $v \in \widehat{W} \cap V$. Since W is normal in $\widehat{W}$, one has $v\, r_{a,0}\, v^{-1} r_{a,0} \in V \cap W$ for every $a \in \Pi$, whence $v - r_a(v) = a(v)a^\vee \in V \cap W$, which proves the second relation.

Definition (6.2.21). — One says that φ is discrete *if Γ_a is a discrete subset of $\mathbf{R}$ for every $a \in \Phi$.*

We have already seen (6.2.16) that if φ is discrete and if $0 \in \Gamma'_a$, then Γ'_a is a group isomorphic to $\mathbf{Z}$.

Proposition (6.2.22). — The set $\Phi' = \{a \in \Phi \mid \Gamma'_a \neq \emptyset\}$ is a root system containing Φ^{nm}. Assume φ discrete. Then W is an affine Weyl group, Σ is the corresponding affine root system, and $\mathscr{E}$ is an échelonnage of Φ' by Σ.

The first assertion is immediate. Assume φ discrete. Replacing it if necessary by an equipollent valuation, we may assume it special. Each $\widetilde{\Gamma}_a$ (for $a \in \Phi'$) is then a group of the form $e_a\mathbf{Z}$ with $e_a > 0$ (6.2.16), and $V \cap W$ is, by (6.2.20), a lattice in V. Since W is generated by orthogonal reflections, it follows that W is an affine Weyl group ([5], chap. VI, §2, no. 5, prop. 8). Moreover, the reflections $r_{a,k}$ for $a \in \Phi'$ and $k \in \Gamma'_a$ form a generating system of W, invariant under inner automorphisms. Consequently they are all the reflections of W ([5], chap. V, §3, no. 2, cor. to th. 1), and Σ is exactly the affine root system associated with W. That $\mathscr{E}$ is an échelonnage of Φ' by Σ in the sense of (1.4.1) is then obvious.

(6.2.23) Assume Φ irreducible and φ discrete, such that $0 \in \Gamma'_a$ for every $a \in \Phi^{\mathrm{nm}}$ (6.2.15). One then sees easily that there exists one and only one real number $e > 0$ such that one and only one of the following conditions is satisfied (the notation is that of 1.4):

1) The échelonnage $\mathscr{E}$ is of type A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4 or G_2; one has $\Phi' = e \cdot {}^v\Sigma$ and $\Gamma_a = \Gamma'_a = \mathbf{Z}e$ for every $a \in \Phi'$.

2) $\mathscr{E}$ is of type $B\text{-}C_n$ $(n \geq 3)$, $C\text{-}B_n$ $(n \geq 2)$ or F_4^{I} (resp. G_2^{I}); Φ' is the root system obtained by multiplying the short roots of ${}^v\Sigma$ by $e\sqrt{2}$ (resp. $e\sqrt{3}$) and the long roots of ${}^v\Sigma$ by e; one has p. 128 $\Gamma_a = \Gamma'_a = \mathbf{Z}e$ if a is a short root of Φ' and $\Gamma_a = \Gamma'_a = 2\mathbf{Z}e$ (resp. $3\mathbf{Z}e$) if a is a long root of Φ'.

3) $\mathscr{E}$ is of type $B\text{-}BC_n$ $(n \geq 3)$, $\Phi' = \Phi$ is of type BC_n, one has $\Phi^{\mathrm{red}} = e \cdot {}^v\Sigma$, $\Gamma'_a = \mathbf{Z}e$ for $a \in \Phi^{\mathrm{red}}$ and $\Gamma_a = \Gamma'_a = 2\mathbf{Z}e$ for $a \in \Phi - \Phi^{\mathrm{red}}$.

4) $\mathscr{E}$ is of type $C\text{-}BC_n^X$ $(n \geq 1)$ with $X = \mathrm{I}, \mathrm{II}, \mathrm{III}$ or IV; $\Phi' = \Phi$ is of type BC_n, $\Phi^{\mathrm{nm}} = e \cdot {}^v\Sigma$, one has $\Gamma'_a = \Gamma_a = \mathbf{Z}e$ for $a \in \Phi^{\mathrm{red}} \cap \Phi^{\mathrm{nm}}$; for $a \in \Phi^{\mathrm{red}} \cap \frac{1}{2}\Phi$ one has

if $X = \mathrm{I}$	$\Gamma'_a = (1/2)\mathbf{Z}e$	$\Gamma_{2a} = 2\mathbf{Z}e$
if $X = \mathrm{II}$	$\Gamma'_a = (1/2)\mathbf{Z}e$	$\Gamma_{2a} = \mathbf{Z}e$
if $X = \mathrm{III}$	$\Gamma'_a = \mathbf{Z}e + \frac{1}{2}e$	$\Gamma_{2a} = 2\mathbf{Z}e$
if $X = \mathrm{IV}$	$\Gamma'_a = \mathbf{Z}e$ or $\mathbf{Z}e + \frac{1}{2}e$	$\Gamma_{2a} = \mathbf{Z}e.$

Remarks (6.2.24). — 1) If $\mathscr{E}$ is of type $C\text{-}BC_n^{\mathrm{IV}}$, one can, by replacing φ by an equipollent valuation, reduce to the case where $\Gamma'_a = \mathbf{Z}e$ for $a \in \Phi^{\mathrm{red}} \cap \frac{1}{2}\Phi$.

2) One sees that the datum of $\mathscr{E}$ determines Φ' and the system of the Γ'_a up to a multiplicative constant (and up to a "translation" on the Γ'_a); likewise, the datum of Φ' and of the Γ'_a determines $\mathscr{E}$.

6.3. Lemmas concerning rank one. —
We keep the hypotheses and notation of 6.2 (see (6.2.4)). We are going to prove results which involve only one root $a \in \Phi$ and its integral multiples (positive or negative), and which therefore in fact express properties of valued root data of rank one. These results will be generalized in 6.4: the lemmas (6.3.2), (6.3.3), (6.3.6), (6.3.9) and (6.3.11) are particular cases of proposition (6.4.9), and lemma (6.3.7) is a particular case of proposition (6.4.28), in view of (6.4.25).

Lemma (6.3.1). — Let $a \in \Phi$, $u \in U_a$ and $u' \in U_{-a}$ be such that $\varphi_a(u) + \varphi_{-a}(u') > 0$. There exists one and only one triple $(u_1, t, u'_1) \in U_a \times T \times U_{-a}$ such that $u'u = u_1 t u'_1$. Moreover one has $t \in H$, $\varphi_a(u_1) = \varphi_a(u)$ and $\varphi_{-a}(u'_1) = \varphi_{-a}(u')$.

One cannot have $u'u \in M_a U_{-a}$, for one would then have $u \in u'^{-1} M_a U_{-a}$ and $\varphi_a(u) + \varphi_{-a}(u') = 0$ by (V 5). The existence of (u_1, t, u'_1) then follows from (6.1.2) (7), and its uniqueness from (DR 6). To prove the rest of the lemma it suffices to treat the case $u \neq 1$, $u' \neq 1$, which we assume from now on. One then has $u_1 \neq 1$ and there exist $u'_2, u'_3 \in U_{-a}$ and $m \in M_a$ such that $u_1 = u'_2 m u'_3$, with $\varphi_{-a}(u'_2) = -\varphi_a(u_1)$. One then has

$$u = (u'^{-1} u'_2) \cdot mt \cdot (t^{-1} u'_3 t u'_1) \in U_{-a}\, mt\, U_{-a}$$

whence $\varphi_a(u) = -\varphi_{-a}(u'^{-1} u'_2)$, since $mt \in M_a$. It follows that p. 129

$$\varphi_{-a}(u'^{-1} u'_2) < \varphi_{-a}(u') = \varphi_{-a}(u'^{-1}).$$

Consequently $\varphi_{-a}(u'^{-1} u'_2) = \varphi_{-a}(u'_2)$ and

$$\varphi_a(u) = -\varphi_{-a}(u'_2) = \varphi_a(u_1).$$

Replacing a by $-a$, u by u'^{-1} and u' by u^{-1}, one obtains the equality $\varphi_{-a}(u'_1) = \varphi_{-a}(u')$.

Finally, m and mt both belong to $M_{a,k}$ with $k = \varphi_a(u) = \varphi_a(u_1)$; one therefore has $\nu(t) = \nu(m^{-1} mt) = r_{a,k}^2 = 1$ and $t \in H$.

Lemma (6.3.2). — Let $a \in \Phi$ and $k, \ell \in \widetilde{\mathbf{R}}$ with $k + \ell > 0$. The product $U_{a,k} H U_{-a,\ell}$ is a group.

This follows from (6.3.1), since H normalizes all the U_α.

Lemma (6.3.3). — Let $a \in \Phi$, $k \in \Gamma_a$ and $m \in M_{a,k}$. Let $U_{-a,-k+}$ be the subgroup which is the union of the $U_{-a,\ell}$ for $\ell > -k$. Then

$$L_{a,k} = (U_{a,k} \cdot H \cdot U_{-a,-k+}) \cup (U_{a,k} \cdot mH \cdot U_{a,k})$$

is the group generated by $H \cup U_{a,k} \cup U_{-a,-k}$.

Since $M_{a,k} \subset U_{a,k}U_{-a,-k}U_{a,k}$, it is clear that $L_{a,k}$ is contained in the group generated by H, $U_{a,k}$ and $U_{-a,-k}$, a group which is also generated by H, $U_{a,k}$ and m, since $mU_{a,k}m^{-1} = U_{-a,-k}$ by (6.2.10). In view of (6.3.1) one has

$$L_{a,k}H = L_{a,k}U_{a,k} = L_{a,k}$$

and it suffices to prove that $L_{a,k}m \subset L_{a,k}$. Since

$$U_{a,k}HU_{-a,-k}m = U_{a,k}mHU_{a,k}$$

it suffices to show that $U_{a,k}mHU_{a,k}m = U_{a,k}HU_{-a,-k}$ is contained in $L_{a,k}$. But, in view of (6.2.2) (3), one has

$$U_{-a,-k} \subset U_{-a,-k+} \cup U_{a,k}mHU_{a,k}$$

whence the lemma.

Remark (6.3.4). — Let $a \in \Phi$ and let T_a be the inverse image under ν of the group of translations with vector proportional to $a^\vee$. Let L'_a be the group generated by U_a and U_{-a}, and let L''_a be the group generated by L'_a and T_a. Put $T'_a = L'_a \cap T_a$. Let $k \in \Gamma_a$ and let $m \in M_{a,k}$. One has:

(47) $$L'_a = U_aT'_a \cup U_amT'_aU_a$$

(48) $$L''_a = U_aT_a \cup U_amT_aU_a.$$

Indeed, one has $mT_a = T_am = T_am^{-1}$, since $\nu(m)$ is the reflection $r_{a,k}$. Consequently

$$U_amT_aU_am = U_aT_aU_{-a} \subset U_aT_a \cup \Big(\bigcup_{r\in\Gamma_a} U_aT_aM_{a,r}U_a\Big).$$

But $T_aM_{a,r} = mT_a$, and this shows that the right-hand side of (2) is stable under right multiplication by U_a, T_a and m, from which one deduces at once that it is equal to L''_a. Finally, (1) follows immediately from (2).

(6.3.5) Let $r, s \in \mathbf{R}$, with $r + s > 0$, and let $u \in \varphi_a^{-1}(r)$ and $u' \in \varphi_{-a}^{-1}(s)$. Put $u = u'_1mu'_2$ and $u' = u_1m'u_2$, with $m \in M_{a,r}$, $m' \in M_{-a,s}$, $u'_i \in U_{-a,-r}$ and $u_i \in U_{a,-s}$. We shall evaluate the commutator $(u', u) = u'uu'^{-1}u^{-1}$. One has p. 130

(49) $$(u', u) \in u_1 \cdot m'(u_2, u)m'^{-1} \cdot m'um'^{-1} \cdot U_a$$

(50) $$(u', u) \in U_{-a} \cdot mu'^{-1}m^{-1} \cdot m(u', u'_2)m^{-1} \cdot u_1'^{-1}.$$

In view of (6.2.10) one has

$$m'um'^{-1} \in U_{-a,r+2s} \qquad \text{and} \qquad mu'^{-1}m^{-1} \in U_{a,s+2r}.$$

Suppose now that u belongs to U_{2a}. One then has $(u_2, u) = 1$ and one deduces from (1) that

$$(u', u) \in U_{a,-s} \cdot U_{-a,r+2s} \cdot U_a$$

whence, in view of (6.3.2), the relation

(3) $$(u', u) \in U_{-a,r+2s} \cdot H \cdot U_a.$$

On the other hand one also has $u'_2 \in U_{-2a}$, whence $(u', u'_2) = 1$, and one deduces likewise from (2) the relation

(4) $$(u', u) \in U_{-a} \cdot H \cdot U_{a,2r+s}.$$

In view of (DR 6), the two relations (3) and (4) imply “by intersection” the relation

$$(5) \qquad (u', u) \in U_{-a,r+2s} \cdot H \cdot U_{a,2r+s}$$

and it is immediate that (5) remains valid for $u \in U_{2a,2r}$ and $u' \in U_{-a,s}$.

Lemma (6.3.6). — Let $k, \ell \in \mathbf{R}$ be such that $k + \ell > 0$. The product

$$U_{-a,\ell} \cdot H \cdot U_{a,k} \cdot U_{2a,k-\ell}$$

is a group.

It is clear that this product is stable under left multiplication by $U_{-a,\ell}$, H and $U_{a,k}$ (in view of (6.3.2)). It is also stable under left multiplication by an element u of $U_{2a,k-\ell}$: indeed, for $u' \in U_{-a,\ell}$ one has, in view of (6.3.5) (5),

$$uu' = u'(u'^{-1}, u)u \in u'\, U_{-a,\, 2\ell+(1/2)(k-\ell)}\, HU_{a,k}u$$

and $2\ell + (1/2)(k - \ell) > \ell$.

Lemma (6.3.7). — Let $k, \ell \in \mathbf{R}$ with $k + \ell > 0$.

(i) If $u \in U_{a,k}$ and $u' \in U_{-a,\ell}$, one has

$$(u', u) \in U_{-2a,k+3\ell} \cdot U_{-a,k+2\ell} \cdot H \cdot U_{a,2k+\ell} \cdot U_{2a,3k+\ell}.$$

(ii) If $u \in U_{2a,2k}$ and $u' \in U_{-a,\ell}$, one has

$$(u', u) \in U_{-2a,2k+4\ell} \cdot U_{-a,2k+3\ell} \cdot H \cdot U_{a,2k+\ell}.$$

Let us prove (i). We may assume $u \neq 1 \neq u'$. Put $r = \varphi_a(u)$ and $s = \varphi_{-a}(u')$ and take up again the notation of (6.3.5). One has $(u_2, u) \in U_{2a,r-s}$ and $m'(u_2, u)m'^{-1} \in U_{-2a,r-s+4s}$, and relation (6.3.5) (1) implies

$$(u', u) \in U_{a,-s} \cdot U_{-2a,r+3s} \cdot U_{-a,r+2s} \cdot U_a$$

whence, in view of lemma (6.3.6), p. 131

$$(u', u) \in U_{-2a,k+3\ell} \cdot U_{-a,k+2\ell} \cdot H \cdot U_a.$$

One proves in the same way, using (6.3.5) (2), that

$$(u', u) \in U_{-a} \cdot H \cdot U_{a,2k+\ell} \cdot U_{2a,3k+\ell}$$

whence (i) (in view of (DR 6)).

If moreover $u \in U_{2a}$, one deduces from (6.3.5) (1), taking into account the relations $(u_2, u) = 1$ and $m'um'^{-1} \in U_{-2a,2r+4s}$, that $(u', u) \in U_{a,-s}U_{-2a,2r+4s}U_a$, whence, in view of (6.3.6),

$$(u', u) \in U_{-2a,2k+4\ell} \cdot U_{-a,2k+3\ell} \cdot H \cdot U_a.$$

On the other hand, (6.3.5) (5) implies

$$(u', u) \in U_{-a,k+2\ell} \cdot H \cdot U_{a,2k+\ell}$$

whence (ii).

(6.3.8) Let $a \in \Phi$, $u \in U_a$, $v \in U_{-a}$. By abuse of language, one says that the commutator (v, u) *admits an H-component* if there exists $h \in H$ such that $(v, u) \in U_{-a}hU_a$; this element $h \in H$ (which is then unique by (DR 6)) is called *the H-component of the commutator* (v, u).

Lemma (6.3.9). — Let $k, k', \ell, \ell' \in \mathbf{R}$ be such that $k' \leq 2k$, $\ell' \leq 2\ell$, $k \leq k' + \ell$, $\ell \leq \ell' + k$, $k' + \ell' > 0$. For $u \in U_{a,k} \cup U_{2a,k'}$ and $u' \in U_{-a,\ell} \cup U_{-2a,\ell'}$, the commutator (u', u) admits an H-component. If X denotes the group generated by the H-components of these commutators, then the group generated by $U_{-2a,\ell'} \cup U_{-a,\ell} \cup U_{a,k} \cup U_{2a,k'}$ is the product

$$U_{-2a,\ell'} \cdot U_{-a,\ell} \cdot X \cdot U_{a,k} \cdot U_{2a,k'}.$$

The first assertion follows immediately from the hypotheses made on k, k', ℓ and ℓ' and from lemma (6.3.2). To prove the second, it suffices to show that the product under consideration is stable under left multiplication by each of its factors, which follows easily from (6.3.7) and from the relation $uv = v(v^{-1}, u)u$.

Remark (6.3.10). — Lemma (6.3.9) implies in particular that the products considered in (6.3.7) (i) and (ii) are groups, and therefore contain respectively the commutator groups $(U_{a,k}, U_{-a,\ell})$ and $(U_{2a,2k}, U_{-a,\ell})$.

Lemma (6.3.11). — Let $a \in \Phi$ be such that $2a \in \Phi$ and let $k, k' \in \mathbf{R}$ with $k' \in \Gamma_{2a}$ and $k' < 2k$. Let $m \in M_{2a,k'}$ and put

$$L_{a,k,k'} = (U_{2a,k'}U_{a,k}HU_{-a,k-k'}U_{-2a,-k'+}) \cup (U_{2a,k'}U_{a,k}mHU_{a,k}U_{2a,k'}).$$

Then $L_{a,k,k'}$ is the subgroup generated by $U_{2a,k'} \cup U_{a,k} \cup H \cup U_{-2a,-k'}$.

One may assume $k' = 0$. Taking into account the relation $mU_{a,k}m^{-1} = U_{-a,k}$ and (6.3.9), one sees, arguing as in (6.3.3), that it suffices to show that $L_{a,k,0}m \subset L_{a,k,0}$, and even that $mHU_{a,k}U_{2a,0}m = HU_{-a,k}U_{-2a,0}$ is contained in $L_{a,k,0}$. Since one obviously has $HU_{-a,k}U_{-2a,0+} \subset L_{a,k,0}$, it suffices to show that $HU_{-a,k}u \subset L_{a,k,0}$ for $u \in \varphi_{2a}^{-1}(0)$. One then has $u \in U_{2a,0}HmU_{2a,0}$. By (6.3.9) (where one takes $k' = 0$, $\ell = k$ and $\ell' = 2k$) one has $HU_{-a,k}U_{2a,0} \subset U_{2a,0}U_{a,k}HU_{-a,k}$, whence p. 132

$$HU_{-a,k}u \subset U_{2a,0}U_{a,k}HU_{-a,k}mU_{2a,0} = U_{2a,0}U_{a,k}HmU_{a,k}U_{2a,0} \subset L_{a,k,0}$$

which completes the proof.

6.4. The groups U_f. Extended valuations.

We keep the notation of 6.2 and 6.3.

(6.4.1)

6.4. The groups U_f. Extended valuations. — Let $\widetilde{\mathbf{R}}$ be the union of $\mathbf{R} \times \{0, 1\}$ and of an element denoted ∞ not belonging to $\mathbf{R} \times \{0, 1\}$. We endow $\widetilde{\mathbf{R}}$ with the following law of composition: for $r, s \in \mathbf{R}$ one puts

$$(r, 0) + (s, 0) = (r + s, 0)$$
$$(r, 0) + (s, 1) = (s, 1) + (r, 0) = (r + s, 1)$$
$$(r, 1) + (s, 1) = (r + s, 1)$$
$$(r, 0) + \infty = (r, 1) + \infty = \infty + (r, 0) = \infty + (r, 1) = \infty.$$

We also endow $\widetilde{\mathbf{R}}$ with the following order relation: for $r, s \in \mathbf{R}$, the relations $(r, 0) \le (s, 0)$, $(r, 0) \le (s, 1)$ and $(r, 1) \le (s, 1)$ are equivalent to $r \le s$, the relation $(r, 1) \le (s, 0)$ is equivalent to $r < s$, and one has $u \le \infty$ for every $u \in \widetilde{\mathbf{R}}$. One verifies immediately that this makes $\widetilde{\mathbf{R}}$ a *totally ordered commutative monoid* and that the map $r \mapsto (r, 0)$ is an isomorphism of ordered monoids from $\mathbf{R}$ onto a submonoid of $\widetilde{\mathbf{R}}$. From now on we shall identify an element $r \in \mathbf{R}$ with its image $(r, 0)$, and we shall denote by $r+$ the element $(r, 1)$ of $\widetilde{\mathbf{R}}$. Note that one has $r+ = \mathrm{Inf}\{s \in \mathbf{R} \mid s > r\}$. For $t \in \mathbf{R}_+^*$ and $r \in \mathbf{R}$ one puts $t(r+) = (tr)+$. For $r \in \widetilde{\mathbf{R}}$ one puts $0r = 0$.

On the other hand, one sees easily that every bounded-below subset of $\widetilde{\mathbf{R}}$ admits an infimum and that every non-empty subset of $\widetilde{\mathbf{R}}$ admits a supremum.

We then extend the definition of $U_{a,k}$ to $k \in \widetilde{\mathbf{R}}$ (and $a \in \Phi$) by putting, as in (6.3.3),

$$U_{a,r+} = \bigcup_{s \in \mathbf{R},\ s > r} U_{a,s}$$

for $r \in \mathbf{R}$.

(6.4.2) This paragraph is devoted to the study of certain subgroups of G generalizing the subgroups denoted P_Ω and U_Ω in §5.2: in (5.2.30), U_Ω was defined, for $\Omega \subset A$, as the group generated by the U_α for $\alpha \supset \Omega$. In other words, in the case of example (6.2.3) e), U_Ω is generated by the $U_{a,f_\Omega(a)}$, where the function $f_\Omega : \Phi \to \widetilde{\mathbf{R}}$ is defined by

(51) $$f_\Omega(a) = \inf\{k \in \mathbf{R} \mid \Omega \subset \alpha_{a,k}\} \qquad (a \in \Phi).$$

As for P_Ω, it is equal to $H \cdot U_\Omega$.

The generalization is then immediate: let f be a map from Φ into the monoid $\widetilde{\mathbf{R}}$. *Throughout the rest of this work we shall denote by U_f the subgroup of G generated by the union of the subgroups $U_{a,f(a)}$ for $a \in \Phi$.* Moreover we shall put: p. 133

$$U_f^+ = U^+ \cap U_f, \quad U_f^- = U^- \cap U_f, \quad H_f = H \cap U_f, \quad N_f = N \cap U_f,$$
$$U_{f,a} = U_a \cap U_f \qquad (\text{for } a \in \Phi).$$

When $a \in \Phi$ and $2a \notin \Phi$, it will sometimes be convenient to assign a value to $f(2a)$: unless expressly stated otherwise, we shall take $f(2a) = 2f(a)$. Recall that we have already made the convention $U_{2a,k} = \{1\}$ in this case for $k \in \mathbf{R}$; we extend it of course to $k \in \widetilde{\mathbf{R}}$.

We denote by $U_f^{(a)}$ the subgroup generated by the union of the $U_{ia,f(ia)}$ for $i \in \{\pm 1, \pm 2\}$, and we put, for $a \in \Phi$:

$$N_f^{(a)} = N \cap U_f^{(a)} \qquad \text{and} \qquad H_f^{(a)} = H \cap U_f^{(a)}.$$

It is clear that the subgroups U_f, H_f and N_f are normalized by H. Consequently, *for every subgroup X of H, the product $X \cdot U_f$ is a group*; in particular, the subgroups $P_f = H \cdot U_f$ are the natural generalization of the subgroups P_Ω of §5.

(6.4.3) If Ω is a non-empty subset of A, the function f_Ω defined by (6.4.2) (1) is "concave" in the following sense:

Definition. — A function $f : \Phi \to \widetilde{\mathbf{R}}$ (resp. $f : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$) is said to be concave *if, for every finite family (a_i) of elements of Φ (resp. $\Phi \cup \{0\}$) such that $\sum_i a_i \in \Phi$ (resp. $\sum_i a_i \in \Phi \cup \{0\}$), one has*

(C) $$f\Big(\sum_i a_i\Big) \le \sum_i f(a_i).$$

(6.4.4) Note that every linear function, that is to say every function of the form $a \mapsto \lambda(a)$ with $\lambda \in V$, is obviously concave: it coincides moreover with the function f_Ω where $\Omega \subset A$ is the part reduced to the point of A corresponding to λ in the identification of V with A obtained by the choice of the origin φ in A. However, not every concave function, even one with values in $\mathbf{R}$, is necessarily of the form f_Ω for a subset $\Omega \subset A$. Besides, the results which we are going to establish about the groups U_f will apply to functions more general than the concave functions (cf. (6.4.8)).

Proposition (6.4.5). — In order that a function $f : \Phi \to \widetilde{\mathbf{R}}$ be concave, it is necessary and sufficient that the two following conditions be satisfied:

(C 1) $\quad f(a) + f(b) \ge f(a+b) \quad$ *for $a, b, a+b \in \Phi$;*

(C 2) $\quad f(a) + f(-a) \ge 0 \quad$ *for $a \in \Phi$.*

That (C 1) and (C 2) are necessary is obvious. Assume them satisfied and let $a_1, \ldots, a_n \in \Phi$ be such that $a = \sum_i a_i \in \Phi$. Let us show by induction on n that $f(a) \le \sum_i f(a_i)$: this follows from (C 1) for $n \le 2$. If $n > 2$, there exists an index $j \in [1, n]$ such that the scalar product

$(a \mid a_j)$ is positive, so that $a - a_j$ is either zero or a root a'. In the first case, choose an index $k \in [1, n]$ with $k \neq j$; one then has, in view of the induction hypothesis and of (C 2), p. 134

$$\sum_i f(a_i) = \Big(\sum_{i \neq j,k} f(a_i)\Big) + f(a_k) + f(a_j) \geq f(-a_k) + f(a_k) + f(a_j) \geq f(a_j) = f(a).$$

In the second case, applying the induction hypothesis to the family $(a_i)_{i \neq j}$, one obtains

$$\sum_i f(a_i) \geq f(a') + f(a_j) \geq f(a)$$

which completes the proof.

Proposition (6.4.6). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a concave function and let $g : \Phi \to \mathbf{R} \cup \{+\infty\}$ be a function majorising f. Put

$$f_0 = \inf\Big(\sum_i t_i f(a_i)\Big)$$

where the infimum is extended over the set of pairs consisting of a finite family (a_i) of elements of Φ and a finite family (t_i) of positive real numbers such that $\sum_i t_i a_i = 0$ and $\sum_i t_i = 1$. Then one has $f_0 \geq 0$, and for every real number $c \leq f_0$ there exists a linear form λ on V^* such that the restriction of $\lambda + c$ to Φ is majorised by f. Moreover, if $f(a) + f(-a) > 0$ for every $a \in \Phi$, one has $f_0 > 0$ and there exist $\mu \in V$ and $k \in \mathbf{R}$, $k > 0$, such that the restriction of $\mu + k$ to Φ is majorised by g.

Let (a_i) be a finite non-empty family of elements of Φ. Since Φ generates a vector subspace over the field of rationals of dimension equal to the dimension of V^*, the families (t_i) of rational numbers such that $\sum_i t_i a_i = 0$ are dense in the set of families (t_i) of real numbers such that $\sum_i t_i a_i = 0$. To show that $f_0 \geq 0$ it therefore suffices to show that a relation $\sum_i t_i a_i = 0$ implies $\sum_i t_i f(a_i) \geq 0$ as soon as the t_i are positive rational numbers, or even only positive integers. In other words, we are reduced to showing that if (a_i) is a finite non-empty family of elements of Φ such that $\sum_i a_i = 0$, then $\sum_i f(a_i) \geq 0$. Choose an index i_0; one then has

$$\sum_i f(a_i) = \Big(\sum_{i \neq i_0} f(a_i)\Big) + f(a_{i_0}) \geq f(-a_{i_0}) + f(a_{i_0}) \geq 0 \tag{52}$$

by (C) and (C 2).

To prove the existence of λ, it suffices to show that the intersection of the half-spaces $L_a = \{x \in V \mid a(x) + c \leq f(a)\}$, for $a \in \Phi$ and $f(a) < \infty$, is non-empty. Now it is known ([4], chap. II, §2, exerc. 21) that, in order for the intersection of a finite family $\mathscr{F}$ of convex subsets of a real vector space of dimension n to be non-empty, it suffices that the intersection of every subfamily of $n + 1$ elements of $\mathscr{F}$ be non-empty. Moreover, if the elements of $\mathscr{F}$ are half-spaces, repeated application of this result shows that it suffices that the intersection of every finite subfamily consisting of $r + 1$ half-spaces whose corresponding linear forms constitute a system of rank r (with $1 \leq r \leq n$) be non-empty. In other words, it suffices for us to verify that if $a_1, \ldots, a_r$ are linearly independent roots and if $a_{r+1} = \sum_{1 \leq i \leq r} t_i a_i \in \Phi$ (with $t_i \in \mathbf{R}$), p. 135 then $\bigcap_{1 \leq i \leq r+1} L_{a_i}$ is non-empty. Now, if this intersection were empty, that would mean that the relations $a_i(x) \leq f(a_i) - c$ for $1 \leq i \leq r$ imply $\sum_{1 \leq i \leq r} t_i a_i(x) > f(a_{r+1}) - c$. This is possible only if $t_i \leq 0$ for $1 \leq i \leq r$. For $a \in \Phi$ with $f(a) < \infty$, let $f^-(a) \in \mathbf{R}$ be such that $f(a) = f^-(a)$ or $f^-(a)+$. Choose $x \in A$ such that $a_i(x) + c = f^-(a_i)$ for $1 \leq i \leq r$, which is legitimate since the a_i are independent. By definition of f_0 one has:

$$f^-(a_{r+1}) - \sum_i t_i f^-(a_i) \geq \Big(1 - \sum_i t_i\Big)c$$

whence $a_{r+1}(x) + c = \sum_i t_i f^-(a_i) + (1 - \sum_i t_i)c \leq f^-(a_{r+1}) \leq f(a_{r+1})$, which completes the proof of the existence of λ.

Now suppose that $f(a) + f(-a) > 0$ for every $a \in \Phi$. In view of (1) one has $f_0 \geq 0+$ and there exists $\lambda \in V$ such that $f \geq \lambda$. Since f is concave, the set Ψ of the $a \in \Phi$ such that $f(a) = \lambda(a)$ is a closed subset of Φ such that $\Psi \cap (-\Psi) = \emptyset$. There therefore exists $\nu \in V$ such that $\nu(a) > 0$ for $a \in \Psi$ ([5], chap. VI, §1, no. 7, prop. 22). Put $h = \inf\{g(a) - \lambda(a) \mid a \notin \Psi\}$, $d = \sup\{\nu(a) \mid a \in \Phi\}$ and $\delta = \inf\{\nu(a) \mid a \in \Psi\} > 0$. Then g majorises the restriction to Φ of the function $\alpha h + \lambda - \beta\nu$ as soon as the real numbers $\alpha, \beta > 0$ satisfy the inequalities $\alpha h/\delta \leq \beta \leq (1-\alpha)/d$.

Proposition (6.4.7). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a concave function. Then f possesses the two following properties:

(QC 1) For every root $a \in \Phi$ one has

$$U_f^{(a)} = U_{-2a,f(-2a)} \cdot U_{-a,f(-a)} \cdot U_{a,f(a)} \cdot U_{2a,f(2a)} \cdot N_f^{(a)}.$$

(QC 2) If $a, b \in \Phi$ are not proportional, the commutator group $(U_{a,f(a)}, U_{b,f(b)})$ is contained in the group generated by the $U_{pa+qb,\, f(pa+qb)}$ for $p, q \in \mathbf{N}^*$ and $pa + qb \in \Phi$.

(QC 2) follows from axiom (V 3) of valuations (6.2.1), taking into account condition (C), which implies $f(pa+qb) \leq pf(a) + qf(b)$ for $a, b, pa+qb \in \Phi$ with $p, q \in \mathbf{N}^*$.

Let us prove (QC 1). Since f is concave, we have, in view of (C 1), the relations $f(-2a) \leq 2f(-a)$, $f(2a) \leq 2f(a)$, $f(a) \leq f(2a) + f(-a)$ and $f(-a) \leq f(-2a) + f(a)$; from these relations and from (C 2) one deduces

$$0 \leq f(2a) + f(-2a) \leq 2\big(f(a) + f(-a)\big).$$

If $f(2a) + f(-2a) > 0$, property (QC 1) follows immediately from (6.3.9) (which shows moreover that $N_f^{(a)} = H_f^{(a)}$). If $f(a) + f(-a) = 0$, it follows from the preceding relations that $f(2a) = 2f(a)$ and $f(-2a) = 2f(-a)$, and $U_f^{(a)}$ is generated by $U_{a,f(a)} \cup U_{-a,f(-a)}$; property (QC 1) then follows at once from (6.3.3) if $f(a) \in \Gamma_a$, and from (6.3.2) in the contrary case.

Finally suppose that $f(a) + f(-a) > 0$ and that $f(2a) + f(-2a) = 0$. To simplify the notation we shall assume that $f(2a) = f(-2a) = 0$, which is legitimate up to replacing φ by an equipollent valuation. One then has $k = f(a) = f(-a) > 0$. If $0 \in \Gamma_{2a}$, (QC 1) follows from (6.3.11); otherwise, the group $U_f^{(a)}$ is the union of the groups generated by the unions p. 136
$U_{-2a,r} \cup U_{-a,k} \cup U_{a,k} \cup U_{2a,0}$ for $0 < 2r \leq k$, and it suffices once again to apply (6.3.9).

Definition (6.4.8). — A function $f : \Phi \to \widetilde{\mathbf{R}}$ is said to be quasi-concave *if it satisfies conditions (QC 1) and (QC 2) of (6.4.7).*

We have just seen that every concave function is quasi-concave. In certain cases the converse is true. Let us take up again, for example, the notation of example (6.2.3) b): G is thus the group of rational points of a connected semi-simple algebraic group $\mathscr{G}$ defined and split over a valued field K. Suppose moreover that $\mathscr{G}$ is simple. The Chevalley commutation relations then imply that every quasi-concave function is concave, except when $\mathscr{G}$ is of type B_n, C_n or F_4 and the characteristic of the residue field of K is equal to 2, or when $\mathscr{G}$ is of type G_2 and the characteristic of the residue field of K is equal to 2 or to 3. In these exceptional cases it is easy to construct quasi-concave functions which are not concave.

Proposition (6.4.9). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a quasi-concave function.

(i) $U_{f,a} = U_{a,f(a)} \cdot U_{2a,f(2a)}$ for every $a \in \Phi$.

(ii) The product map $\prod_{a \in \Phi^{\pm\,\mathrm{red}}} U_{f,a} \to U_f^{\pm}$ is bijective, whatever the order in which the factors of the product are arranged.

(iii) One has $U_f = U_f^- \cdot U_f^+ \cdot N_f$.

(iv) N_f is the group generated by the $N_f^{(a)}$ for $a \in \Phi$.

For $a \in \Phi$, put $X_a = U_{a,f(a)} \cdot U_{2a,f(2a)}$. In view of (QC 2) and (6.1.6), the product of the X_a for $a \in \Phi^{\pm\,\mathrm{red}}$, arranged in an arbitrary order, is a group, which we shall denote $X^{\pm}$. On the other hand let Y be the group generated by the $N_f^{(a)}$ for $a \in \Phi$.

We shall first show that the product $X^- X^+ Y$ is independent of the choice of the Weyl chamber D used to define Φ^+ and Φ^-. For this, it suffices to show that this product does not change if one replaces D by the chamber $r_a(D)$, where r_a is the reflection associated with a root a simple for D. Denote then by X'_a (resp. X'_{-a}) the product of the X_b for $b \in \Phi^{+\,\mathrm{red}}$, $b \neq a$ (resp. $b \in \Phi^{-\,\mathrm{red}}$, $b \neq -a$). In view of (QC 2) and (6.1.6), $X'_{\pm a}$ is a group and is normalized by X_a and X_{-a}, hence by the group they generate, that is to say by $U_f^{(a)} = X_{-a} X_a N_f^{(a)}$ ((QC 1)). Consequently one has

$$\begin{aligned} X^- X^+ Y &= X'_{-a} X_{-a} X_a X'_a Y = X'_{-a} X'_a X_{-a} X_a Y = X'_{-a} X'_a X_a X_{-a} N_f^{(a)} Y \\ &= X'_{-a} X_a X'_a X_{-a} Y \end{aligned}$$

which proves our assertion.

It follows at once that $X^- X^+ Y$ is stable under left multiplication by all the X_a for $a \in \Phi^{\mathrm{red}}$, and consequently that

$$U_f = X^- X^+ Y.$$

Now let $x^+ \in X^+$, $x^- \in X^-$ and $y \in Y$. If $x^- x^+ y \in U^-$, one has $x^+ y \in U^-$. Since $Y \subset N$, one deduces that $y = 1$ ((6.1.15) c)), whence $x^+ \in U^+ \cap U^- = \{1\}$ (DR 6). One concludes that $U_f^- = X^-$, and one shows in the same way that $U_f^+ = X^+$: this proves (ii), and (i) follows from it by intersection with U_a. Finally, if $x^- x^+ y \in N$, one has, again by (6.1.15) c), $y = x^- x^+ y$, which proves (iv); and (iii) then follows from the relation $U_f = X^- X^+ Y$. p. 137

Proposition (6.4.10). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a quasi-concave function. For $a \in \Phi$ put

$$f'(a) = \inf\{k \in \Gamma'_a \mid k \geq f(a) \ \text{ or } \ a/2 \in \Phi \ \text{ and } \ k \geq 2f(a/2)\}.$$

Let Φ_f be the set of the $a \in \Phi$ such that $f'(a) + f'(-a) = 0$. Then Φ_f is a root system in the space V_f^* which it generates, and the map which to $n \in N_f$ associates the restriction of ${}^v\nu(n)$ to V_f^* is a surjective homomorphism from N_f onto the Weyl group of Φ_f, with kernel H_f.

In view of (6.4.9) (i) one has, for $a \in \Phi$,

$$f'(a) = \inf\{k \in \Gamma'_a \mid U_{a,k} \subset U_f\}.$$

Consequently the function f' is in a certain sense "covariant" under N_f. More precisely, let $n \in N_f$, $a \in \Phi$ and $b = {}^v\nu(n)(a)$. Then

$$f'(b) = f'(a) + a\big(\nu(n)^{-1}(\varphi) - \varphi\big). \tag{53}$$

This shows in particular that Φ_f is stable under ${}^v\nu(N_f)$. It therefore suffices for us to prove that ${}^v\nu(N_f)$ is generated by the reflections r_a for $a \in \Phi_f$. In view of (6.4.9) (iv), this follows from the following lemma:

Lemma (6.4.11). — Let f and f' be as in (6.4.10) and let $a \in \Phi$.

(i) One has $f'(a) + f'(-a) \geq 0$ and $2f'(a) + f'(-2a) \geq 0$.

(ii) The following conditions are equivalent:

a) $N_f^{(a)} \not\subset H$;

b) there exists $k \in \mathbf{R}$ such that $\nu(N_f^{(a)}) = \{1, r_{a,k}\} = \{1, r_{2a,2k}\}$;

c) ${}^v\nu(N_f^{(a)}) = \{1, r_a\} = \{1, r_{2a}\}$;

d) a or $2a$ belongs to Φ_f.

Let us first prove the equivalence of a), b) and c). It is clear that b) or c) implies a). On the other hand, it is known (6.3.4) that $\nu(N_f^{(a)})$ consists of reflections $r_{a,k}$ and of translations $ka^\vee$ with $k \in \mathbf{R}$. If there existed $k > 0$ and $n \in N_f^{(a)}$ such that $\nu(n) = ka^\vee$, one would have, for every integer $p > 0$:

$$n^p U_{a,f(a)} n^{-p} = U_{a,f(a)-2pk} \subset U_f$$

which is possible only if $f(a) = \infty$; and one sees likewise that one would have $f(2a) = \infty$. But this is absurd, for it implies $U_{f,a} \subset U_{-a}$ and $N_f^{(a)} = \{1\}$. Consequently $\nu(N_f^{(a)})$ contains only reflections (besides 1), and cannot contain two distinct reflections whose product would be a non-zero translation. Whence the equivalence of a), b) and c).

Now suppose $f'(a) + f'(-a) < 0$. There then exist $k \in \Gamma'_a$ and $h \in \Gamma'_{-a}$ such that

$$f'(a) \leq k, \qquad f'(-a) \leq h \qquad \text{and} \qquad k + h < 0.$$

The group U_f then contains $U_{a,k} \cup U_{-a,-k}$ and $U_{a,-h} \cup U_{-a,h}$, hence $M_{a,k}$ and $M_{a,-h}$, and $\nu(N_f^{(a)})$ p. 138
contains the two distinct reflections $r_{a,k}$ and $r_{a,-h}$. But we have just seen that this is impossible, whence the first assertion of (i). Likewise, if $2f'(a) + f'(-2a) < 0$, there exist $k \in \Gamma'_a$ and $h \in \Gamma'_{-2a}$ such that $f'(a) \leq k$, $f'(-2a) \leq h$ and $2k + h < 0$, and one sees as above that $\nu(N_f^{(a)})$ contains the two distinct reflections $r_{2a,2k}$ and $r_{2a,-h}$, which completes the proof of (i). Note moreover that an analogous argument shows that if $2f'(a) + f'(-2a) = 0$, then a and $2a$ belong to Φ_f.

If $a \in \Phi_f$, one has $k = f'(a) = -f'(-a) \in \mathbf{R}$ by the formulae defining addition in $\widetilde{\mathbf{R}}$ (6.4.1). Consequently $k \in \Gamma'_a$ and $M_{a,k} \subset N_f^{(a)}$. One sees likewise that if $2a \in \Phi_f$, there exists $k \in \Gamma'_{2a}$ such that $M_{2a,k} \subset N_f^{(a)}$. This shows that d) implies a).

Finally, suppose a), b) and c) satisfied. Up to replacing φ by an equipollent valuation, we may assume that $r_{a,0} \in \nu(N_f^{(a)})$. By (6.4.10) (1) one then has

$$f'(a) = f'(-a) \qquad \text{and} \qquad f'(2a) = f'(-2a).$$

In view of (i) it follows that $f'(a) \geq 0$ and $f'(2a) \geq 0$. Finally, if $f'(a)$ and $f'(2a)$ were both strictly positive, $U_{f,a}$ would be contained in the union of the groups generated by $U_{a,r} \cup U_{-a,-r}$ for $r \in \mathbf{R}$, $r > 0$, and one would have $N_f^{(a)} \subset H$ by (6.3.2), which contradicts c). Consequently d) is indeed satisfied.

The proof of the lemma and that of proposition (6.4.10) are complete.

Remarks (6.4.12). — a) Let us keep the notation of (6.4.10). For $a \in \Phi$ put

$$f''(a) = \inf\{k \in \Gamma_a \mid k \geq f(a) \ \textit{or} \ a/2 \in \Phi \ \textit{and} \ k \geq 2f(a/2)\}.$$

One has $f''(a) \leq f'(a)$, and $f''(a) = f'(a)$ if $\Gamma'_a = \Gamma_a$, for example if $2a \notin \Phi$. One shows exactly as above that (6.4.11) (i) is also valid for f''; in other words one has, for $a \in \Phi$:

$$f''(a) + f''(-a) \geq 0 \qquad \textit{and} \qquad 2f''(a) + f''(-2a) \geq 0. \tag{54}$$

On the other hand it is clear that $U_f = U_{f''}$. On the contrary, one does not always have $U_f = U_{f'}$. More precisely, for $a \in \Phi$ one has

$$U_{a,f'(a)} \cdot U_{2a,f'(2a)} \subset U_{f,a} \subset U_{a,\inf\{f'(a),\frac{1}{2}f'(2a)\}} \tag{55}$$

the first inclusion being an equality as soon as $\Gamma'_a \neq \emptyset$.

b) Suppose φ discrete and take up again the notation of (6.2.22) sqq. Let Ω be a subset of the affine space A and consider the corresponding concave function f_Ω and group $U_\Omega = U_{f_\Omega}$ (6.4.2). It is immediate that, for every $a \in \Phi$, $f'_\Omega(a)$ is the smallest real number k such that the half-space $\alpha_{a,k} = \{x \in A \mid a(x - \varphi) + k \geq 0\}$ is an affine root of the system Σ associated with a *by the échelonnage $\mathscr{E}$ and containing* Ω*. Consequently,* the root system $\Phi_\Omega = \Phi_{f_\Omega}$ consists of the roots $a \in \Phi$ for which there exists $k \in \mathbf{R}$ such that $(\alpha_{a,k}, a) \in \mathscr{E}$ and $\Omega \subset \partial\alpha_{a,k}$.

Suppose in particular that Ω *is a facet* F *of* A*. Then* Φ_F *is exactly* the root system attached p. 139
to F by $\mathscr{E}$ in the sense of (1.4.2) and, by (1.4.5) c), the Dynkin diagram of Φ_F is obtained by deleting from the Dynkin diagram of the échelonnage $\mathscr{E}$ the vertices not belonging to the type X of F.

(6.4.13) In what follows we shall define certain normal subgroups of H and study the intersection H_f of H with a subgroup U_f. Let $k \in \widetilde{\mathbf{R}}$; we shall denote by $H_{(k)}$ the set of the $h \in H$ such that

$$(h, U_{a,r}) = \{(h,u) \mid u \in U_{a,r}\} \subset U_{a,r+k} \cdot U_{2a,2r+k}$$

for every $a \in \Phi$ and every $r \in \mathbf{R}$ (or $r \in \widetilde{\mathbf{R}}$). One verifies immediately that the $H_{(k)}$ form a decreasing family of normal subgroups of H, and even of N. One has $H_{(k)} = H$ for $k \leq 0$, and $H_{(\infty)}$ is, in view of (6.1.13), the centralizer of the subgroup of G generated by the U_a for $a \in \Phi$.

(6.4.14) Let again $k \in \widetilde{\mathbf{R}}$. We shall define another subgroup of H, denoted $H_{[k]}$. For $k \leq 0$, $H_{[k]}$ is the subgroup generated by the union of the intersections with H of the subgroups generated by $U_{a,r} \cup U_{-a,-r}$ for $a \in \Phi$ and $r \in \mathbf{R}$. For $0 < k < \infty$, $H_{[k]}$ is the subgroup generated by the H-components (6.3.8) of the commutators (u, u') and (u', u) with $u \in U_{a,r}$ and $u' \in U_{-a,s}$, $r + s = k$, or else $u \in U_{a,r}$ and $u' \in U_{-2a,s}$, $2r + s = k$ (with a running through Φ). Here again, the $H_{[k]}$ form a decreasing family of normal subgroups of H, and even of N. Finally, one puts $H_{[\infty]} = \bigcap_k H_{[k]}$.

(6.4.15) It is not impossible that the condition

$$\text{(Pr)} \qquad H_{[k]} \subset H_{(k)} \qquad \text{for every } k \in \widetilde{\mathbf{R}}$$

is always satisfied. We shall see later that it is so in very general cases, for example as soon as all the irreducible components of Φ are of rank > 1, and also in the case of semi-simple algebraic groups over a local field.

Note that the $H_{[k]}$ are contained in the group G' generated by the U_a for $a \in \Phi$. Consequently, if $H_{[\infty]} \subset H_{(\infty)}$, then $H_{[\infty]}$ is contained in the center of G'.

Examples (6.4.16). — a) Let us take up again the notation of (6.2.3) a) ($G = \mathrm{SL}_2(K)$), and let $k > 0$. One sees easily that $H_{(k)}$ consists of the matrices $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathrm{SL}_2(K)$ with $\omega(xuy^{-1} - u) \geq \omega(u) + k$ for every $u \in K$, and that $H_{[k]}$ is generated by the matrices of the form $\begin{pmatrix} x & 0 \\ 0 & ux^{-1}u^{-1} \end{pmatrix}$ with $x, u \in K^*$ and $\omega(x-1) \geq k$. One deduces at once that condition (Pr) is satisfied. If K is commutative, one has

$$H_{(k)} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in K^* \text{ and } \omega(x^2 - 1) \geq k \right\}$$

$$H_{[k]} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in K^* \text{ and } \omega(x - 1) \geq k \right\}.$$

b) Let us now take up again the notation of (6.2.3) b) (G is thus the group of rational points p. 140
of a connected semi-simple algebraic group $\mathscr{G}$ defined and split over the commutative valued field K). Let $k > 0$. One verifies immediately that

$$H_{(k)} = \{t \in T \mid \omega(a(t) - 1) \geq k \text{ for every } a \in \Phi\}.$$

On the other hand, $H_{[k]}$ is generated by the images under the various homomorphisms ζ_a (for $a \in \Phi^+$) of the analogous subgroups of $\mathrm{SL}_2(K)$. *Now, if $a, b \in \Phi$ and $x \in K^*$, one has*

$$b\left(\zeta_a\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}\right)\right) = x^{n_{a,b}}, \qquad \text{with} \quad n_{a,b} = 2\frac{(b\,|\,a)}{(a\,|\,a)},$$

which shows that $H_{[k]} \subset H_{(k)}$.

Suppose moreover that $\mathscr{G}$ is simply connected and let P be the group of weights of $\mathscr{G}$. It is known that the map which to $t \in T$ associates the map $p \mapsto p(t)$ from P into K^* is an isomorphism of T onto $\mathrm{Hom}(P, K^*)$. *For $a \in \Phi$, $p \in P$ and $x \in K^*$ one has*

$$p\left(\zeta_a\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}\right)\right) = x^n \quad \text{with } n \in \mathbf{Z},$$

from which one deduces that $\omega(p(t) - 1) \geq k$ for every $t \in H_{[k]}$. Conversely, let $(p_a)_{a \in \Pi}$ be the family of fundamental weights associated with the basis Π of Φ. One has

$$t = \prod_{a \in \Pi} \zeta_a\left(\begin{pmatrix} p_a(t) & 0 \\ 0 & p_a(t)^{-1}\end{pmatrix}\right) \qquad (t \in T)$$

from which one deduces that

$$H_{[k]} = \{t \in T \mid \omega(p(t) - 1) \geq k \ \text{ for every weight } p \in P\}.$$

If $\mathscr{G}$ is not simply connected, $H_{[k]}$ is the canonical image of the corresponding subgroup of the simply connected covering of $\mathscr{G}$.

Proposition (6.4.17). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a quasi-concave function. Put

$$d(f) = \inf_{a \in \Phi}\big(\inf(f(a) + f(-a),\ f(-2a) + 2f(a))\big).$$

The group $H_f = H \cap U_f$ is contained in $H_{[d(f)]}$. Moreover, H_f is generated by the $H_f^{(a)} = H \cap N_f^{(a)} = H \cap U_f^{(a)}$ for $a \in \Phi$.

Let us prove the last assertion. Let H' be the subgroup generated by the $H_f^{(a)}$. It is obviously a normal subgroup of N_f. Let Δ be a basis of Φ_f, and let N_1 be the subgroup of N_f generated by the $N_f^{(a)} \cap M_a^0$ for $a \in \Delta$. In view of (6.4.10) and (6.4.9) (iv) one has $N_f = N_1 H'$, and (6.1.10) implies $H' = H_f$.

To complete the proof of the proposition, it therefore suffices for us to show that $H_f^{(a)} \subset H_{[d(f)]}$ for every $a \in \Phi$. Let $a \in \Phi$; put, with the notation of (6.4.12) a),

$$r = \inf\big(f''(a), \tfrac{1}{2}f''(2a)\big) \qquad \text{and} \qquad s = \inf\big(f''(-a), \tfrac{1}{2}f''(-2a)\big).$$

In view of (6.4.12) (1), one has $r + s \geq 0$ and $U_f^{(a)}$ is contained in the group generated by $U_{a,r}$ and $U_{-a,s}$. One deduces that $H_f \subset H_{[0]}$, whence the desired result when $d = d(f) \leq 0$. Suppose now $d > 0$; for $\varepsilon \in \{\pm 1, \pm 2\}$ put

$$r_\varepsilon = \inf\Big\{\sum_i f(\varepsilon_i a)\ \Big|\ \varepsilon_i \in \{\pm 1, \pm 2\},\ \sum_i \varepsilon_i = \varepsilon\Big\}.$$

It is easy to see that p. 141

$$r_{-2} \leq 2r_{-1}, \quad r_2 \leq 2r_1, \quad r_1 \leq r_2 + r_{-1}, \quad r_{-1} \leq r_{-2} + r_1 \quad \text{and} \quad r_2 + r_{-2} \geq d > 0.$$

One may therefore apply (6.3.9), which shows that, if X denotes the group generated by the H-components of the commutators (u, u') for $u \in U_{a,r_1} \cup U_{2a,r_2}$ and $u' \in U_{-a,r_{-1}} \cup U_{-2a,r_{-2}}$, the product

$$Y = U_{-2a,r_{-2}} \cdot U_{-a,r_{-1}} \cdot X \cdot U_{a,r_1} \cdot U_{2a,r_2}$$

is a group which contains $U_f^{(a)}$, since $r_\varepsilon \le f(\varepsilon a)$. Now one sees easily that $r_1 + r_{-1} \ge d$, $r_2 + r_{-2} \ge d$, $2r_1 + r_{-2} \ge d$ and $2r_{-1} + r_2 \ge d$. Consequently $H_f^{(a)} \subset X \subset H_{[d]}$.

(6.4.18) We shall now study the conditions under which a group of the form U_f (or more generally $X \cdot U_f$ with $X \subset H$) normalizes another group of the same form. We shall need the following notation: if f and g are two functions on Φ with values in $\widetilde{\mathbf{R}}$, one denotes by $J(f,g)$ the set of the triples (a,ε,η) with $a \in \Phi$, $\varepsilon,\eta \in \{1,2\}$ and $\varepsilon g(\eta a) + \eta f(-\varepsilon a) > 0$, and one denotes by $H_{f,g}$ the subgroup of H generated by the H-components of the commutators (u',u), with $u' \in U_{-\varepsilon a, f(-\varepsilon a)}$, $u \in U_{\eta a, g(\eta a)}$ and $(a,\varepsilon,\eta) \in J(f,g)$. It is immediate that this group is contained in $H_{[k]}$ with

$$k = d(f,g) = \inf\{\varepsilon g(\eta a) + \eta f(-\varepsilon a) \mid (a,\varepsilon,\eta) \in J(f,g)\}.$$

Proposition (6.4.19). — Let $f, g : \Phi \to \widetilde{\mathbf{R}}$ be two quasi-concave functions such that

$$(56) \qquad f(pa+qb) \le pf(a) + qg(b) \qquad \text{for } p,q \in \mathbf{N}^* \text{ and } a, b, pa+qb \in \Phi.$$

Let X be a subgroup of H. In order that U_g normalize the group $X \cdot U_f$ it is necessary and sufficient that the two following conditions be fulfilled:

$$(57) \qquad (X, U_{g,a}) \subset U_{f,a} \qquad \text{for every } a \in \Phi;$$

$$(58) \qquad H_{f,g} \subset X \cdot U_f.$$

It is clear that these conditions are necessary. Assume them satisfied. It suffices to show that the commutator of an element of a generating system of U_g with an element of a generating system of XU_f is contained in XU_f. Taking (2) into account, it therefore suffices to show that $(U_{a,f(a)}, U_{b,g(b)}) \subset XU_f$ for $a, b \in \Phi$. If $b \notin -\mathbf{R}_+ a$, this follows from axiom (V 3) of valuations (6.2.1) and from condition (1). Otherwise, there exists $c \in \Phi$ such that $a = -\varepsilon c$ and $b = \eta c$, with $\varepsilon, \eta \in \{1,2\}$; if $(c,\varepsilon,\eta) \in J(f,g)$, the inclusion to be established follows from (1) and (3); if $(c,\varepsilon,\eta) \notin J(f,g)$, in other words if $\eta f(-\varepsilon c) + \varepsilon g(\eta c) \le 0$, one has, in view of condition (1),

$$f(b) = f\big(\eta(-\varepsilon c) + (\varepsilon+1)(\eta c)\big) \le \eta f(-\varepsilon c) + (\varepsilon+1) g(\eta c) \le g(\eta c) = g(b)$$

whence $U_{b,g(b)} \subset U_{b,f(b)}$, which completes the proof.

Corollary (6.4.20). — Let $f, g : \Phi \to \widetilde{\mathbf{R}}$ be as in (6.4.19). Suppose moreover that for every pair (a,b) of roots such that $a \in -\mathbf{R}_+ b$ one has $g(a) = \infty$ or $f(b) = \infty$. Then U_g normalizes U_f.

Indeed, one then has $H_{f,g} = \{1\}$.

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Corollary (6.4.21). — Let f and g be as in (6.4.19). Suppose moreover that f and g are infinite on Φ^-. Then U_g normalizes U_f.

This follows from (6.4.20).

Corollary (6.4.22). — Let $f, g : \Phi \to \widetilde{\mathbf{R}}$ be as in (6.4.19). Put

$$r = \sup\big(\{f(a) - g(a) \mid a \in \Phi\} \cup \{f(2a) - 2g(a) \mid a, 2a \in \Phi\}\big)$$

$$s = \inf\big(\{f(a) + g(-a) \mid a \in \Phi\} \cup \{f(2a) + 2g(-a),\ 2f(a) + g(-2a) \mid a, 2a \in \Phi\}\big).$$

If X is a subgroup of H such that $H_{[s]} \subset X \subset H_{(r)}$, then U_g normalizes $X \cdot U_f$.

Indeed, one has $H_{f,g} \subset H_{[s]}$ and

$$(H_{(r)}, U_{a,g(a)}) \subset U_{a,g(a)+r} \cdot U_{2a,2g(a)+r} \subset U_{a,f(a)} \cdot U_{2a,f(2a)}.$$

Proposition (6.4.23). — Let $f : \Phi \to \widetilde{\mathbf{R}}$ be a concave function. For $a \in \Phi$ put

$$\begin{aligned} f^*(a) &= f(a) && \text{if } f(a) + f(-a) > 0 \\ f^*(a) &= f(a)+ && \text{if } f(a) + f(-a) = 0. \end{aligned}$$

Then the function $f^* : \Phi \to \widetilde{\mathbf{R}}$ is concave. Let moreover X be a subgroup of H and X^* a normal subgroup of X, satisfying the two conditions:

$$H_{f,f^*} \subset X^*; \tag{59}$$

$$(X^*, U_{f,a}) \subset U_{f^*,a} \qquad \text{for every } a \in \Phi. \tag{60}$$

Then $X^* \cdot U_{f^*}$ is a normal subgroup of $X \cdot U_f$.

Finally, let $\overline{G}$ be the quotient group $XU_f/X^*U_{f^*}$ and let $\overline{U}_a$ (for $a \in \Phi \cup 2\Phi$) be the canonical image of $U_{a,f(a)}$ in $\overline{G}$, and $\overline{T}$ that of $X \cdot H_f$. The group $\overline{U}_a$ is not contained in $\overline{U}_{2a}$ if and only if $a \in \Phi_f$, and $(\overline{T}, (\overline{U}_a)_{a \in \Phi_f})$ is a generating root datum of type Φ_f in $\overline{G}$.

Let us first show that, for $p, q \in \mathbf{N}^*$, $a, b, pa + qb \in \Phi$, one has

$$f^*(pa + qb) \le pf(a) + qf^*(b). \tag{61}$$

Indeed, if (3) is not satisfied, one necessarily has the following relations, since $f(pa + qb) \le pf(a) + qf(b)$:

$$\begin{aligned} f^*(pa+qb) &\ne f(pa+qb) && \text{hence } f(pa+qb) = -f(-pa-qb) \in \mathbf{R}; \\ f^*(b) &= f(b) && \text{hence } f(b) + f(-b) > 0; \\ f(pa+qb) &= pf(a) + qf(b) && \text{hence } f(b) \in \mathbf{R}. \end{aligned}$$

One deduces that

$$pf(a) + (q-1)f(b) + f(-pa-qb) = -f(b) < f(-b)$$

which contradicts the concavity of f (condition (C) of (6.4.3)). Whence (3).

Since $f^*(a) + f^*(-a) \ge f(a) + f(-a) \ge 0$, one deduces at once from (3) and from (6.4.5) that f^* is concave; and conditions (1), (2) and (3) imply, in view of (6.4.19), that U_f normalizes $X^*U_{f^*}$, whence the second assertion, since X normalizes U_{f^*} and X^*.

Suppose that $\overline{U}_a \not\subset \overline{U}_{2a}$, in other words that p. 143

$$U_{a,f(a)} \not\subset U_{a,f^*(a)} \cdot U_{2a,f(2a)}. \tag{62}$$

This implies $f(a) \ne f^*(a)$, hence $f(a) + f(-a) = 0$, whence $f(2a) = 2f(a)$ and

$$U_{a,f(a)} \not\subset U_{a,f(a)+} \cdot U_{2a}.$$

Consequently one has $f(a) = f'(a) \in \Gamma'_a$ and $f(-a) = -f(a) \in \Gamma'_{-a}$ (6.2.16), whence $f(-a) = f'(-a)$ and $a \in \Phi_f$.

Conversely, if $a \in \Phi_f$, one has $f'(a)+f'(-a) = 0$; since $f'(a) \ge f(a)$ and $f(a)+f(-a) \ge 0$, one has $f(a) = -f(-a) = f'(a) \in \mathbf{R}$, whence $f^*(a) > f(a)$ and $f'(a) \in \Gamma'_a$. One deduces (4), whence $\overline{U}_a \not\subset \overline{U}_{2a}$.

Finally, the verification of the last assertion is immediate.

(6.4.24) It is clear that $H_{f,f^*} \subset H_{[0+]}$ and one has $(H_{(0+)}, U_{f,a}) \subset U_{f^*,a}$ for every $a \in \Phi$. Consequently conditions (1) and (2) of (6.4.23) are satisfied if one has

$$H_{[0+]} \subset X^* \subset H_{(0+)}.$$

Of course, this is possible only if $H_{[0+]} \subset H_{(0+)}$, for example if φ satisfies condition (Pr) of (6.4.15). But in every case there exist subgroups X^* satisfying the conditions imposed in (6.4.22), for example H_{f,f^*} itself. To show this we shall need the following lemma:

Lemma (6.4.25). — Let $a \in \Phi$, $r \in \Gamma_a$ and $k \in \widetilde{\mathbf{R}}$, with $k > 0$. Let $u \in U_{-a,-r}$, $v \in U_{a,r+k} \cup U_{2a,2r+k}$ and let h be the H-component of the commutator (v, u).

(i) If $b \in \Phi$, $b \notin \mathbf{R}a$, one has

$$(h, U_{b,s}) \subset U_{b,s+k} \cdot U_{2b,2s+k} \qquad \text{for every } s \in \widetilde{\mathbf{R}}.$$

(ii) $(h, U_{a,r}) \subset U_{a,r+k} \cdot U_{2a,2r+k}$.

(iii) $(h, U_{-a,-r}) \subset U_{-a,-r+k} \cdot U_{-2a,-2r+k}$.

Up to replacing φ by an equipollent valuation, we may assume, in order to simplify the notation, that $r = 0$.

Let $b \in \Phi$, $b \notin \mathbf{R}a$, and $s \in \widetilde{\mathbf{R}}$; let X be the group generated by $U_{-a,0} \cup U_{a,k} \cup U_{2a,k}$ and Y the group generated by the $U_{pa+qb,\,t(p,q)}$ with $p \in \mathbf{Z}$, $q \in \mathbf{N}^*$, $pa + qb \in \Phi$ and $t(p,q) = qs$ if $p < 0$ and $t(p,q) = k + qs$ if $p \geq 0$. Axiom (V 3) of valuations shows that X normalizes Y and that the commutator of an element of $U_{-a,0} \cup U_{a,k} \cup U_{2a,k}$ with an element of $U_{b,s}$ belongs to Y. One therefore has $(X, U_{b,s}) \subset Y$, so that, taking (6.1.6) into account,

$$(H \cap X, U_{b,s}) \subset Y \cap U_b = U_{b,k+s} \cdot U_{2b,k+2s}$$

which proves (i).

Let us prove (iii). Let $x \in U_{-a,0}$ and let Z be the group $U_{-2a,k} \cdot U_{-a,k} \cdot H \cdot U_{a,k} \cdot U_{2a,k}$ (6.3.9). In view of (6.3.7) there exist $y_2 \in U_{-2a,k}$, $y_1 \in U_{-a,k}$, $z_1 \in U_{a,k}$ and $z_2 \in U_{2a,k}$ such that $(v, u) = y_2 y_1 h z_1 z_2$. On the other hand, v and the commutators (v^{-1}, x), (v^{-1}, xu), (x, y_i) p. 144
and (x, z_i) all belong to Z. Consequently one has

$$xvu = v(v^{-1}, x)xu \in Zxu \qquad \text{and} \qquad xuv = v(v^{-1}, xu)xu \in Zxu$$

whence $x(v, u)x^{-1} \in Z$. But $xy_ix^{-1} = (x, y_i)y_i$ and $xz_ix^{-1} = (x, z_i)z_i$ also belong to Z. Consequently one has $xhx^{-1} \in Z$ and $(h, x) \in Z \cap U_{-a} = U_{-a,k} \cdot U_{-2a,k}$, whence (iii).

Finally, (ii) follows from (iii) and from the following lemma:

Lemma (6.4.26). — Let $a \in \Phi$, $h \in H$, $r \in \Gamma_a$ and $k \in \widetilde{\mathbf{R}}$ with $k > 0$. If one has $(h, U_{-a,-r}) \subset U_{-a,-r+k} \cdot U_{-2a,-2r+k}$, then one also has $(h, U_{a,r}) \subset U_{a,r+k} \cdot U_{2a,2r+k}$.

One may suppose $r = 0$. Since $0 \in \Gamma_a$, the group $U_{a,0}$ is generated by $\varphi_a^{-1}(0)$ and, since h normalizes $U_{a,k} \cdot U_{2a,k}$, it suffices to show that $(h, x) \in U_{a,k} \cdot U_{2a,k}$ for $x \in \varphi_a^{-1}(0)$. Put $x = x'mx''$ with $x', x'' \in U_{-a,0}$ and $m \in M_{a,0}$. Then

$$\begin{aligned}(h, x) &= h \cdot x' \cdot mh^{-1}m^{-1} \cdot m(h, x'')m^{-1} \cdot x'^{-1} \\ &\in H \cdot U_{-a,0} \cdot H \cdot U_{a,k} \cdot U_{2a,k} \cdot U_{-a,0} \subset U_{-a,0} \cdot H \cdot U_{a,k} \cdot U_{2a,k}\end{aligned}$$

whence the lemma.

Proposition (6.4.27). — Let f, f^* be as in (6.4.23). The subgroup H_{f,f^*} is normal in H and one has $(H_{f,f^*}, U_{f,b}) \subset U_{f^*,b}$ for every $b \in \Phi$.

The first assertion is obvious, since H normalizes all the $U_{a,k}$. Since H_{f,f^*} and $U_{f,b}$ normalize $U_{f^*,b}$, it suffices to prove that, if h is the H-component of the commutator of an element $u \in U_{-\varepsilon a, f(-\varepsilon a)}$ and an element $v \in U_{\eta a, f^*(\eta a)}$ with $a \in \Phi^{\mathrm{red}}$, $\varepsilon, \eta \in \{1, 2\}$ and $\varepsilon f^*(\eta a) + \eta f(-\varepsilon a) > 0$, and if $x \in U_{b,f(b)}$ with $b \in \Phi$, then $(h, x) \in U_{b,f^*(b)}$. This is obvious if $f^*(b) = f(b)$ or if $f(b) \notin \Gamma_b$. If $b \notin \mathbf{R}a$, it follows from (6.4.25) (i). Suppose that $f(a) \in \Gamma_a$, $f(a) + f(-a) = 0$ and $b \in \{\pm a, \pm 2a\}$; then $u \in U_{-a,f(-a)}$ and $v \in U_{a,f(a)+} \cup U_{2a,2f(a)+}$ (note that, if $2a \in \Phi$, one then has $f(2a) = 2f(a) = -f(-2a)$), and our assertion follows from (6.4.25) (ii) and (iii) (taking $r = f(a)$ and $k = 0+$).

It remains for us to examine only the case where $f(a) + f(-a) > 0$, $f(2a) + f(-2a) = 0$, $f(2a) \in \Gamma_{2a}$ and $b = \pm 2a$. Put $f(2a) = 2r$. One has $f(a) > r$, otherwise the concavity of f

would imply $f(-a) \leq f(-2a) + f(a) \leq \frac{1}{2}f(-2a) \leq -f(a)$. Likewise, $f(-a) > -r$, and one has $u \in U_{-a,-r}$ and $v \in U_{a,r+} \cup U_{2a,2r+}$. By (6.4.25) (ii) and (iii) one therefore has

$$(h, U_{\pm 2a,\pm 2r}) \subset (h, U_{\pm a,\pm r}) \cap U_{\pm 2a} \subset U_{\pm a,\pm r+} \cap U_{\pm 2a} = U_{\pm 2a,\pm 2r+}$$

which completes the proof.

Proposition (6.4.28). — Let $f, g, h : \Phi \to \widetilde{\mathbf{R}}$ be three quasi-concave functions. Put $f_1 = \inf(f, h)$ and $g_1 = \inf(g, h)$ and suppose that, for $p, q \in \mathbf{N}^*$ and $a, b, pa + qb \in \Phi$, one has

$$h(pa + qb) \leq pf_1(a) + qg_1(b). \tag{63}$$

Let moreover X and Y be two subgroups of H such that

$$(X, U_{g,a}) \cup (Y, U_{f,a}) \cup (H_{f_1,g_1}, U_{f,a} \cup U_{g,a}) \subset U_{h,a} \tag{64}$$

for every $a \in \Phi$.

Throughout, Φ is the root system of the root datum $(T, (U_a))$, $\widetilde{\mathbf{R}}$ denotes $\mathbf{R}$ enlarged by the symbols $r+$, and for a function $f : \Phi \to \widetilde{\mathbf{R}}$ we write U_f for the subgroup generated by the $U_{a,f(a)}$, $a \in \Phi$, and $P_f = H \cdot U_f$. Commutator subgroups are written (X, Y), as in the original.

Page 145 completes the proof of (6.4.28), whose conclusion is that $(X, Y) \cdot H_{f_1,g_1} \cdot U_h$ is a group containing the commutator group $(X \cdot U_f, Y \cdot U_g)$. The argument runs in three steps: an elementary inequality showing $h(-\varepsilon a) = f_1(-\varepsilon a)$ and $h(\eta a) = g_1(\eta a)$ under the stated sign condition; the verification, via (6.4.19), that U_f and U_g normalize $H_{f_1,g_1} \cdot U_h$; and the inclusion $(U_f, U_g) \subset H_{f_1,g_1} \cdot U_h$, obtained from (V 3) and (6.3.9). The general case then follows from the commutator identity

$$(xu, yv) = xy \cdot (y^{-1}, u) \cdot (u, v) \cdot (v, x^{-1}) \cdot x^{-1}y^{-1}.$$

Corollary (6.4.29). — *Let Δ be a subset of the basis Π of Φ and let Ψ be the set of positive roots that are not linear combinations of elements of Δ. Let $f, g : \Phi \to \widetilde{\mathbf{R}}$ be quasi-concave functions, and suppose f (resp. g) is infinite on $\complement\Psi$ (resp. on $-\Psi$). For $a \in \Phi$ put*

$$h(a) = \inf\Big\{\textstyle\sum_{1\leq i\leq m} f(a_i) + \sum_{1\leq j\leq n} g(b_j)\Big\},$$

the infimum being taken over all pairs of non-empty finite sequences (a_i), (b_j) of elements of Φ with $\sum_i a_i + \sum_j b_j = a$. Then $h : \Phi \to \widetilde{\mathbf{R}}$ is concave and $(U_f, U_g) \subset U_h$.

Proof. h is concave and satisfies (6.4.28)(1) with $f_1 = \inf(f, h)$, $g_1 = \inf(g, h)$. Examining the two cases $a \in -\Psi$ and $a \in (\pm\Phi^+ \cap \complement\Psi)$ shows f_1 or g_1 is infinite at a; hence $H_{f_1,g_1} = \{1\}$ and (6.4.28) applies.

Corollary (6.4.30). — *Let $f_*, g_* : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$ be functions whose restrictions to Φ are quasi-concave, and let $h_* : \Phi \cup \{0\} \to \widetilde{\mathbf{R}} \cup \{-\infty\}$ be given by*

$$h_*(a) = \inf\Big\{\textstyle\sum_i f_*(a_i) + \sum_j g_*(b_j)\Big\},$$

the infimum over pairs of non-empty finite sequences (a_i), (b_j) in $\Phi \cup \{0\}$ with $a = \sum_i a_i + \sum_j b_j$; the order on $\widetilde{\mathbf{R}} \cup \{-\infty\}$ is defined by $-\infty < k$ for all $k \in \widetilde{\mathbf{R}}$.

If $h_*(0) \neq -\infty$, then $h_*(a) \neq -\infty$ for every $a \in \Phi$ and the restriction h of h_* to Φ is concave. Suppose in addition that $H_{[h_*(0)]} \subset H_{(h_*(0))}$ (for instance that φ satisfies condition (Pr)), and let X (resp. Y) be a subgroup of $H_{(f_*(0))}$ (resp. $H_{(g_*(0))}$). Then

$$(X \cdot U_f, Y \cdot U_g) \subset (X, Y) \cdot H_{[h_*(0)]} \cdot U_h.$$

Proof. One checks that, in the notation of (6.4.28), $H_{f_1,g_1} \subset H_{[h_*(0)]}$ and that conditions (1) and (2) of (6.4.28) hold.

Corollary (6.4.31). — *Let f and g be linear functions on Φ and put $k = \inf\{g(a) - f(a) \mid a \in \Phi^+\}$. Then $(U_f^+, U_g^+) \subset U_{f+k}^+$.*

Proof. Apply (6.4.29) with $\Delta = \emptyset$ (so $\Psi = \Phi^+$ and $\complement\Psi = -\Psi = \Phi^-$) to the functions f^+, g^+ equal to ∞ on Φ^- and to f, g on Φ^+. Linearity of f gives $h(a) = \inf\{f(a) + \sum_j (g(b_j) - f(b_j))\} \geq f(a) + k$ for $a \in \Phi^+$.

The authors remark that g need not be assumed linear: quasi-concavity of g suffices.

Corollary (6.4.32). — *Let L be a half-line with origin 0 in V^*, contained in D_0. For $x \in A$ let U_x^+ denote the subgroup generated by the $U_{a,-a(x-\varphi)}$ for $a \in \Phi^+$. There is a constant k with $0 < k \leq 1$ such that*

$$(U_x^+, U_y^+) \subset U_{x+k(y-x)}^+$$

for all points x, y of A with $x - y \in L$.

Proof. Take $v \in L$ and set $k = \inf a(v)/\sup a(v)$, the bounds over $a \in \Phi^+$; then $0 < k \leq 1$. Write $x = y+tv$ with $t \in \mathbf{R}_+$ and apply (6.4.31): the resulting function satisfies $h(b) \geq -b(x)+b(ktv) = -b(x+k(y-x))$.

If Φ has rank one and type BC_1, then $k = 1/2$. If Φ has type A_1, then $(U_x^+, U_y^+) = \{1\}$ for all $x, y \in A$.

(6.4.33) The subgroups $H_{[k]}$ and $H_{(k)}$ and condition (Pr) of (6.4.15) are now studied in more detail.

Proposition (6.4.33). — *For all $k, \ell \in \widetilde{\mathbf{R}}$, the commutator group $(H_{(k)}, H_{(\ell)})$ is contained in $H_{(k+\ell)}$. If $H_{[k]} \subset H_{(k)}$, then for $a \in \Phi$ and $m \in M_a^0$ one has $(m, H_{(k)}) \subset H_{[k]}$.*

Proof. For the first assertion one may assume $k, \ell > 0$; setting $X_y = U_{a,x+y} \cdot U_{2a,2x+y} \subset U_{a,x}$ for $y \in \widetilde{\mathbf{R}}_+$, (V 3) gives $(X_y, X_z) \subset X_{y+z}$, and a computation with the commutators $v = (u^{-1}, h^{-1})$, $v' = (u^{-1}, h'^{-1})$ modulo $X_{k+\ell}$ yields $((h, h'), u) \in X_{k+\ell}$. For the second, write $m = u'uu''$ and use the group $Y = U_{-2a,-2r+k}U_{-a,-r+k}H_{[k]}U_{a,r+k}U_{2a,2r+k}$ of (6.3.9), normalized by $U_{a,r}$ and $U_{-a,-r}$ by (6.4.19); then $(m, h) \in Y \cap H = H_{[k]}$.

Proposition (6.4.34). — *Let $k \in \widetilde{\mathbf{R}}$ with $k > 0$. An element h of H belongs to $H_{(k)}$ if and only if*

(1) $$(h, U_{a,r}) \subset U_{a,r+k} \cdot U_{2a,2r+k} \qquad \textit{for all } r \in \mathbf{R},$$

for every $a \in \Pi$.

Proof. If (1) holds for a root a it holds for $2a$; so it suffices to show that the condition "(1) holds for the elements of Π" is independent of the choice of Π. This reduces to two assertions: (2) if (1) holds for a it holds for $-a$ — this is Lemma (6.4.26); and (3) if (1) holds for distinct $a, b \in \Pi$ it holds for $c = r_a(b)$ — proved using (6.1.8), (6.4.21) and (6.4.9).

Proposition (6.4.35). — *If Φ has no irreducible component of rank one, then condition (Pr) is satisfied.*

Proof. Lemma (6.4.25) gives condition (6.4.34)(1) for every root $a \notin \mathbf{R}b$, and the hypothesis on Φ allows one to choose the basis Π with $b \notin \mathbf{R}\Pi$; then apply (6.4.34).

Remark (6.4.36). — *A more elaborate argument shows that (Pr) holds as soon as Φ contains no irreducible component of type BC_1. The authors know of no example in which (Pr) fails.*

Proposition (6.4.37). — *Suppose (Pr) is satisfied and let $k, \ell \in \widetilde{\mathbf{R}}_+$. Then*

(1) $$(H_{(k)}, H_{[\ell]}) \subset H_{[k+\ell]},$$

(2) $$(H_{[k]}, H_{[\ell]}) \subset H_{[k+\ell]}.$$

Proof. Only (1) is needed. Assume $k > 0$; since $H_{[k+\ell]}$ is normal in H it suffices to treat commutators with a generating system of $H_{[\ell]}$, i.e. with $U_f \cap H$ for a suitable f supported on $\pm a, \pm 2a$. Apply (6.4.30) with $X = \{1\}$, $Y = H_{(k)}$, and conclude with (6.3.9), which gives $U_h \cap H \subset H_{[2k+\ell]} \subset H_{[k+\ell]}$.

Definition (6.4.38). — *An* extension *of the valuation φ is a map $\varphi_0 : H \to \mathbf{R}_+ \cup \{\infty\}$ satisfying:*

(P 1) for every $k \in \mathbf{R}$ the preimage $U_{0,k} = \varphi_0^{-1}([k,\infty])$ is a subgroup of H, and $H_{[k]} \subset U_{0,k} \subset H_{(k)}$;

(P 2) for $k, \ell \in \mathbf{R}$, $(U_{0,k}, U_{0,\ell}) \subset U_{0,k+\ell}$.

φ is extendable *if it admits an extension. An* extended valuation *of the root datum $(T, (U_a))$ is a pair (ψ, ψ_0) consisting of a valuation ψ of $(T, (U_a))$ and an extension ψ_0 of ψ.*

(P 2) implies each $U_{0,k}$ is normal in $U_{0,0} = H$; together with (6.4.33), (P 1) shows $U_{0,k}$ is even normal in N. Both conditions remain valid if k or ℓ is allowed the value ∞. Condition (P 1) may be made explicit: for all $a \in \Phi$ and $h, k \in \mathbf{R}$, $U_{0,k}$ is a group and

(P 1′) $(U_{a,h}, U_{0,k}) \subset U_{a,h+k} \cdot U_{2a,2h+k}$;

(P 1″) the H-components of the commutators (u, u') and (u', u), for $u \in U_{a,r}$ and $u' \in U_{-a,s}$ (resp. $u' \in U_{-2a,s}$) with $r + s = k$ (resp. $2r + s = k$), lie in $U_{0,k}$.

Proposition (6.4.39). — *The valuation φ is extendable if and only if condition (Pr) of (6.4.15) is satisfied.*

Proof. Necessity is clear. Conversely, under (Pr) the two functions

$$\varphi_{[0]}(h) = \sup\bigl(\{k \in \mathbf{R} \mid h \in H_{[k]}\} \cup \{0\}\bigr), \qquad \varphi_{(0)}(h) = \sup\{k \in \mathbf{R} \mid h \in H_{(k)}\}$$

are extensions of φ, by (6.4.37).

(6.4.40) From here on the authors set $U_0 = H$. This makes the notation $U_{a,k}$ coherent for $a \in \Phi \cup \{0\}$ and $k \in \mathbf{R}$, and is justified by the fact that H plays, relative to the element 0 of V^*, a role analogous to that of the groups U_a relative to the roots $a \in \Phi$. The next proposition exhibits this parallelism between 0 and the roots with respect to an extended valuation.

Proposition (6.4.41). — *For $a \in \Phi \cup \{0\}$ let ψ_a be a map from U_a to $\mathbf{R} \cup \{\infty\}$, with $\psi_0(u) \geq 0$ for all $u \in U_0$. The pair $((\psi_a)_{a\in\Phi}, \psi_0)$ is an extended valuation of the root group datum $(T, (U_a))$ if and only if the maps ψ_a satisfy conditions (V 0), (V 1), (V 2), (V 4), (V 5) of (6.2.1) (with φ replaced by ψ) together with:*

(V 1 bis) for every $k \in \mathbf{R}$, the set $U_{0,k} = \psi_0^{-1}([k,\infty])$ is a group;

(VP) let $a, b \in \Phi \cup \{0\}$ and $k, \ell \in \mathbf{R}$; let Ψ be the set of elements of $\Phi \cup \{0\}$ of the form $pa + qb$ with $p, q \in \mathbf{N}^*$, and for $c \in \Psi$ put $k(c) = \inf\{pk + q\ell \mid p, q \in \mathbf{N}^*,\ pa + qb = c\}$. Then the commutator group $(U_{a,k}, U_{b,\ell})$ is contained in the product of the groups $U_{c,k(c)}$ for $c \in \Psi$ and $c/2 \in \Psi$, taken in a suitable order.

Proof. Necessity is straightforward. Conversely (VP) for $b \notin -\mathbf{R}_+ a$ is just (V 3), so $\psi = (\psi_a)$ is a valuation; (VP) with $a = b = 0$ gives (P 2), with $a = 0$, $b \in \Phi$ gives $U_{0,k} \subset H_{(k)}$, and with $a \in \Phi$, $b = -a$ or $b = -2a$ gives $H_{[r]} \subset U_{0,r}$ via (6.3.9).

Definition (6.4.42). — *Let φ_0 be an extension of φ. For a function $f : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$, let U_f again denote the subgroup of G generated by the $U_{a,f(a)}$ for $a \in \Phi \cup \{0\}$ (for $k = r+$ with $r \in \mathbf{R}$, $U_{0,k}$ denotes the union of the $U_{0,s}$ for $s \in \mathbf{R} \cup \{\infty\}$, $s > r$). One says f is* quasi-concave *(relative to (φ, φ_0)) if its restriction to Φ is quasi-concave and $H_{f|\Phi}$ is contained in $U_{0,f(0)}$.*

If $f : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$ is concave (6.4.4) then it is quasi-concave, by (6.4.17), since $f(0) \leq d(f|\Phi)$. If f is quasi-concave then $U_f \cap U_0 = U_{0,f(0)}$.

Proposition (6.4.43). — *Let φ_0 be an extension of φ and let $f, g : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$ be quasi-concave functions such that $f(pa + qb) \leq pf(a) + qg(b)$ for $p, q \in \mathbf{N}^*$ and $a, b, pa + qb \in \Phi \cup \{0\}$. Then U_f is normalized by U_g.*

Proof. One has $H_{f|\Phi,\, g|\Phi} \subset U_{0,f(0)}$ and

$$(U_{0,f(0)}, U_{a,g(a)}) \subset (H_{(f(0))}, U_{a,g(a)}) \subset U_{a,g(a)+f(0)} U_{2a,2g(a)+f(0)} \subset U_{a,f(a)} U_{2a,f(2a)};$$

now apply (6.4.19) with $X = U_{0,f(0)}$.

Proposition (6.4.44). — *Let φ_0 be an extension of φ, let $f, g : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$ be quasi-concave, and let $h : \Phi \cup \{0\} \to \widetilde{\mathbf{R}} \cup \{-\infty\}$ be defined by*

$$h(a) = \inf\Big\{\textstyle\sum_i f(a_i) + \sum_j g(b_j)\Big\},$$

the infimum over pairs of non-empty finite sequences (a_i), (b_j) in $\Phi \cup \{0\}$ with $a = \sum_i a_i + \sum_j b_j$. If $h(0) \neq -\infty$, then h is concave and $(U_f, U_g) \subset U_h$.

Proof. A special case of (6.4.30), taking $X = U_{0,f(0)}$ and $Y = U_{0,g(0)}$.

Proposition (6.4.45). — *Suppose φ is extendable (i.e. (Pr) holds). Let $f : \Phi \to \widetilde{\mathbf{R}}$ be quasi-concave, and suppose there exist a linear function $\lambda \in V$ and $k \in \widetilde{\mathbf{R}}$ with $k > 0+$ and $f \geq \lambda + k$ (cf. (6.4.6)). Let H' be a subgroup of $H_{(k)}$. Put $Z = H' \cdot U_f$ and, for $n \in \mathbf{N}^*$,*

$$Z_n = Z \cap \big(\mathscr{C}^n(H') \cdot H_{[nk]} \cdot U_{\lambda+nk}\big).$$

Then:

(1) *$(Z_n, Z_{n'}) \subset Z_{n+n'}$ for all $n, n' \in \mathbf{N}^*$;*
(2) *$\mathscr{C}^n(Z) \subset Z_n$ for all $n \in \mathbf{N}^*$;*
(3) *$\mathscr{C}^\infty(Z) \subset H_{(\infty)}$ — recall that this is the centralizer of the group generated by the U^a, $a \in \Phi$ (6.4.13);*
(4) *if $H' \subset H_{[k]}$, then $\mathscr{C}^\infty(Z)$ is contained in $H_{[\infty]}$ and in the center of Z. If moreover $H_{[\infty]} = \{1\}$, then Z is pronilpotent.*

Here, for a group X, $\mathscr{C}^n(X)$ denotes the n-th term of the descending central series, so $\mathscr{C}^1(X) = X$ and $\mathscr{C}^{n+1}(X) = (\mathscr{C}^n(X), X)$, and $\mathscr{C}^\infty(X) = \bigcap_{n\geq 1} \mathscr{C}^n(X)$.

Proof. Apply (6.4.30) with f_*, g_* the restrictions of $\lambda + nk$, $\lambda + n'k$ and $X = \mathscr{C}^n(H')H_{[nk]}$, $Y = \mathscr{C}^{n'}(H')H_{[n'k]}$; (1) follows from $h_* \geq \lambda + (n + n')k$ together with $(\mathscr{C}^n(H'), \mathscr{C}^{n'}(H')) \subset \mathscr{C}^{n+n'}(H')$ and (6.4.37). Then (2) and (3) follow using (6.4.33), (6.4.9) and (DR 6), and (4) from (6.4.15).

Corollary (6.4.46). — *Suppose φ is extendable and discrete, and let $e \in \mathbf{R}_+^*$ be such that $\Gamma_a \subset \mathbf{Z}e$ for every $a \in \Phi$. Suppose moreover $H_{[\infty]} = \{1\}$, and resume the hypotheses and notation of (6.4.23). Then $H_{f,f^*} \subset H_{[e]}$. If $X^* \subset H_{[e]}$, then $X^* \cdot U_{f^*}$ is a pronilpotent normal subgroup of $X \cdot U_f$.*

Proof. By (6.4.24), $H_{f,f^*} \subset H_{[0+]} = H_{[e]}$. Setting $g(a) = \inf\{t \in \mathbf{Z}e \mid t \geq f^*(a)\}$ gives $U_g = U_{f^*}$ with g quasi-concave, and (6.4.6) supplies λ and $k \leq e$ with $g > \lambda + k$; apply (6.4.45).

Remarks (6.4.47). — *(1) The hypothesis $\Gamma_a \subset \mathbf{Z}e$ in (6.4.46) is not very restrictive: by (6.2.23), every discrete valuation is equivalent (6.2.5) to one with this property.*

(2) With the hypotheses and notation of (6.4.23) and (6.4.46) one may take $X = H$ and $X^* = H_{[e]}$. The group $H \cdot U_f$ is then an extension of $\overline{G} = HU_f/H_{[e]}U_{f^*}$, which carries a generating root datum $(H/H_{[e]}, (\overline{U}_a))$ of type Φ_f — making it akin to a reductive algebraic group — by the pronilpotent group $H_{[e]} \cdot U_{f^*}$. This situation recurs later for semisimple algebraic groups over a local field: $\overline{G}$ is then the group of rational points of a reductive algebraic group

over the residue field k, and $\mathrm{Ker}(H\cdot U_f \to \overline{G})$ *the group of rational points of a prounipotent proalgebraic group over k, the "prounipotent radical" of $H\cdot U_f$.*

Proposition (6.4.48). — *Let $g : \Phi \cup \{0\} \to \widetilde{\mathbf{R}}$ be a concave function with $g(0) > 0$, let $f : \Phi \to \widetilde{\mathbf{R}}$ be a quasi-concave function majorising the restriction of g to Φ, and let X be a subgroup of H containing H_g. Let $(a_1, \ldots, a_m)$ be the elements of Φ^{red} in an arbitrary order and let $j \in \mathbf{N}$ with $0 \le j \le m$. Let*

$$\tau_j : X \times \prod_{1\le i\le m} U_{f,a_i} \longrightarrow X \cdot U_f$$

be the map $(x, u_1, \ldots, u_m) \mapsto u_1 \cdots u_j \, x \, u_{j+1} \cdots u_m$.

(i) τ_j is injective.

(ii) If, for every $a \in \Phi$, the group U_a — given the topological group structure for which the family $(U_{a,k})_{k\in\mathbf{R}}$ is a fundamental system of neighborhoods of 1 — is complete, then τ_j is bijective.

Proof. One reduces to $j = m$ with Φ irreducible. Rank one follows from (DR 6), (6.4.9), (6.4.10) (completeness is not needed there). For rank ≥ 2, φ is extendable by (6.4.35), so (6.4.45) is available: with $Z = H_{[k]}U_{\lambda+k}$, $Z_n = H_{[nk]}U_{\lambda+nk}$, an induction on n shows $u_i^{-1}u_i' \in Z_n \cap U_{a_i}$, giving (i) on letting $n \to \infty$. For (ii) a second induction produces approximations $u_i^{(n)}$ that are Cauchy by construction; their limits v_i satisfy $(v_1 \cdots v_m)^{-1}u \in X$, so u lies in the image of τ_m.

6.5. Discrete valuations and double Tits systems. —

In this subsection the notation of 6.2 — in particular that of (6.2.4) — and of 6.4 (see (6.4.2) and (6.4.3) for the meaning of P_f and f_Ω) is retained, and the valuation φ is assumed *discrete*. By (6.2.22) one may then also use the notation of 1.3 and assume $A = \mathbf{A}$, $W = \mathbf{W}$, $\Sigma = \mathbf{\Sigma}$.

Theorem (6.5). — *Let G' be the subgroup of G generated by $N' = \nu^{-1}(\mathbf{W})$ and the U_a $(a \in \Phi)$. Put $T' = T \cap N'$ and $B = P_{f_C}$.*

(i) $B \cap N' = H$ and $N'/H = \mathbf{W}$. The quadruple $(G', B, N', \mathbf{S})$ is a double Tits system. If this system is substituted for the one written (G, B, N, S) in §4, the groups written $\mathfrak{B}_D^0$ and $\mathfrak{B}_D$ in 4.1 are respectively HU^+ and $T'U^+$ (for $D = D_0$).

(ii) The injection of G' into G is a (B, N')-adapted homomorphism of connected type, for which the objects written $\widehat{N}$, $\widehat{\nu}$ and $\widehat{W}$ in 4.1 are respectively N, ν and $\widehat{W}$.

Proof. The theorem is an almost immediate consequence of (5.2.19) and (5.2.31); the authors also give an independent proof based on (6.4.9), for readers who have not read §5. That (i) implies (ii) follows from $G = T\cdot G' = N\cdot G'$ together with (6.2.10)(iii). For (i): the first two relations come from (6.4.10) and the definition of N'; axiom (T 1) of (1.2.6) follows from $U_a \subset T'BT'$, axiom (T 2) from $(\mathbf{W}, \mathbf{S})$ being a Coxeter system (1.3.4), and (T 4) from $U_a \cap sBs^{-1} = U_{\alpha_+}$, $U_a \cap B = U_\alpha$ by (6.4.9). Axiom (T 3) follows from the two inclusions $sB \subset BsU_\alpha$ and $(U_\alpha sHU_\alpha \cup U_{(\alpha^*)_+}HU_\alpha)wB \subset BswB \cup BwB$. The identification of $\mathfrak{B}_D^0$ and $\mathfrak{B}_D$ uses (4.1.5) and $B \subset U^-HU^+$; finally (6.1.12) applied to $(T', (U_a))$ in G' shows $(G', T'U^+)$ is a Tits system, whence the double Tits system.

Remark (6.5, 1). — *Let $\Omega \subset A$, $\Omega \neq \emptyset$. The subgroup written P_Ω in 4.1 is none other than P_{f_Ω}: this follows from (5.2.19), and is also a consequence of the more general result proved in (7.1.11).*

Remark (6.5, 2). — *Let (V_n) be a decreasing sequence of subgroups of G forming a fundamental system of neighborhoods of 1 for a complete topological group structure on G. Suppose moreover that B and the U_a are closed subgroups and that the topology induced on each U_a is the one* *defined by the $U_{a,k}$. Then the condition of (6.4.48)(ii) is satisfied, and so are the hypotheses of (2.8.2): it suffices to consider an increasing sequence of bounded convex subsets (Ω_n) of A with non-empty interior such that $f_{\Omega_n}(a) \geq \inf\{k \in \mathbf{R} \mid U_{a,k} \subset V_n\}$ for all $a \in \Phi$; one checks at once that $P_{\Omega_n} \subset H \cdot V_n$.*

7. The building of a valued root datum

The notation of 6.2–6.4 is retained. It was seen in (6.5) that when φ is discrete it defines a Tits system of affine type, hence a building on which G acts. In this section a metric space $\mathscr{I}$ on which G acts is constructed *without* assuming φ discrete; it is identified with the building of G when φ is discrete, and in general retains some of its structures and properties. Although $\mathscr{I}$ is no longer a polysimplicial complex in the non-discrete case, its chambers and facets are still defined.

Before defining $\mathscr{I}$ in 7.3, certain subgroups of G are studied, closely related to particular cases of the groups U_f of 6.4, which generalize the fixers and stabilizers studied in §4 for the building of a Tits system of affine type. In particular subgroups attached to the chambers of A, written $B_{x,D}$, are introduced; they play the role of the group B of such a system.

When φ is discrete, most results of §§7–8 reduce to special cases of results already established in §§3, 4, 5, but are recovered here by a different method. The main novelty of the approach is that it applies equally to non-discrete valuations, in particular to *dense* valuations, i.e. those for which $\Gamma_a = \varphi_a(U_a^*)$ is dense in $\mathbf{R}$ for every root $a \in \Phi$. (When Φ is irreducible this is equivalent to density of Γ_a in $\mathbf{R}$ for a *single* root a.)

If D is a vector chamber of Φ, one writes Φ_D^+ (resp. Φ_D^-) for the set of roots $a \in \Phi$ positive (resp. negative) on D, Π_D for the corresponding basis of Φ, and U_D^+ (resp. U_D^-) for the subgroup of G generated by the U_a with $a \in \Phi_D^+$ (resp. $a \in \Phi_D^-$).

(7.1.1) Let Ω be a non-empty subset of the affine space A (6.2.6). In (6.4.3) it was assigned a concave function $f_\Omega : \Phi \to \mathbf{R}$ by

$$f_\Omega(a) = \inf\{k \in \mathbf{R} \mid a(x) + k \geq 0 \text{ for all } x \in \Omega\}, \qquad a \in \Phi.$$

From now on A is identified with the vector space V by the choice of the origin φ; this allows the elements of V^*, in particular the roots $a \in \Phi$, to be regarded as functions on A. Likewise an element $x \in A$ defines a linear function $a \mapsto -a(x)$ on Φ, which is sometimes again written x. To the concave function f_Ω one associates, as in (6.4.2), the subgroup U_{f_Ω} generated by the union of the $U_{a,f_\Omega(a)}$, $a \in \Phi$, and the subgroup $P_{f_\Omega} = H \cdot U_{f_\Omega}$; these are written more simply U_Ω and P_Ω. Since $U_{a,f_\Omega(a)}$ is the union of the $U_{a,k}$ for $k \in \Gamma_a$, $k \geq f_\Omega(a)$, one sees that U_Ω is generated by the union of the subgroups U_α, where α runs over the affine roots (6.2.6) containing Ω (cf. (5.2.30)).

(7.1.2) Clearly U_Ω and P_Ω depend only on the intersection of the affine roots containing Ω, that is, on the set

$$\mathrm{cl}(\Omega) = \bigcap_{\substack{a\in\Phi,\ k\in\Gamma_a \\ k\geq f_\Omega(a)}} \{x \in A \mid a(x) + k \geq 0\}.$$

The set $\mathrm{cl}(\Omega)$, again called the *enclosure* of Ω, is a closed convex subset of A; one says Ω is *closed* if $\Omega = \mathrm{cl}(\Omega)$ (cf. (2.4.4)).

(7.1.3) It was seen in (6.4.10) that the subgroup $N_{f_\Omega} = N \cap U_\Omega$ has the following property: its image $\nu(N_{f_\Omega})$ in $\widehat{W} = \nu(N) = N/H$ is the subgroup generated by the reflections $r_{a,k}$ for $a \in \Phi$

and $k \in \Gamma'_a$ with $\Omega \subset \{x \in A \mid a(x) + k = 0\}$, and hence is identified with the Weyl group of the root system $\Phi_\Omega = \Phi_{f_\Omega}$ consisting of the roots $a \in \Phi$ for which there exists $k \in \Gamma'_a$ with $a(\Omega) = \{-k\}$. One sets

$$N_\Omega = H \cdot N_{f_\Omega} = H \cdot (N \cap U_\Omega) = N \cap P_\Omega.$$

When Ω reduces to a point x one writes U_x, P_x, etc. Finally, the map sending $\Omega \subset A$, $\Omega \neq \emptyset$, to U_Ω (resp. P_Ω, N_Ω, Φ_Ω) is decreasing for inclusion.

(7.1.4) As in (1.3.11), a *sector* of A is a set $x + D$ of translates of a point $x \in A$ (the *vertex* of the sector) by the elements of a vector chamber D (the *direction* of the sector). One has $U_{x+D} \subset U_D^+$ and $N_{x+D} = H$. If $\Omega \subset A$ contains a sector of direction D one therefore also has $U_\Omega \subset U_D^+$ and $N_\Omega = H$. More generally, for every non-empty subset Ω of A,

(1) $$P_\Omega \cap U_D^+ = U_{\Omega+D}, \qquad P_\Omega \cap U_D^- = U_{\Omega-D},$$

whence, by (6.4.9),

(2) $$P_\Omega = N_\Omega \cdot U_{\Omega+D} \cdot U_{\Omega-D}$$

for every vector chamber D.

Proposition (7.1.5). —

7.1. The subgroups P_Ω and $\widehat{P}_\Omega$. — *Let Ω and Ω' be non-empty subsets of A and let D be a vector chamber.*

(i) If $\Omega' \subset \Omega + \overline{D}$, then $P_\Omega \cdot P_{\Omega'} \subset N_\Omega \cdot U_D^- \cdot U_{\Omega'+D} \cdot N_{\Omega'}$.

(ii) If $\Omega' \subset \Omega + \overline{D}$ and $\Omega \subset \Omega' - \overline{D}$, then $P_\Omega \cdot P_{\Omega'} = N_\Omega \cdot U_{\Omega-D} \cdot U_{\Omega'+D} \cdot N_{\Omega'}$.

Proof. By (7.1.4)(2), $P_\Omega P_{\Omega'} = N_\Omega U_{\Omega-D} U_{\Omega+D} U_{\Omega'+D} U_{\Omega'-D} N_{\Omega'}$. If $\Omega' \subset \Omega + \overline{D}$ then $U_{\Omega'+D} \supset U_{\Omega+D}$, which absorbs a factor and gives (i) since $U_{\Omega-D} U_{\Omega'-D} \subset U_D^-$. If moreover $\Omega \subset \Omega' - \overline{D}$ then $U_{\Omega'-D} \subset U_{\Omega-D}$, giving (ii).

Corollary (7.1.6). — *Let x be a point of A and Ω a non-empty subset of A. There is a vector chamber D with $P_\Omega \cdot P_x \subset N_\Omega \cdot U_D^- \cdot U_{x+D} \cdot N_x$.*

Proof. Choose $y \in \Omega$ and D with $x - y \in \overline{D}$; apply (7.1.5)(i).

Corollary (7.1.7). — *Let x and y be two points of A. There is a vector chamber D with $P_y \cdot P_x = N_y \cdot U_{y-D} \cdot U_{x+D} \cdot N_x$.*

Proof. Choose D with $x - y \in \overline{D}$; apply (7.1.5)(ii).

(7.1.8) Let again $\Omega \subset A$, $\Omega \neq \emptyset$. Clearly $\nu(n)(x) = x$ for all $x \in \Omega$ and $n \in N_\Omega$. Write $\widehat{N}_\Omega$ for the *fixer* of Ω in N:

$$\widehat{N}_\Omega = \{n \in N \mid \nu(n)(x) = x \text{ for all } x \in \Omega\}.$$

One has $N_\Omega \subset \widehat{N}_\Omega$, and $\widehat{N}_\Omega$ normalizes P_Ω since N permutes the affine roots (6.2.10). Consequently

$$\widehat{P}_\Omega = \widehat{N}_\Omega \cdot P_\Omega = \widehat{N}_\Omega \cdot U_\Omega$$

is a group having P_Ω and U_Ω as normal subgroups. For every vector chamber D, $\widehat{P}_\Omega = \widehat{N}_\Omega \cdot U_{\Omega+D} \cdot U_{\Omega-D}$, and one deduces immediately from (6.1.15) that

(1) $$\widehat{P}_\Omega \cap U_D^+ = U_{\Omega+D}, \quad \widehat{P}_\Omega \cap U_D^- = U_{\Omega-D}, \quad \widehat{P}_\Omega \cap N = \widehat{N}_\Omega.$$

The map $\Omega \mapsto \widehat{P}_\Omega$ is decreasing for inclusion. Moreover, for all $n \in N$ and all non-empty $\Omega \subset A$,

$$nP_\Omega n^{-1} = P_{\nu(n)(\Omega)}, \qquad n\widehat{P}_\Omega n^{-1} = \widehat{P}_{\nu(n)(\Omega)}.$$

One checks at once that in the case of a discrete valuation the notations $\widehat{N}_\Omega$ and $\widehat{P}_\Omega$ agree with those of §4.

Proposition (7.1.9). — *Let Ω be a non-empty subset of A. If the valuation φ is dense, then $\widehat{N}_{\mathrm{cl}(\Omega)} = \widehat{N}_\Omega$ and $\widehat{P}_{\mathrm{cl}(\Omega)} = \widehat{P}_\Omega$.*

Proof. Since $U_{\mathrm{cl}(\Omega)} = U_\Omega$ (7.1.2), the first equality implies the second. The fixed-point set L of $\widehat{N}_\Omega$ in A is an affine subspace whose direction L^0 is the fixed-point subspace of ${}^v\nu(\widehat{N}_\Omega)$ in ${}^vW$, hence the intersection of the kernels of the roots containing it. Thus L is an intersection of closed half-spaces equipollent to affine roots and containing Ω; density of φ makes each such half-space an intersection of affine roots, so L is closed and $\mathrm{cl}(\Omega) \subset L$.

When φ is discrete the equalities of (7.1.9) still hold if Ω contains a non-empty open set, but need not hold in general.

Remark (7.1.10). — *Let $x \in A$; then $\widehat{N}_x = \nu^{-1}(\widehat{W}_x)$, where $\widehat{W}_x$ is the stabilizer of x in $\widehat{W} = \nu(N)$. When G is generated by H and the U_a (the "simply connected" case) and φ is discrete, $\widehat{W} = W$ is the affine Weyl group of the root system ${}^v\Sigma$ (1.3.8) and is generated by the reflections in the walls of A. It is known that $\widehat{W}_x$ is then generated by the reflections in the walls of A passing through x (1.3.5). Hence $\widehat{N}_x = N_x$ and $\widehat{P}_x = P_x$.*

This is no longer necessarily correct when φ is not discrete, even assuming G generated by the U_a and all Γ_a groups.

Examples.

1. Suppose $\Phi = \{a, -a\}$ of type A_1 and φ special. Then W is the group of transformations $x \mapsto \varepsilon x + \gamma a^\vee$ with $\varepsilon = \pm 1$ and $\gamma \in \widetilde{\Gamma}_a$, where $\widetilde{\Gamma}_a$ is the group generated by Γ_a (6.2.20). The points of A lying on a wall are those of the form $\frac{1}{2}\gamma a^\vee$ with $\gamma \in \Gamma_a$, whereas the points stabilized by some non-trivial element of W are the $\frac{1}{2}\gamma a^\vee$ with $\gamma \in \widetilde{\Gamma}_a$. So if Γ_a is not a group there are points $x \in A$ with $\widehat{N}_x \neq N_x$, even assuming $\widehat{W} = W$.
2. Let K be a field with a valuation ω, let $\Gamma = \omega(K^*)$, and suppose $\Gamma/2\Gamma$ is a vector space of dimension > 1 over $\mathbf{F}_2$. Take for G the group of K-rational points of a simple algebraic group split over K of type G_2 (necessarily simply connected), with the valued root datum defined in (6.2.3) b). Let $\lambda, \mu \in \Gamma$ have images in $\Gamma/2\Gamma$ linearly independent over $\mathbf{F}_2$, and put $x = \frac{1}{2}(\lambda a^\vee + \mu b^\vee)$, where a, b are the two roots of a basis of the root system of G. One checks easily that the symmetry about x lies in the stabilizer of x in $\widehat{W} = W$, yet no wall of A passes through x. Hence $\widehat{N}_x \neq N_x$.

Proposition (7.1.11). — *Let $\Omega \subset A$, $\Omega \neq \emptyset$. Then $\widehat{P}_\Omega = \bigcap_{x \in \Omega} \widehat{P}_x$.*

Proof. One first proves $\widehat{P}_\Omega \cap \widehat{P}_x = \widehat{P}_{\Omega \cup \{x\}}$, using the decomposition $g = nvu$ of (7.1.8) together with (6.1.15), (DR 6), (7.1.8)(1) and (6.4.9); induction then gives $\widehat{P}_{\{x_1,\dots,x_k\}} = \widehat{P}_{x_1} \cap \cdots \cap \widehat{P}_{x_k}$. It remains to treat an increasing directed family (Ω_i) with union Ω: one may assume all Ω_i contain a common point x, so $\widehat{N}_x/H$ is finite; passing to a cofinal subfamily and using (DR 6), the components of g are independent of i, and lie in H, $\bigcap_i U_{\Omega_i + D} = U_{\Omega + D}$ and $\bigcap_i U_{\Omega_i - D} = U_{\Omega - D}$ respectively.

Corollary (7.1.12). — *Let Ω and Ω' be non-empty subsets of A. Then $\widehat{P}_\Omega \cap \widehat{P}_{\Omega'} = \widehat{P}_{\Omega \cup \Omega'}$.*

Remark (7.1.13). — *For non-empty $\Omega, \Omega' \subset A$ one has $f_{\mathrm{cl}(\Omega \cup \Omega')} = \sup(f_\Omega, f_{\Omega'})$. One might try to generalize (7.1.12) by showing that $U_f \cap U_g \subset HU_{\sup(f,g)}$ whenever f and g are concave on Φ — which would imply that $\sup(f, g)$ is concave as well. But this assertion is in general false. For example, take for G the group of K-rational points of a split simple algebraic group of* *type G_2 over a field K with a non-improper valuation, equipped with the root datum of (6.2.3) b). Let (a,b) be a basis of Φ with b the long root, and put, for $c \in \Phi$,*

$$f(c) = 0 \ \text{ if } c = \pm a,\ \pm(2b+3a), \qquad f(c) = 0+ \ \text{ otherwise};$$
$$g(c) = 0 \ \text{ if } c = \pm(b+a),\ \pm(b+3a), \qquad g(c) = 0+ \ \text{ otherwise}.$$

One checks easily that f and g are concave and that $U_f \cap U_g \not\subset HU_{\sup(f,g)}$: indeed $N_{\sup(f,g)} \subset H$, whereas $N_f \cap N_g$ contains an element whose image in ${}^vW$ equals -1.

7.2. Chambers and facets of A. —

(7.2.1) Let X be a set and $\mathscr{F}$ a filter base on X; the *germ* of $\mathscr{F}$ is the filter generated by $\mathscr{F}$. If $Y \subset X$, one says the germ γ of $\mathscr{F}$ is *contained in* Y if some element of γ (or of $\mathscr{F}$) is contained in Y. More generally, if there is $Z \in \gamma$ with $Y \cap Z = \emptyset$ one puts $\gamma \cap Y = \emptyset$; otherwise the intersections $Z \cap Y$, $Z \in \gamma$, form a filter base on X whose germ is written $\gamma \cap Y$, and it is contained in Y.

If X is a topological space, the *closure* of γ, written $\overline{\gamma}$, is the germ associated with the filter base formed by the closures of the elements of γ (or of $\mathscr{F}$). One says γ *has non-empty interior* if the interiors of the elements of γ are non-empty; these then form a filter base whose germ is called the *interior* of γ. One says γ is *open* if it equals its interior.

(7.2.2) Let $\mathscr{F}$ be a filter base on the affine space A. The family of the U_Ω for $\Omega \in \mathscr{F}$ is increasing and directed for inclusion, so its union is a subgroup of G depending only on the germ γ of $\mathscr{F}$; it is written U_γ. The subgroups N_γ, P_γ, $\widehat{N}_\gamma$, $\widehat{P}_\gamma$ and the root system Φ_γ are defined analogously.

(7.2.3) Let D be a vector chamber. The sectors of A of direction D form a filter base; the associated germ is written $\gamma(D)$ and called the *sector germ* of direction D of A. Immediately,

$$U_{\gamma(D)} = U_D^+ \qquad \text{and} \qquad P_{\gamma(D)} = \widehat{P}_{\gamma(D)} = H \cdot U_D^+.$$

(7.2.4) Let x be a point of A and E a non-empty convex cone with vertex 0 in V, open in the vector subspace it spans. Write $\gamma(x,E)$ for the germ of the filter base formed by the subsets X of A with the following two properties: X is an intersection of affine roots and of complements of affine roots, and there is a neighborhood Y of x in A such that $X \cap (x+E) \supset Y \cap (x+E)$. One puts

$$U_{x,E} = U_{\gamma(x,E)}, \qquad P_{x,E} = P_{\gamma(x,E)}, \qquad \text{etc.}$$

A *facet* of A is a germ of the form $\gamma(x,E)$.[3] For a germ $\gamma(x,E)$ to be a facet it is necessary and sufficient that, for every root $a \in \Phi_x$, the convex cone E either be contained in the hyperplane $\{v \in V \mid a(v) = 0\}$ or not meet that hyperplane.

(7.2.4 (continued)) This is in particular the case when E is a vector facet of Φ; one then puts $F_{x,E} = \gamma(x,E)$. If E is a vector chamber, the facet $F_{x,E}$ is called a *chamber* of A and is also written $C_{x,E}$.

Two facets $F_{x,E}$ and $F_{x',E'}$ are equal if and only if they are contained in the same affine roots, equivalently if and only if $P_{x,E} = P_{x',E'}$. When φ is dense this forces $x = x'$. In all cases $F_{x,E} = F_{x,E'}$ if and only if the canonical projections of E and E' into the dual of the vector subspace of V spanned by Φ_x lie in a single vector facet of Φ_x. The preimage E'' of this vector facet in V — also called, by abuse of language, a vector facet of Φ_x in V — is therefore the largest convex cone of V with $F_{x,E} = F_{x,E''}$, and $E'' \mapsto F_{x,E''}$ is a bijection from the set of vector facets of Φ_x in V onto the set of facets of the form $F_{x,E}$. The vector facet E'' is called

[3] *Translator's note.* The scan of this sentence was not fully legible; the delimiting clause on the admissible cones E should be checked against the original before the definition is relied upon.

the *direction* of $F_{x,E''}$ at x, and the *dimension* of $F_{x,E''}$ is the dimension of the vector subspace spanned by E''. One checks easily that this dimension does not depend on the choice of x, and that the set of $X \in F_{x,E''}$ which are convex subsets of $x + E''$ all of whose points are internal is a base of the filter $F_{x,E''}$. A facet F is a chamber if and only if $\dim F = \dim V$.

Every facet F is contained in the closure of at least one chamber C; one then has $P_C \subset P_F$. More generally, a facet F is contained in the closure of a facet F' if and only if $P_F \supset P_{F'}$.

(7.2.5) When φ is discrete one recovers, up to a canonical identification, the notions of chamber and facet already seen: a facet (resp. chamber) in the sense of (7.2.4) is precisely the filter of all subsets of A containing a facet (resp. chamber) in the sense of §2.

When φ is dense one sees immediately that $\gamma(x, E)$ is the filter of all subsets X of A for which there is a neighborhood Y of x in A with $X \cap (x + E') \supset Y \cap (x + E')$, where E' is the direction of $\gamma(x, E)$ at x.

In all cases, the *center* of a facet $F_{x,E}$ is the limit along the filter $\gamma(x, E)$ of the barycenters of the compact subsets belonging to $\gamma(x, E)$. When φ is discrete one recovers the barycenter of the facet $F_{x,E}$; when φ is dense, the center of $F_{x,E}$ is x.

(7.2.6) Let $x \in A$ and let D be a vector chamber. Since the chamber $C_{x,D}$ is open in A (in the sense of (7.2.1)), every element of $\widehat{N}_{x,D}$ fixes all points of a non-empty open subset of A and hence belongs to H. Therefore $\widehat{P}_{x,D} = P_{x,D}$, and one puts

$$B_{x,D} = P_{x,D} = \widehat{P}_{x,D}.$$

One checks immediately that $U_{x,D} = U_{f_{x,D}}$, where $f_{x,D} : \Phi \to \widetilde{\mathbf{R}}$ is defined by

$$\text{(1)} \qquad \begin{cases} f_{x,D}(a) = -a(x) & \text{if } a \in \Phi_D^+, \\ f_{x,D}(a) = (-a(x))+ & \text{if } a \in \Phi_D^-. \end{cases}$$

One has $\Phi_{x,D} = \emptyset$, $N_{x,D} = H$ and, taking account of $f_{x,D}(2a) = 2f_{x,D}(a)$ when $a, 2a \in \Phi$, (6.4.9) gives

$$\text{(2)} \qquad B_{x,D} = H \cdot U_{x,D} = \prod_{a \in \Phi_D^{+\mathrm{red}}} U_{a,-a(x)} \cdot H \cdot \prod_{a \in \Phi_D^{-\mathrm{red}}} U_{a,-a(x)+}.$$

More generally, if Δ is another vector chamber,

$$\text{(3)} \qquad B_{x,D} = \prod_{a \in \Phi_\Delta^{+\mathrm{red}}} U_{a,f_{x,D}(a)} \cdot H \cdot \prod_{a \in \Phi_\Delta^{-\mathrm{red}}} U_{a,f_{x,D}(a)}.$$

Recall (6.4.9) that (2) and (3) are in fact decompositions of $B_{x,D}$ as a (set-theoretic) direct product, and that the position of the factor H, placed here "in the middle", is immaterial.

More generally still, let $F = \gamma(x, E)$ be a facet. Then $U_F = U_{f_F}$, where $f_F : \Phi \to \widetilde{\mathbf{R}}$ is defined by

$$\begin{aligned} f_F(a) &= -a(x) && \text{if } a \text{ is positive or zero on } E, \\ f_F(a) &= -a(x)+ && \text{if } a \text{ is strictly negative on } E. \end{aligned}$$

(7.2.7) Let $x \in A$ and consider the function $f_x^* : \Phi \to \widetilde{\mathbf{R}}$ associated as in (6.4.23) with the linear function $f_x : a \mapsto -a(x)$. One has $f_x^*(a) = -a(x)+$ for all $a \in \Phi_x$. Resuming the notation of (6.4.18) and putting $H_x^* = H_{f_x, f_x^*}$ — a normal subgroup of H — it was seen in (6.4.23) and (6.4.27) that

$$P_x^* = H_x^* \cdot U_{f_x^*}$$

is a normal subgroup of $P_x = H \cdot U_{f_x}$; moreover the quotient group $\overline{G}_x = P_x/P_x^*$ carries a root datum $(\overline{T}, (\overline{U}_a)_{a \in \Phi_x})$ of type Φ_x, where $\overline{T}$ (resp. $\overline{U}_a$) is the canonical image of H (resp. $U_{a,-a(x)}$)

in $\overline{G}_x$. One then sees easily that, for a vector chamber D, *the subgroup $B_{x,D}$ is the preimage in P_x of the minimal parabolic subgroup $\overline{T}\,\overline{U}_D^+$ of $\overline{G}_x$* (6.1.12), generated by $\overline{T}$ and the $\overline{U}_a$ for $a \in \Phi_x \cap \Phi_D^+$. More generally the $P_{x,E}$ are the preimages of the parabolic subgroups of $\overline{G}_x$ associated with the various vector facets of Φ_x.

Note also that P_x^* is a normal subgroup of $\widehat{P}_x$ and that the canonical injection of P_x/P_x^* into $\widehat{P}_x/P_x^*$ is adapted to the Tits system of $\overline{G}_x$ ((1.2.13) and (6.1.12)).

7.3. Iwasawa decomposition and Bruhat decomposition. —

This subsection proves that the subgroups $B_{x,D}$ attached to the chambers of A give rise to "Iwasawa decompositions" $G = U_D^+ \cdot N \cdot B_{x,D}$ and to "Bruhat decompositions" $G = B_{x,D} \cdot N \cdot B_{x,D}$. When φ is discrete this simply recovers earlier results ((1.2.7) and (5.1.3)) by another method. Note however that the "Schiffmann formula" (7.3.3) has not yet been proved even in the discrete case.

Proposition (7.3.1). — *Let D and D' be two vector chambers and let x be a point of A.*

(i) $G = U_D^+ \cdot N \cdot B_{x,D'}$.

(ii) More precisely, the natural map from $\widehat{W} = N/H$ to the set of double cosets $U_D^+\backslash G/B_{x,D'}$ is bijective.

Proof. Put $U = U_D^+$, $B = B_{x,D'}$, $Z = U \cdot N \cdot B$ and $f(a) = f_{x,D'}(a)$; write B_a for the subgroup generated by $U_{a,f(a)}$, $U_{-a,f(-a)}$ and H, so that $B_a = HU_{a,f(a)}U_{-a,f(-a)} = HU_{-a,f(-a)}U_{a,f(a)}$ by (6.3.2). The proof of (i) has three steps.

(1) For $a \in \Phi$, the group L_a generated by U_a, U_{-a} and T is contained in $Z_a = U_a \cdot T \cdot \{1, m_a\} \cdot B_a$, where m_a is any element of $M_a^0 \subset N$ (cf. (6.1.2)(2)). By (6.1.2)(7) it suffices that $m_a u \in Z_a$ for $u \in U_a$, which is clear if $\varphi_a(u) \geq f(a)$ and follows from the decomposition $u = v'mv''$ and (6.2.2)(V 5 bis) otherwise.

(2) For a a simple root for D: $U_{-a} \cdot Z \subset Z$. One uses (DR 2) — the subgroup U'_a of U^+ generated by the U_b, $b \in \Phi_D^+$, $b \neq a$, is normalized by U_a and U_{-a}, with $U^+ = U'_a \cdot U_a$ (6.1.6) — together with step (1), reducing to $un \in Z$ for $n \in N$, $u \in U_{-a}$, which follows by applying (1) to the root $-b$ where $b = -{}^v\nu(n^{-1})(a)$.

(3) Hence Z is stable under left multiplication by the U_{-a} for $a \in \Pi_D$, and evidently also by the U_a for $a \in \Phi_D^+$ and by T, hence by all of G (6.1.4). This proves (i).

(7.3.1 (4)) If $n, n' \in N$ satisfy $n' \in U_D^+ nB$, put $\Delta = {}^v\nu(n^{-1})(D)$. By (7.2.6) (3), $n'^{-1}n \in B\,U_\Delta^+ = H\,(U_\Delta^- \cap B)\,U_\Delta^+$, so that $n'^{-1}n \in H$ by (6.1.15). This gives (ii).

Corollary **(7.3.2)**. — *(i) The natural map $N/N_x \to U_D^+\backslash G/P_x$ (resp. $N/\widehat{N}_x \to U_D^+\backslash G/\widehat{P}_x$) is bijective.*

(ii) If x is a special point of A (6.2.13)*, the natural map $T/H \to U_D^+\backslash G/P_x$ is bijective.*

Proof. For (i) it is enough, given (7.3.1) (i), to see that $n' \in U_D^+ nP_x$ forces $n \in n'N_x$ (and likewise with hats). With $\Delta = {}^v\nu(n^{-1})(D)$ one has $n'^{-1}n \in P_x\,U_\Delta^+ = N_x\,(U_\Delta^- \cap P_x)\,U_\Delta^+$ by (7.1.4) and (7.1.8), and the claim follows from (6.1.15). If x is special, N is the semi-direct product of N_x by T, whence (ii); note that then $\widehat{P}_x = P_x$. □

Proposition **(7.3.3)**. — *Let $x \in A$, let D be a vector chamber, $a \in \Phi_D^-$, $v \in U_a$ and $n \in N$. If $v \in U_D^+ nP_x$, then either*

$$\varphi_a(v) \geqslant -a(x), \quad v \in \widehat{P}_x, \quad n \in \widehat{N}_x,$$

or else

$$a\big(\nu(n)(x) - x\big) > 0 \quad \textit{and} \quad \varphi_a(v) = -a(x) - \tfrac{1}{2}\,a\big(\nu(n)(x) - x\big)$$

("Schiffmann's formula").

Proof. Suppose $k = \varphi_a(v) < -a(x)$. By (6.2.2) (3) one may write $v = u'mu''$ with $u', u'' \in U_{-a,-k}$ and $m \in M_{a,k}$. As $U_{-a,-k} \subset \widehat{P}_x \cap U_D^+$, (7.3.2) gives $m \in n\widehat{N}_x$, whence $\nu(n)(x) = \nu(m)(x) = x - ka^\vee - a(x)a^\vee$ and $a(\nu(n)(x) - x) = -2k - 2a(x)$. □

This is the analogue of a formula of G. Schiffmann for real semi-simple Lie groups of real rank one ([33]).

Theorem **(7.3.4)**. — *Let $x, x' \in A$ and let D, D' be vector chambers.*

(i) $G = B_{x,D}\, N\, B_{x',D'}$.

(ii) More precisely, the canonical map of N into the set of double cosets modulo $B_{x,D}$ and $B_{x',D'}$ induces a bijection $\widehat{W} = N/H \xrightarrow{\sim} B_{x,D}\backslash G/B_{x',D'}$.

(7.3.5) *Notation for the proof.* For $x, x' \in A$ and vector chambers D, D': a hyperplane of A through x and x' parallel to a wall of Φ cannot meet the sector $x' + D'$, so there is a unique vector chamber $D(x; x', D')$ such that the sector $x + D(x; x', D')$ contains the intersection of a neighborhood of x' with $x' + D'$. Let $w(x, D; x', D')$ be the unique element of ${}^vW$ with

$$(1) \qquad D(x; x', D') = w(x, D; x', D') \,.\, D$$

and let $\lambda(x, D; x', D')$ be its length in ${}^vW$ with respect to the reflections in the walls of D.

Lemma **(7.3.6)**. — *Let $x, x' \in A$, let D, D' be vector chambers and let L be an open half-line with origin o in V, contained in D. Let $g \in G$ and $n \in N$ with $g \in B_{x,D}\, n\, B_{x',D'}$.*

(i) One of the following holds:

(i 1) *$g \in B_{z,D}\, n\, B_{x',D'}$ for every $z \in x + L$;*

(i 2) *there are a point $y \in x + L$ and an element $n' \in N$ with*

$$g \in B_{z,D}\, n\, B_{x',D'} \quad \text{for all } z \in]x, y[, \qquad g \in B_{y,D}\, n'\, B_{x',D'},$$
$$\lambda\big(y, D; n'.x', {}^v\nu(n')(D')\big) > \lambda\big(x, D; n.x', {}^v\nu(n)(D')\big).$$

(ii) $g \in B_{z,D}\, N\, B_{x',D'}$ for every $z \in x + L$.

(7.3.7) *How the lemma gives the theorem.* Let $g \in G$. By (7.3.1) there are $n \in N$, $u \in U_D^-$ with $g \in unB_{x',D'}$; write $u = \prod_{a \in \Phi_D^{+\mathrm{red}}} u_a$, $u_a \in U_{-a}$. Since $a(v) > 0$ for $a \in \Phi_D^{+\mathrm{red}}$, $v \in L$, one finds $y \in x - L$ with $-a(y) + \varphi_{-a}(u_a) > 0$ for all such a; then $u_a \in U_{-a,a(y)+}$ and $u \in B_{y,D}$. Hence $g \in B_{y,D}\, n\, B_{x',D'}$ and (7.3.6) (ii) (applied with y in place of x and $z = x$) yields $g \in B_{x,D}\, N\, B_{x',D'}$, i.e. (i).

For (ii), put $B = B_{x,D}$, $B' = B_{x',D'}$ and suppose $n' \in BnB'$ with $n, n' \in N$. Then $1 \in Bn''B''$ with $n'' = nn'^{-1}$, $B'' = n'B'n'^{-1}$, so $n'' \in BB''$. Put $x'' = n'.x'$, $D'' = {}^v\nu(n')(D')$ (so $B'' = B_{x'',D''}$), $\Delta = D(x; x'', D'')$ and $\gamma = (x + \Delta) \cap C_{x'',D''}$; then $B'' = P_\gamma$. By (7.1.5), $BB'' \subset U_\Delta^+ H\, U_\Delta^-$, and since $N \cap U_\Delta^+ H U_\Delta^- = H$ (6.1.15 (a)) one gets $n'' \in H$, i.e. $n' \in nH$.

(7.3.8) *Proof of the lemma.* Assume (i); then (ii) follows by iterating: in case (i 1) there is nothing to do, and in case (i 2) one applies (i) again with y, n' in place of x, n. Since lengths in ${}^vW$ are bounded, after finitely many steps one lands in case (i 1). One obtains points $y_0 = y, y_1, \ldots, y_k$ with $y_{i+1} \in y_i + L$ and elements $n_1 = n', n_2, \ldots, n_{k+1}$ of N such that $g \in B_{z,D}\, n\, B_{x',D'}$ for $z \in [x, y_0[$, $g \in B_{z,D}\, n_i\, B_{x',D'}$ for $z \in [y_{i-1}, y_i[$, and $g \in B_{z,D}\, n_{k+1}\, B_{x',D'}$ for $z = y_k$ or $z \in y_k + L$; whence (ii).

There remains (i). Replacing g by gn^{-1}, x' by $n.x'$ and D' by ${}^v\nu(n).D'$ one reduces to $n = 1$, i.e. $g \in B_{x,D}B_{x',D'}$. Put $\Delta = D(x; x', D')$ and

$$\Psi = \Phi_D^{-\mathrm{red}} \cap \Phi_\Delta^-, \qquad \Psi' = \Phi_D^{-\mathrm{red}} \cap \Phi_\Delta^+.$$

By (6.4.9) every $b \in B_{x,D}$ is uniquely $b = \prod_{a\in\Phi_D^{+\mathrm{red}}} u_a \cdot \prod_{a\in\Psi} u_a \cdot \prod_{a\in\Psi'} u_a \cdot h$ with $u_a \in U_{a,f_{x,D}(a)}$, $h \in H$.

For $a \in \Phi_D^+$ and $z \in x + L$ one has $a(z) > a(x)$, hence $U_{a,f_{x,D}(a)} \subset B_{z,D}$. By definition of Δ one has $x' \in x + \overline{D}$ (with $\overline{D} = \overline{\Delta}$) and there is $v \in D'$ with $x' + v \in x + \Delta$; so for $a \in \Phi_\Delta^+$, $a(x') \geqslant a(x)$ and $a(x') + a(v) > a(x)$, whence $a(x') > a(x)$ when $a \in \Phi_\Delta^+ \cap \Phi_{D'}^-$. It follows that $U_{a,f_{x,D}(a)} \subset B_{x',D'}$ for all $a \in \Phi_\Delta^+$, in particular for $a \in \Psi'$. Multiplying g by an element lying in every $B_{z,D}$, $z \in x + L$, one reduces to $g \in uB_{x',D'}$ with $u = \prod_{a\in\Psi} u_a$, $u_a \in U_{a,-a(x)+}$.

If $u = 1$, (i 1) holds. (This is the case when $\Delta = -D$, i.e. when $\lambda(x, D; x', D')$ is maximal, since then $\Psi = \varnothing$.)

If $u \neq 1$, there is a unique $y \in x + L$ with $a(y) \geqslant -\varphi_a(u_a)$ for all $a \in \Psi$ and with equality for at least one $a \in \Psi$. Then

(1) $$u \in U_{y-\Delta} \cap U_{y-D} \subset U_y, \qquad u \notin U_{y^*},$$

y^* being the concave function attached to the linear function y as in (6.4.23). By (6.4.23) and (6.4.27) there is a subgroup $H^* \subset H$ with $H^*U_{y^*}$ normal in HU_y and with $\overline{G} = HU_y/H^*U_{y^*}$ carrying a generating root datum $(\overline{T}, (\overline{U}_a)_{a\in\Phi_y})$ of type Φ_y, where $\overline{T}$ is the image of H, $\overline{U}_a$ the image of $U_{a,y(a)}$, and $\Phi_y = \{a \in \Phi : a(y) \in \Gamma'_a\}$. Putting $\Delta' = D(y; x', D')$, (6.1.15) provides $u' \in U_{y+D}$, $u'' \in U_{y+\Delta'}$ and $n' \in N_y$ with

(2) $$u \in U_{y^*}\, u'\, n'\, u''.$$

Since $U_{y+D} \cup U_{y^*} \subset B_{y,D}$ and $U_{y+\Delta'} \subset B_{x',D'}$ (because $C_{x',D'} \cap (y + \Delta') \neq \varnothing$), one gets $u \in B_{y,D}n'B_{x',D'}$ and $g \in B_{y,D}n'B_{x',D'}$: the second relation of (i 2). For $z \in]x, y[$ one has $a(z) > -\varphi_a(u_a)$ for all $a \in \Psi$, so $u \in B_{z,D}$ and $g \in B_{z,D}B_{x',D'}$: the first relation.

It remains to prove

(3) $$\lambda\big(y, D; n'.x', {}^v\nu(n')(D')\big) > \lambda(x, D; x', D').$$

Step 1.

(4) $$\lambda(y, D; x', D') \geqslant \lambda(x, D; x', D').$$

Indeed, let $\mathscr{M}$ be the set of hyperplanes of V that are kernels of elements of Φ. The length of $w \in {}^vW$ is the number of members of $\mathscr{M}$ separating D and $w(D)$ ([5], ch. VI, §1, no. 6). Hence $\lambda(z, D; x', D')$ is, for $z \in A$, the cardinality of the set $\mathscr{M}_z$ of $M \in \mathscr{M}$ such that $z + M$ separates $z + D$ from a neighborhood of x in $x' + \overline{D'}$. For fixed M, the set of $z \in A$ with $M \in \mathscr{M}_z$ is the open half-space $x' + M + D$ or the closed one $x' + M + \overline{D}$, and its intersection with $x + L$ is a half-line. Hence $\mathscr{M}_y \supset \mathscr{M}_x$, giving (4). In fact this shows more: $w' = w(y, D; x', D')$ dominates $w = w(x, D; x', D')$, i.e. $w' = ww''$ for some $w'' \in {}^vW$ with $\ell(w') = \ell(w) + \ell(w'')$.

Step 2.

(5) $$\lambda(y, D; x', D') > \lambda(x, D; x', D') \quad \text{whenever } n' \in H.$$

Equality in (4) forces $w' = w$, i.e. $\Delta' = \Delta$. The image $\overline{u}$ of u in $\overline{G}$ lies in $\overline{U}_D^- \cap \overline{U}_\Delta^-$, while the images $\overline{u}', \overline{u}''$ of u', u'' lie in $\overline{U}_D^+$, $\overline{U}_{\Delta'}^+$ respectively. If $n' \in H$ and $\Delta = \Delta'$ then $\overline{u} \in \overline{U}_D^- \cap \overline{U}_\Delta^- \cap \overline{U}_D^+\overline{T}\overline{U}_\Delta^+$, whence $\overline{u} = 1$ by (DR 6) and (6.1.6); so $u \in U_{y^*}$, contradicting (1).

Step 3. By (4) and (5), (3) will follow from

(6) $$\lambda\big(y, D; n'(x'), {}^v\nu(n')(D')\big) \geqslant \lambda(y, D; x', D'), \text{ with equality only if } n' \in H.$$

Since $n' \in N_y$ one checks that $w\big(y, D; n'.x', {}^v\nu(n')(D')\big) = {}^v\nu(n')\, w(y, D; x', D')$. Setting $t = {}^v\nu(n')$ and $w' = w(y, D; x', D')$, (6) is equivalent to

(7) $$\ell(tw') \geqslant \ell(w'), \text{ with equality only if } t = 1.$$

Identify the Weyl group ${}^vW'$ of Φ_y with a subgroup of ${}^vW$, and identify a vector chamber of Φ_y (a simplicial cone in a quotient of V) with its inverse image in V. Let E be the vector chamber of Φ_y containing D; let R (resp. R') be the reflections in the walls of D (resp. E), generating ${}^vW$ (resp. ${}^vW'$), and w_0 (resp. w'_0) the longest element. Then $w_0(D) = -D$, $w'_0(E) = -E$, so $w'_0(E) \supset w_0(D)$. Let $\overline{B} = \overline{T}\prod_{a\in\Phi_y\cap\Phi_D^+}\overline{U}_a$ be the minimal parabolic of $\overline{G}$ attached to E. Since $w'(D) = \Delta'$ one has $\overline{U}_{y+\Delta'} \subset w'\overline{B}w'^{-1}$, and (1), (2) give

$$\bar{u} \in w'_0\,\overline{B}\,w'^{-1}_0, \tag{8}$$

$$\bar{u} \in \overline{B}\,t\,w'\,\overline{B}\,w'^{-1}. \tag{9}$$

Take a reduced decomposition $(r_k, \dots, r_1)$ of $w'^{-1}w_0$ in ${}^vW$ w.r.t. R, so $w_0 = w'r_k\cdots r_1$ and $\ell(w') = \ell(w_0) - k$ ([5], ch. VI, §1, no. 6, cor. 3 to prop. 17). For $1 \leqslant i \leqslant k$ put $w_i = w'r_k\cdots r_{i+1}$, so $w_k = w'$ and $w_i = w_{i+1}r_i$; the chambers $w_i(D)$, $w_{i+1}(D)$ are adjacent and the reflection s_i in their common wall is $w_{i+1}w_i^{-1} = w_ir_iw_i^{-1}$. Let w'_i be the unique element of ${}^vW'$ with $w'_i(E) \supset w_i(D)$; this is consistent with w'_0, and $w'_k = w_k = w'$ since $w' \in {}^vW'$. The chambers $w'_i(E)$, $w'_{i+1}(E)$ are adjacent or equal according as $s_i \in {}^vW'$ or not; in the first case $s_i = w'_{i+1}w'^{-1}_i$.

By (6.1.15 (a)) there is for each $i = 0, \dots, k$ a unique $\overline{w}'_i \in {}^vW'$ with $\overline{B}\,\bar{u}\,w'_i\,\overline{B} = \overline{B}\,\overline{w}'_i\,\overline{B}$, and $\overline{w}'_0 = w'_0$ by (8), $\overline{w}'_k = tw'$ by (9). Let $I = \{0, \dots, k-1\}$ and $I_1 = \{i \in I : w'_{i+1} = w'_i\}$. Clearly

$$i \in I_1 \implies w'_{i+1} = w'_i \text{ and } \overline{w}'_{i+1} = \overline{w}'_i, \tag{10}$$

and we saw that

$$i \notin I_1 \implies w'_{i+1}w'^{-1}_i = w_{i+1}w_i^{-1} = s_i. \tag{11}$$

For $i \notin I_1$, $w'^{-1}_{i+1}w'_i = w'^{-1}_i s_i w'_i$ is a reflection in a wall of E, i.e. an element $r'_i \in R'$; by (6.1.15) and axiom (T 3) of Tits systems,

$$\overline{B}\overline{w}'_{i+1}\overline{B} = \overline{B}\bar{u}w'_{i+1}\overline{B} = \overline{B}\bar{u}w'_ir'_i\overline{B} \subset \overline{B}\overline{w}'_i\overline{B}r'_i\overline{B} \subset \overline{B}\overline{w}'_i\overline{B} \cup \overline{B}\overline{w}'_ir'_i\overline{B}.$$

Hence $I - I_1$ splits into $I_2 = \{i : \overline{w}'_{i+1} = \overline{w}'_i\}$ and $I_3 = \{i : \overline{w}'_{i+1} = \overline{w}'_ir'_i\}$.

Finally put $\overline{w}_i = \overline{w}'_iw'^{-1}_iw_i$ $(i = 0, \dots, k)$; then $\overline{w}_0 = w_0$ and $\overline{w}_k = tw'\,w'^{-1}w' = tw'$. For $i \in I$ set $d_i = \overline{w}^{-1}_{i+1}\overline{w}_i = w^{-1}_{i+1}w'_{i+1}\overline{w}'^{-1}_{i+1}\overline{w}'_iw'^{-1}_iw_i$. Then $d_i = r_i$ if $i \in I_1$ (by (10)); $d_i = w^{-1}_{i+1}s_iw_i = 1$ if $i \in I_2$ (by (11)); $d_i = w^{-1}_{i+1}w_i = r_i$ if $i \in I_3$. Consequently

$$w_0 = \overline{w}_0 = \overline{w}_k(\overline{w}_k^{-1}\overline{w}_{k-1})\cdots(\overline{w}_1^{-1}\overline{w}_0) = tw'\prod_{i\in I_1\cup I_3} r_i,$$

whence $\ell(tw') \geqslant \ell(w_0) - \mathrm{Card}(I_1 \cup I_3) \geqslant \ell(w_0) - k = \ell(w')$, with equality only if $I_2 = \varnothing$, which forces $w_0 = tw_0$ and $t = 1$. This proves (7), the lemma (7.3.6), and therefore the theorem (7.3.4). □

Proposition **(7.3.9)**. — *Let F be a facet of A and let L be a subgroup of G containing H and such that $L \cap U_a = P_F \cap U_a$ for every $a \in \Phi$. Let $N_F^\dagger$ be the stabilizer of F in N and put $P_F^\dagger = N_F^\dagger P_F$. Then $P_F^\dagger$ is a subgroup of G and $P_F \subset L \subset P_F^\dagger$.*

Proof. The first assertion holds because $N_F^\dagger$ normalizes P_F; and $P_F \subset L$ since P_F is generated by H and the $U_a \cap P_F$. As P_F contains some $B_{x,D}$, (7.3.4) shows that if $L \not\subset P_F^\dagger$ there is $n \in N \cap L$ with $n.F \neq F$. Then some affine root α contains exactly one of F, $\nu(n).F$; replacing if need be n by n^{-1} and α by $\nu(n^{-1}).\alpha$, assume $\alpha \supset F$ and $\alpha \not\supset \nu(n).F$. Then $U_\alpha \subset L$ and $U_{\nu(n^{-1}).\alpha} = n^{-1}U_\alpha n \subset L$, which by (7.2.6) and (6.4.9) forces $\nu(n^{-1}).\alpha \supset F$ — absurd. □

7.4. The building of G. —

(7.4.1) Consider the following relation between two elements (g, x) and (h, y) of $G \times A$:

$$\text{(R)} \qquad \text{there exists } n \in N \text{ such that } y = \nu(n)(x) \text{ and } g^{-1}hn \in \widehat{P}_x.$$

Since $\widehat{P}_x = P_x\widehat{N}_x$ and $\widehat{N}_x$ is the stabilizer of x in N, (R) is unchanged if $\widehat{P}_x$ is replaced by P_x.

(R) is an equivalence relation. Let $g, h, k \in G$, $x, y, z \in A$, $m, n \in N$ with $y = \nu(m)(x)$, $z = \nu(n)(y)$, $g^{-1}hm \in \widehat{P}_x$, $h^{-1}kn \in \widehat{P}_y$. By (7.1.8), $\widehat{P}_y = m\widehat{P}_x m^{-1}$; hence

$$h^{-1}gm^{-1} = m(m^{-1}h^{-1}g)m^{-1} \in \widehat{P}_y, \qquad x = \nu(m^{-1})(y),$$
$$g^{-1}knm = g^{-1}hm \cdot m^{-1}h^{-1}knm \in \widehat{P}_x\, m^{-1}\widehat{P}_y m = \widehat{P}_x, \qquad z = \nu(nm)(x).$$

The first gives symmetry, the second transitivity; reflexivity is clear.

Definition **(7.4.2)**. — *The* building *of G (equipped with the root datum $(T, (U_a))$ and with the valuation φ of it) is the set ${}^{\varphi}\mathscr{I} = \mathscr{I}$ obtained as the quotient of $G \times A$ by the equivalence relation* (R).

The building ${}^{\varphi}\mathscr{I}$ clearly depends only on the equipollence class of φ.

The map j sending $x \in A$ to the class of $(1, x)$ is injective: if $j(x) = j(y)$ there is $n \in N$ with $y = \nu(n)(x)$ and $n \in \widehat{P}_x$; as $N \cap \widehat{P}_x = \widehat{N}_x$ (7.1.8), $y = x$. *We identify A with its image in $\mathscr{I}$ via j from now on.*

Left translation of G on itself is compatible with (R) and gives an action $(g, x) \mapsto g.x$ of G on $\mathscr{I}$ for which the class of (g, x) equals $g.x$. For $n \in N$ this extends the action of n on A given by ν: $n.x = \nu(n)(x)$. Clearly $\mathscr{I} = G.A$.

Finally, one checks easily that for φ discrete this construction recovers, up to isomorphism, the building of G attached to the Tits system of G (6.5).

Proposition **(7.4.3)**. — *Let ψ be a valuation equivalent to φ* (6.2.5)*. There is one and only one map $i : {}^{\varphi}\mathscr{I} \to {}^{\psi}\mathscr{I}$ with the properties:*

a) $i(g.x) = g.i(x)$ for $x \in {}^{\varphi}\mathscr{I}$, $g \in G$;

b) the restriction of i to A is an affine map of $A = \varphi + V$ onto $\psi + V$.

If $\lambda : \Phi \to \mathbf{R}_+^*$ is defined by $\psi = \lambda\varphi + V$, then $i(\varphi) = \lambda\varphi$.

Proof. One has $\lambda(\varphi + V) = \psi + V$. Let $i_1 : A \to \psi + V$ be the affine map $x \mapsto \lambda x$. Clearly

$$(1) \qquad {}^{\varphi}P_x = {}^{\psi}P_{i_1(x)} \qquad (x \in A)$$

and, by (6.2.5) (1),

$$(2) \qquad i_1(n.x) = n.i_1(x) \qquad (x \in A,\ n \in N),$$

whence

$$(3) \qquad {}^{\varphi}\widehat{N}_x = {}^{\psi}\widehat{N}_{i_1(x)} \quad \text{and} \quad {}^{\varphi}\widehat{P}_x = {}^{\psi}\widehat{P}_{i_1(x)} \qquad (x \in A).$$

So i_1 extends uniquely to a map i with (a) and (b).

Conversely let i satisfy (a). By the definition of the building,

$$(4) \qquad {}^{\varphi}\widehat{P}_x = {}^{\psi}\widehat{P}_{i(x)} \qquad \text{for all } x \in A.$$

Let x be a special point of A and $y \in A$, $y \neq x$. There is an affine root α containing x but not y; by (7.1.8) (1) and (6.4.9), $U_\alpha \subset {}^{\varphi}\widehat{P}_x$ and $U_\alpha \not\subset {}^{\varphi}\widehat{P}_y = {}^{\psi}\widehat{P}_{\lambda y}$. By (4), $i(x) \neq \lambda y$; hence $i(x) = \lambda x$ for every special point x. If i also satisfies (b), then $i|A = i_1$, whence uniqueness and the last assertion. □

Proposition **(7.4.4)**. — *Let Ω be a non-empty subset or a facet of A. The fixer of Ω in G (for the action of G on $\mathscr{I}$) is equal to $\widehat{P}_\Omega$.*

(The fixer of a germ is by definition the union of the fixers of its elements.) This follows from the definition of $\mathscr{I}$ when Ω is a point, and the general case follows using (7.1.11).

Proposition **(7.4.5)**. — *Let $a \in \Phi$ and $u \in U_a^*$; put $k = \varphi_a(u)$. The set of points of A fixed by u is the affine root*

$$\alpha_{a,k} = \{x \in A \mid a(x) + k \geqslant 0\}.$$

Indeed, for every $x \in A$ one has $\widehat{P}_x \cap U_a = U_{a,-a(x)}$, and $u \in \widehat{P}_x$ is equivalent to $k \geqslant -a(x)$. Note that there are points of $\mathscr{I}$ fixed by u outside A (cf. (7.4.33)).

Corollary **(7.4.6)**. — *For every non-empty closed subset Ω of A, the set of points of A fixed by all elements of $\widehat{P}_\Omega$ is Ω.*

This follows from (7.4.5) when Ω is an affine root; the general case follows since $\mathrm{cl}(\Omega)$ is the intersection of the affine roots containing Ω.

Definition **(7.4.7)**. — *An* apartment *of $\mathscr{I}$ is a transform of A by an element of G.*

Proposition **(7.4.8)**. — *Let $g \in G$. The intersection $A \cap g^{-1}.A$ is a closed subset of A, and there exists $n \in N$ such that $g.x = n.x$ for all $x \in A \cap g^{-1}.A$.*

Proof. We may assume $\Omega = A \cap g^{-1}.A \neq \varnothing$. Let $\mathscr{X}$ be the set of subsets X of Ω with $g^{-1}N \cap \widehat{P}_X \neq \varnothing$. By the very definition of $\mathscr{I}$, $\mathscr{X}$ contains all one-point subsets of Ω. Let $X \in \mathscr{X}$, $y \in \Omega$ and $n_X, n_y \in N$ with $g^{-1}n_X \in \widehat{P}_X$, $g^{-1}n_y \in \widehat{P}_y$. By (7.1.6) there is a vector chamber D with

$$n_X^{-1}n_y \in \widehat{P}_X\widehat{P}_y \subset \widehat{N}_X\, U_D^-\, U_D^+\, \widehat{N}_y$$

(using $\widehat{P}_X = \widehat{N}_X P_X$ and $\widehat{P}_y = P_y\widehat{N}_y$). Hence there are $n'_X \in \widehat{N}_X$, $n'_y \in \widehat{N}_y$ with $n_X'^{-1}n_X^{-1}n_y n'_y \in N \cap U_D^- U_D^+ = \{1\}$. Putting $n = n_X n'_X = n_y n'_y$ we get $g^{-1}n \in \widehat{P}_X \cap \widehat{P}_y = \widehat{P}_{X\cup\{y\}}$. By induction on k, every finite subset $\{x_1, \ldots, x_k\}$ of Ω lies in $\mathscr{X}$.

Now fix $x_0 \in \Omega$ and write Ω as the union of an increasing filtering family of finite subsets $X_i \ni x_0$. There are $n_0, n_i \in N$ with $g^{-1}n_0 \in \widehat{P}_{x_0}$ and $g^{-1}n_i \in \widehat{P}_{X_i} \subset \widehat{P}_{x_0}$. Then $n_i^{-1}n_0 \in \widehat{N}_{x_0}$, and since $\widehat{N}_{x_0}/H$ is finite we may (passing to a cofinal subfamily and multiplying the n_i on the right by elements of H) assume n_i independent of i. Then $g^{-1}n_i \in \bigcap_i \widehat{P}_{X_i} = \widehat{P}_\Omega$ (7.1.11), which gives the last assertion. Moreover there is $n' \in \widehat{N}_\Omega$ with $g^{-1}n_i n' \in P_\Omega = P_{\mathrm{cl}(\Omega)}$ (7.1.2), so $g.x \in A$ for all $x \in \mathrm{cl}(\Omega)$, i.e. Ω is closed. □

Corollary **(7.4.9)**. — *Let Ω be a subset or a facet of A.*

(i) The fixer $\widehat{P}_\Omega$ of Ω acts transitively on the set of apartments containing Ω.

(ii) Suppose Ω is a chamber and let $b \in P_\Omega$. Then $b.x = x$ for all $x \in A \cap b.A$.

Proof. Let $A' = g.A$ be an apartment containing Ω and $n \in N$ with $g^{-1}.x = n.x$ for all $x \in A \cap g.A$. Then $gn \in \widehat{P}_\Omega$ and $A' = gn.A$, whence (i). If Ω is a chamber and $g \in P_\Omega$, then by (7.2.6) $n \in N \cap P_C = H$, whence (ii). □

Compare (7.4.9) with (2.5.8) and (2.5.9).

Corollary **(7.4.10)**. — *The subgroup N (resp. H) is the stabilizer (resp. the fixer) of A in G.*

Proof. Let $g \in G$ with $g.A = A$. By (7.4.8) there is $n \in N$ with $g^{-1}n \in \widehat{P}_A$. But $U_A = \{1\}$, since A is contained in no affine root, so $\widehat{P}_A = \widehat{N}_A = H$; and H is by definition the kernel of ν. □

Compare (7.4.10) with (2.2.5).

(7.4.11) As in (2.2.8), it follows from (7.4.10) that on every apartment A' of $\mathscr{I}$ there is one and only one structure of Euclidean affine space such that, for every $g \in G$ with $A' = g.A$, the map $x \mapsto g.x$ is an isomorphism of Euclidean affine spaces from A onto A'. In particular each apartment A' carries a distance $d_{A'}$.

Moreover (7.4.8) shows at once that if M is a subset of an apartment, the set $g.\,\mathrm{cl}(g^{-1}.M)$ does not depend on the choice of $g \in G$ with $M \subset g.A$. This set is called the *enclosure* of M and denoted $\mathrm{cl}(M)$; it is contained in every apartment containing M. One says that M is *closed* if $\mathrm{cl}(M) = M$.

Definition **(7.4.12)**. — *A* sector *(resp.* sector germ, chamber, facet*) of $\mathscr{I}$ is a transform of a sector (resp. sector germ, chamber, facet) of A by an element of G.*

Of course one identifies a germ of A, defined by a filter base $\mathscr{F}$ on A, with the germ defined by $\mathscr{F}$ in $\mathscr{I}$. Definition (7.4.12) is justified by:

Proposition **(7.4.13)**. — *(i) If a facet X of $\mathscr{I}$ meets an apartment $A' = g.A$ of $\mathscr{I}$, there is a facet Y of A with $X = g.Y$, and X is contained in A'.*

(ii) If a sector (resp. sector germ) X of $\mathscr{I}$ is contained in an apartment $A' = g.A$, there is a sector (resp. sector germ) Y of A with $X = g.Y$.

Proof. (i) Let $x \in A$, let E be a vector facet of Φ_x and $h \in G$ with $X = h.F_{x,E}$; suppose X meets $g.A$. For every convex neighborhood Ω of x in A one has $h.(\Omega \cap (x+E)) \cap g.A \neq \varnothing$. For Ω small enough there is an open half-line E' with origin o in V, $E' \subset E$, with $h.(\Omega \cap (x + E')) \subset g.A$. Then $F_{x,E} = F_{x,E'}$, and by (7.4.8) there is $n \in N$ with $g^{-1}h.y = n.y$ for all $y \in A \cap g^{-1}h.A$. Hence $n^{-1}g^{-1}h \in \widehat{P}_{x,E'} = \widehat{P}_{x,E}$ and $X = h.F_{x,E} = g.F_{x'',E''}$ with $x'' = n.x$, $E'' = {}^{v}\nu(n)(E)$. The proof of (ii) is analogous and simpler. □

In particular, the facets (resp. chambers) of $\mathscr{I}$ meeting A are exactly the facets (resp. chambers) of A.

Corollary **(7.4.14)**. — *Let F, F' be facets of A with F' contained in the closure of F in A* (7.2.1), *and let $g \in G$. Every apartment of $\mathscr{I}$ containing $g.F$ also contains $g.F'$.*

This follows from (7.4.13), since $A \cap h.A$ is a closed subset of A for every $h \in G$ (7.4.8).

Proposition **(7.4.15)**. — *Let Ω, Ω' be subsets or facets of A and let $\widehat{W}_\Omega$ (resp. $\widehat{W}_{\Omega'}$) be the image of $\widehat{N}_\Omega$ (resp. $\widehat{N}_{\Omega'}$) in $\widehat{W}$. The natural map $\widehat{W}_\Omega \backslash \widehat{W} / \widehat{W}_{\Omega'} \to \widehat{P}_\Omega \backslash G / \widehat{P}_{\Omega'}$ is bijective.*

The proof is identical with that of (4.2.1), replacing the reference (2.5.8) by (7.4.9).

Proposition **(7.4.16)**. — *Suppose the valuation φ is dense. Let $x \in A$ and let E be a vector facet.*

(i) The only point of A fixed by $P_{x,E}$ is x.

(ii) If $g \in G$ is such that $gP_{x,E}g^{-1} \subset \widehat{P}_x$, then $g \in \widehat{P}_x$.

Proof. Let $y \in A$, $y \neq x$. As φ is dense there is an affine root containing x but not y; by (7.4.5) this gives (i). For (ii), by (7.3.4) we may assume $g \in N$; but for $n \in N$ one has $nP_{x,E}n^{-1} = P_{n.x,E'}$ with $E' = {}^{v}\nu(n)(E)$, so (ii) follows from (i). □

Corollary **(7.4.17)**. — *If φ is dense, the normalizer of P_x (resp. $\widehat{P}_x$) equals $\widehat{P}_x$, for every $x \in A$.*

Theorem **(7.4.18)**. — *(i) Two chambers (resp. facets, points) of $\mathscr{I}$ lie in a common apartment.*

(ii) A chamber (resp. facet, point) of $\mathscr{I}$ and a sector germ lie in a common apartment.

(iii) Two germs of sectors of $\mathscr{I}$ lie in a common apartment.

Proof. These are geometric translations of (7.3.4), (7.3.1) and (6.1.15) a) respectively. By (7.4.14) it suffices to treat chambers and germs of sectors. Let C, C' be chambers of $\mathscr{I}$; transforming both by one element of G we may assume $C = C_{x,D}$ and $C' = g.C_{x',D'}$ with $x, x' \in A$, D, D' vector chambers, $g \in G$. By (7.3.4), $g = bnb'$ with $b \in B_{x,D}$, $n \in N$, $b' \in B_{x',D'}$; then $C = b.C$ and $C' = bn.C_{x',D'}$ lie in the common apartment $b.A$. One proves (ii) similarly from (7.3.1) and (iii) from (6.1.15); see also (4.2.5). □

Theorem **(7.4.19)**. — *Let C be a chamber of $\mathscr{I}$ and A' an apartment containing C. There is one and only one map ρ from $\mathscr{I}$ into A' such that, for every $b \in B_C$, the restriction of ρ to the* *apartment $b.A'$ coincides with the restriction of $x \mapsto b^{-1}.x$. The restriction of ρ to A' is the identity, and if x is a point of A' adherent to C then $\rho^{-1}(x) = \{x\}$.*

For the proof see (2.3.2) and (2.3.4). As in (2.3.5) this map is denoted $\rho_{A';C}$ and called the *retraction of $\mathscr{I}$ onto A' with center C.*

Proposition **(7.4.20)**. — *(i) There is one and only one distance d on $\mathscr{I}$ whose restriction to every apartment A' of $\mathscr{I}$ is the Euclidean distance of A' (7.4.11). It is G-invariant.*

(ii) Let C be a chamber of $\mathscr{I}$, x a point adherent to C and A' an apartment containing C; let ρ be the retraction of $\mathscr{I}$ onto A' with center C. For all $y, z \in \mathscr{I}$,

$$d(\rho(y), \rho(z)) \leqslant d(y, z).$$

If moreover C, y and z lie in one apartment, then $d(\rho(y), \rho(z)) = d(y, z)$. In particular $d(x, \rho(y)) = d(x, y)$ for all $y \in \mathscr{I}$.

(iii) Let $x, y \in \mathscr{I}$ and $S = \{z \in \mathscr{I} \mid d(x, y) = d(x, z) + d(z, y)\}$. Then S is contained in every apartment A' containing x and y, and coincides with the segment $[xy]$ of the affine space A'; one writes $S = [xy]$. Every isometry of $\mathscr{I}$ — in particular every element of G — fixing x and y fixes every point of $[xy]$.

(iv) Let $x, y, z, z' \in \mathscr{I}$ with $z \in [xy]$, and $\varepsilon \in \mathbf{R}_+$ with $d(x, z') \leqslant d(x, z) + \varepsilon d(x, y)$ and $d(y, z') \leqslant d(y, z) + \varepsilon d(x, y)$. Then

$$d(z, z')^2 \leqslant 4\, d(x, z)\, d(y, z)\, \varepsilon + d(x, y)^2 \varepsilon^2.$$

(v) Let $x, y \in \mathscr{I}$ and $t \in [0, 1]$; write $tx + (1 - t)y$ for the unique $z \in [xy]$ with $d(y, z) = t\, d(x, y)$. The map $(t, x, y) \mapsto tx + (1-t)y$ from $[0, 1] \times \mathscr{I} \times \mathscr{I}$ to $\mathscr{I}$ is continuous. The metric space $\mathscr{I}$ is contractible.

For the proof see nos. (2.5.1)–(2.5.4) and (2.5.14)–(2.5.16). In the proof of (2.5.3) (i) one replaces the use of facets by the following lemma.

Lemma **(7.4.21)**. — *Let C be a chamber, A' an apartment, and x, y two points of A'. There is a monotone sequence $(x_0 = x, x_1, \ldots, x_m = y)$ of points of the segment $[xy]$ of A' such that for each $i = 0, \ldots, m - 1$ the segment $[x_i x_{i+1}]$ of A' and the chamber C lie in one apartment.*

Proof. We may assume $x \neq y$. For each $z \in]xy[$ there are chambers C_z^+, C_z^- contained in A' with $]xz[\cap C_z^- \neq \varnothing$ and $]zy[\cap C_z^+ \neq \varnothing$. Let A_z^+ (resp. A_z^-) be an apartment containing C and C_z^+ (resp. C_z^-). There are then $z^+ \in]zy[$ and $z^- \in]xz[$ with $[zz^+] \subset A_z^+$ and $[z^- z] \subset A_z^-$. One defines x^+, A_x^+ and y^-, A_y^- analogously. There are finitely many points $z_1, \ldots, z_k$ of $[xy]$ such that the $]z_i^- z_i^+[$ together with $[xx^+[$ and $]y^- y]$ form an open cover of $[xy]$. Ordering $x, x^+, z_i^-, z_i, z_i^+, y^-, y$ monotonically gives the required sequence. □

Remark **(7.4.22)**. — *Let C, C' be chambers of A and $g = bnb' \in G$ with $b \in B_C$, $n \in N$, $b' \in B_{C'}$. Since the restriction of $\rho_{A;C}$ to the apartment $b.A \supset g.C'$ coincides with the action of b^{-1}, one has $\rho_{A;C}(g.C') = n.C'$. Likewise $\rho_{A;C}(g.x) = n.x$ as soon as $g \in B_C n \widehat{P}_x$ ($x \in A$, $n \in N$).*

(7.4.23) Let v be an element of length 1 of V, $E = E(v)$ the vector facet containing it and $\widetilde{E} = \widetilde{E}(v)$ the vector subspace generated by E. Put

$$c(v) = \sin(\alpha/2)$$

where α is the smallest of the angles that v makes with its transforms distinct from itself under the various elements of ${}^vW$.

If $\widetilde{E} = V$, put $k(v) = +\infty$; if $\widetilde{E} \neq V$, put

$$k(v) = \sin \beta$$

where β is the smallest of the angles that v makes with the vector subspaces generated by the vector facets not contained in $\widetilde{E}$ and of dimension $\leqslant \dim \widetilde{E}$. Finally let $\delta(v)$ be the distance from v to the complement of E in $\widetilde{E}$, and put

$$h(v) = \inf\{c(v), k(v), \delta(v)\}.$$

Note that $h(v) > 0$.

Proposition **(7.4.24)**. — *Keep the notation of* (7.4.23). *Let $x, y_1, z \in A$ with $x \in [zy_1]$ and $y_1 - z \in \mathbf{R}_+^* v$. Let $g \in \widehat{P}_z$ and $y \in \mathscr{I}$ with $g^{-1}.y \in A$ and*

$$d(y, y_1) + d(g^{-1}.y, y_1) \leqslant 2c(v)\, d(x, y_1).$$

Then $g \in \widehat{P}_x$.

Proof. The set of $t \in [zx]$ with $g \in \widehat{P}_t$ is a segment $[zm]$. Suppose $m \neq x$. Let D be a vector chamber with $v \in \overline{D}$ and ρ the retraction onto A with center the chamber $C_{m,D}$. Let $n \in N$ with $g \in B_{m,D} n B_{m,D}$. Since $g.m = m$ we have $n.m = m$; and by (7.4.22) there is $m' \in]my_1]$ with $\rho(g.p) = n.p$ for $p \in [mm']$. As $\rho(g.[my_1])$ is the segment $[m\, \rho(g.y_1)]$, we get $[m\, \rho(g.y_1)] = n.[my_1]$ and

$$\rho(g.y_1) - y_1 = d(m, y_1)\big({}^v\nu(n)(v) - v\big).$$

Now ${}^v\nu(n)(v) \neq v$: otherwise, since an element of $B_{m,D}$ fixes the points of a neighborhood of m in $m + \overline{D}$, we would have $g.(m + tv) = m + tv$ for all small $t \geqslant 0$, contradicting the definition of m. Hence the length of ${}^v\nu(n)(v) - v$ is at least $2c(v)$ and

$$\begin{aligned} d(\rho(g.y_1), y_1) &\geqslant 2c(v)d(m, y_1) > 2c(v)d(x, y_1) \geqslant d(y, y_1) + d(g^{-1}.y, y_1),\\ d(\rho(g.y_1), y_1) &> d(y, y_1) + d(y, g.y_1) \geqslant d(\rho(y), y_1) + d(\rho(y), \rho(g.y_1))\\ &\geqslant d(y_1, \rho(g.y_1)), \end{aligned}$$

a contradiction. Hence $m = x$ and $g \in \widehat{P}_x$. □

Proposition **(7.4.25)**. — *Let A' be an apartment of $\mathscr{I}$ and γ a sector germ contained in A'. There is one and only one map, denoted $\rho_{A';\gamma}$, from $\mathscr{I}$ into A' with the following property: for every bounded subset M of $\mathscr{I}$ there is a sector $\mathfrak{C}$ of A' belonging to γ such that, for every chamber C contained in $\mathfrak{C}$, one has $\rho_{A';\gamma}(y) = \rho_{A';C}(y)$ for all $y \in M$.*

Proof. We may take $A' = A$ and $\gamma = \gamma(D)$, D a vector chamber; uniqueness is clear. Let $M \subset \mathscr{I}$ be bounded, $y_1 \in A$, $v \in -D$. Put $r = \sup\{d(y_1, y) \mid y \in M\}$ and choose $x \in y_1 - \mathbf{R}_+^* v$ so that the ball of A with center y_1 and radius r lies in the sector $x - D$. Let $y \in M$ and $g \in G$ with $y' = g.y \in A$. Write $g = hnu$ with $h \in \widehat{P}_{y'}$, $n \in N$, $u \in U_D^+$ (7.3.1); then $u.y \in A$. There is $z \in x - \mathbf{R}_+^* v$ with $u \in U_{z+D}$, and by (7.4.19) $u.y = \rho_{A;C}(y)$ for every chamber $C \subset z + D$, whence

$$d(y_1, u.y) \leqslant d(y_1, y) \leqslant r \leqslant c(v)\, d(y_1, x).$$

By (7.4.24), $u \in \widehat{P}_x$, so $u \in U_{x+D}$ and

$$u.y = \rho_{A;C}(y) \tag{1}$$

for every chamber $C \subset x + D$ and every $y \in M$. □

As in 2.9, $\rho_{A';\gamma}$ is called the *retraction of $\mathscr{I}$ onto A' relative to the sector germ γ*. It is the identity on A' and decreases distances.

Proposition **(7.4.26)**. — *Keep the notation of* (7.4.23). *Let $z \in A$ and $g \in \widehat{P}_z$; put $\tau = \sup\{t \in \mathbf{R} \mid g.(z + tv) = z + tv\}$. For every $t \in \mathbf{R}$ with $t \geqslant \tau$,*

$$2c(v)(t - \tau) \leqslant d\big(z + tv,\ g.(z + tv)\big) \leqslant 2(t - \tau).$$

Proof. Let $t \geqslant \tau$. Put $y = y_1 = z + tv$, $d = d(y, g.y)$ and $x = y - \left(\frac{1}{2}c(v)^{-1}d\right)v$. By (7.4.24), $g \in \widehat{P}_x$, whence $\tau \geqslant t - \frac{1}{2}c(v)^{-1}d$: this is the first inequality. For the second, put $u = z + \tau v$; then $d(z + tv, g.(z + tv)) \leqslant d(u, z + tv) + d(u, g.(z + tv)) = 2(t - \tau)$. □

Note that when Φ has rank one, $c(v) = 1$.

Corollary **(7.4.27)**. — *Let D be a vector chamber with $v \in \overline{D}$; let $y \in A$, $u \in U_D^+$ and $\lambda \in \mathbf{R}_+$.*

(i) If $d(y, u.y) \leqslant \lambda$, then $u \in U_{y+\frac{1}{2}c(v)^{-1}\lambda v+D}$.

(ii) If $u \in U_{y+\lambda v+D}$, then $d(y, u.y) \leqslant \sup(0, 2\lambda)$.

Proof. Let $s = \inf\{t \in \mathbf{R} \mid u \in U_{y+tv+D}\}$; since $u \in U_D^+$, $s < \infty$. If $s \leqslant 0$ then $u.y = y$ and there is nothing to prove. If $s > 0$, apply (7.4.26) with $-v$ in place of v, $g = u$, $z = y + sv$, $t = s$: then $\tau = 0$ and $2c(v)s \leqslant d(y, u.y) \leqslant 2s$, which gives (i) and (ii). □

Remark **(7.4.28)**. — *Let $x \in A$, $u \in U_D^+$ and $n \in N$ with $u \in \widehat{P}_x n \widehat{P}_x$. Then $d(x, u.x) = d(x, n.x)$, and* (7.4.27) *gives a two-sided inequality between the "size" of n, measured by $d(x, n.x)$, and that of u, measured by the smallest $\lambda \in \mathbf{R}_+$ with $u \in U_{x+\lambda v+D}$. Note that for $v \in D$ the subgroups $U_{x+\lambda v+D}$, $\lambda \in \mathbf{R}$, form a decreasing filtration of U_D^+ and allow one to define a "valuation" φ_v of U_D^+.*

Proposition **(7.4.29)**. — *Keep the notation of* (7.4.23). *Let $x, x', y_1, y_1' \in A$ be such that $y_1 - x$, $x - x'$ and $x' - y_1'$ lie in $\mathbf{R}_+^* v$. Put $L = x + \widetilde{E} = x' + \widetilde{E}$ and let $y \in \widehat{P}_x.L$, $y' \in \widehat{P}_{x'}.L$ satisfy*

$$d(y, y_1) < h(v)\, d(x, y_1) \qquad \textit{and} \qquad d(y', y_1') < h(v)\, d(x', y_1').$$

Then $(x - E) \cap (x' + E) \subset \mathrm{cl}(\{y, y'\})$.

Proof. Let C be a chamber of A to which y_1' is adherent; put $\rho = \rho_{A;C}$ and choose $g \in \widehat{P}_C$ with $\rho(y) = g.y$. Then

$$d(y, y_1) + d(g.y, y_1) \leqslant 2d(y, y_1) < 2c(v)\, d(x, y_1),$$

so $g \in \widehat{P}_x$ by (7.4.24). Since $\mathrm{cl}(\{x\} \cup C) \supset (y_1' + E) \cap (x - E) \supset (x' + E) \cap (x - E)$, we get $g \in \widehat{P}_C \cap \widehat{P}_{(x'+E)\cap(x-E)}$.

$\rho(y) \in x + E$. There are $z \in L$ and $h \in \widehat{P}_x$ with $\rho(y) = gh.z$; by (7.4.8) there is $n \in \widehat{N}_x$ with $\rho(y) = n.z$, whence $\rho(y) - x = {}^v\nu(n)(z - x)$. As $z - x \in \widetilde{E}$, $\rho(y) - x$ lies in a vector facet of dimension $\leqslant \dim \widetilde{E}$. If $\rho(y) - x \notin \widetilde{E}$ then

$$d(y_1, y) \geqslant d(y_1, \rho(y)) \geqslant k(v) d(y_1, x) \geqslant h(v) d(y_1, x),$$

contrary to hypothesis. Moreover $d(y_1, \rho(y)) \leqslant d(y_1, y) < \delta(v) d(y_1, x)$, which forces $\rho(y) = g.y$ to lie in $x + E$.

Now take $t \in \mathbf{R}_+$ large enough that $g.y \in x + tv - E$; put $z = x + tv$ and $y'' = g.y'$. Arguing as above but with $x', z, y_1', y'', -v, -E$ in place of x, y_1', y_1, y, v, E, and using $d(y_1', y'') = d(y_1', y')$ and $y'' \in \widehat{P}_{x'}.y' \subset \widehat{P}_{x'}.L$, one finds

$$g' \in \widehat{P}_z \cap \widehat{P}_{(x'+E)\cap(z-E)} \subset \widehat{P}_{g.y} \cap \widehat{P}_{(x'+E)\cap(x-E)}$$

with $g'g.y = g.y \in x + E$ and $g'g.y' = g'.y'' \in x' - E$. Then

$$\mathrm{cl}(\{y, y'\}) = g^{-1}g'^{-1}.\mathrm{cl}(\{g'g.y, g'g.y'\}) \supset g^{-1}g'^{-1}.\big((x'+E)\cap(x-E)\big) = (x'+E)\cap(x-E). \quad □$$

Corollary **(7.4.30)**. — *Keep the notation of* (7.4.23) *and let λ be a real number with $0 \leqslant \lambda < 1$. There is a real number $\varepsilon > 0$ such that, for $x, x' \in A$ and $y, y' \in \mathscr{I}$, the relations $x' - x \in \mathbf{R}_+^* v$, $d(x, y) < \varepsilon d(x, x')$, $d(x', y') < \varepsilon d(x, x')$ and $\{y, y'\} \subset \widehat{P}_u.(x + \widetilde{E})$ for every $u \in [xx']$ imply the existence of $z, z' \in [yy'] \cap A$ with $d(z, z') \geqslant \lambda d(x, x')$.*

The derivation from (7.4.29) is very close to that of (2.8.10) from (2.8.9); we leave it to the reader.

Remark **(7.4.31)**. — *If E is a vector chamber, then $L = x + \widetilde{E} = A$ and the conditions $y \in \widehat{P}_x.L$, $y' \in \widehat{P}_{x'}.L$ (resp. $y, y' \in \widehat{P}_u.(x + \widetilde{E})$) of* (7.4.29) *and* (7.4.30) *hold automatically; indeed $\mathscr{I} = \widehat{P}_x.A$ for every $x \in A$ by* (7.4.9) (i) *and* (7.4.18) (i).

Corollary **(7.4.32)**. — *The apartment A is the smallest convex subset of $\mathscr{I}$ stable under T.*

One deduces (7.4.32) from (7.4.30) exactly as (2.8.11) from (2.8.10).

Proposition **(7.4.33)**. — *Let v be an element of length 1 of V contained in the vector chamber D. There is a constant $h > 0$ with the following property: if $x \in A$, $u \in U_{x+D}$ and $\lambda \in \mathbf{R}_+$, then u fixes every point of $\mathscr{I}$ lying in the ball with center $x + \lambda v$ and radius $h\lambda$. If Φ has rank one one may take $h = 1/2$; if Φ is of type A_1 one may take $h = 1$.*

Proof. Let x, u, λ be as stated; put $y = x + \lambda v$ and $c = c(v)^{-1}$. Let $z \in \mathscr{I}$; put $\delta = \lambda^{-1} d(y, z)$ and $t = \rho_{A;\gamma(D)}(z)$. By (7.4.25) (1) there is $u' \in U_D^+$ with $z = u'.t$. Since $d(y, t) \leqslant d(y, z)$,

$$d(y, u'.y) \leqslant d(y, z) + d(u'.t, u'.y) = d(y, z) + d(t, y) \leqslant 2\lambda\delta,$$

so $u' \in U_{y+c\lambda\delta v+D}$ by (7.4.27) (i). Put $x' = y + c\lambda\delta v$. By (6.4.32) there is a constant k depending only on v, with $0 < k \leqslant 1$, such that

$$(U_{x+D}, U_{x'+D}) \subset U_{x'+k(x-x')+D}.$$

Put $r = \delta(v)$ (7.4.23); one checks easily that $t \in x' + k(x - x') + D$ as soon as

$$c\delta - k(1 + c\delta) \leqslant -r^{-1}\delta.$$

Since $k \leqslant 1$ it suffices for this that $\delta \leqslant kr$. Under that condition one also has $t \in x + D$ and

$$u.z = uu'.t = uu'u^{-1}.t = u'(u'^{-1}, u).t = u'.t = z.$$

So one may take $h = kr$. If Φ has rank one then $r = 1$ and one may take $k = 1/2$ (6.4.32); if Φ is of type A_1 one may take $k = 1$. □

Corollary **(7.4.34)**. — *Suppose φ dense. There exist two constants $\sigma, \tau > 0$ such that for any points $x, y \in \mathscr{I}$ there is $g \in G$ satisfying:*

(1) *g fixes every point of $\mathscr{I}$ in the ball with center x and radius $\sigma d(x, y)$;*

(2) $d(y, g.y) \geqslant \tau d(x, y)$.

Proof. For $t \in V$, $t \neq 0$, let $\alpha(t)$ be the smallest of the angles formed by t and the $a^\vee$, $a \in \Phi$, and put $\alpha = \inf_{t\in V, t\neq 0} \cos\alpha(t) > 0$. Let v be an element of length 1 of V contained in a vector chamber D and h the corresponding constant of (7.4.33). We show that $\sigma = \frac{1}{3}h\alpha$ and $\tau = \frac{2}{3}c(v)\alpha$ work.

It suffices to treat $x, y \in A$ with $x - y \in \overline{D}$. Choose $a \in \Phi$ with $\cos\angle(x - y, a^\vee) \geqslant \alpha$; then $a \in \Phi_D^+$. Put $d = d(x, y)$ and choose $z \in [xy]$ with $-a(z) \in \Gamma_a$, $d(x, z) \geqslant \frac{1}{3}d$ and $d(y, z) \geqslant \frac{1}{3}d$ (possible since φ is dense). Let $u \in U_a$ with $\varphi_a(u) = -a(z)$. By (7.4.5) the fixed-point set of u in A is the half-space $\{t \in A \mid a(t) - a(z) \geqslant 0\}$; in particular $u \in U_{x-\lambda v+D}$ with $\lambda = a(x - z)a(v)^{-1} \geqslant \alpha d(x, z) \geqslant \frac{\alpha}{3}d$. By (7.4.33), $u.t = t$ for every $t \in \mathscr{I}$ with $d(x, t) \leqslant h\frac{\alpha}{3}d = \sigma d(x, y)$: this is (1).

Similarly $u \notin U_{y+\mu v+D}$ as soon as $\mu < a(z - y)a(v)^{-1}$, so by (7.4.27) (i)

$$d(y, u.y) \geqslant 2c(v)a(z - y)a(v)^{-1} \geqslant 2c(v)\alpha d(y, z) \geqslant \tau d(x, y),$$

which is (2). □

7.5. The completion of the building. —

(7.5.1) When φ is discrete, the building of G is a complete metric space (2.5.12). This is not so in general. We then write $\widehat{\mathscr{I}}$ for the completion of $\mathscr{I}$. The operations of G, as well as the retractions onto an apartment, extend by continuity to $\widehat{\mathscr{I}}$. From (7.4.20) (iv) one deduces easily that if x_i (resp. y_i) in $\mathscr{I}$ tend to x (resp. y) in $\widehat{\mathscr{I}}$, then for each $t \in [0,1]$ the points $tx_i + (1-t)y_i$ tend to a point $z \in \widehat{\mathscr{I}}$ depending only on t, x, y; one then puts $z = tx + (1-t)y$ and $[xy] = \{tx + (1-t)y \mid t \in [0,1]\}$. By continuity, (7.4.20) (iv) and (v) remain valid with $\mathscr{I}$ replaced by $\widehat{\mathscr{I}}$. It follows at once that $[xy]$ is the unique geodesic of $\widehat{\mathscr{I}}$ joining x to y; more precisely,

$$[xy] = \{z \in \widehat{\mathscr{I}} \mid d(x,z) + d(z,y) = d(x,y)\}.$$

Proposition **(7.5.2)**. — *Let v be an element of length 1 of V contained in a vector chamber D. For every point $x \in \widehat{\mathscr{I}}$ there is a triple $\big(y, (\lambda_p), (u_p)\big)$ satisfying:*

(1) *$y \in A$, $(\lambda_p)_{p\geqslant 1}$ is a decreasing sequence of real numbers tending to 0, and $(u_p)_{p\geqslant 1}$ is a sequence of elements of U_D^+;*

(2) *$u_{p+1} \in u_p U_{y+\lambda_p v+D}$ for every $p \geqslant 1$;*

(3) *$x = \lim_{p\to\infty} u_p.y$.*

Conversely, for every triple $\big(y, (\lambda_p), (u_p)\big)$ satisfying (1) *and* (2)*, the sequence $u_p.y$ converges to an element x of $\widehat{\mathscr{I}}$. In order that $x \in \mathscr{I}$ it is necessary and sufficient that there exist $u \in U_D^+$ with*

$$\text{(4)} \qquad u_p \in u U_{y+\lambda_p v+D} \quad \textit{for every } p \geqslant 1.$$

One then has $x = u.y$.

Proof. Let $x \in \widehat{\mathscr{I}}$ and let ρ be the retraction onto A relative to the germ of sector $\gamma(D)$, extended by continuity to $\widehat{\mathscr{I}}$. Choose points $x_p \in \mathscr{I}$ with $\varepsilon_p = d(x, x_p)$ decreasing to 0. Put $y = \rho(x)$, $y_p = \rho(x_p)$. For each p there is $u_p \in U_D^+$ with $x_p = u_p.y_p$ (7.4.25) (1). Since $d(u_p.y, u_p.y_p) = d(y, y_p) \leqslant d(x, x_p)$, after replacing the x_p by the $u_p.y$ and passing to a subsequence we may assume $x_p = u_p.y$ for all p.

For $q \geqslant p$ we have $d(u_q^{-1}u_p.y, y) = d(u_p.y, u_q.y) \leqslant 2\varepsilon_p$, so by (7.4.27) (i)

$$\text{(2 bis)} \qquad u_q^{-1}u_p \in U_{y+\lambda_p v+D} \qquad (q \geqslant p)$$

with $\lambda_p = c(v)^{-1}\varepsilon_p$. Clearly $\big(y, (\lambda_p), (u_p)\big)$ satisfies (1), (2) and (3).

Conversely let $\big(y, (\lambda_p), (u_p)\big)$ satisfy (1) and (2). Since (λ_p) decreases, (2 bis) holds; then (7.4.27) (ii) gives $d(u_p.y, u_q.y) = d(u_q^{-1}u_p.y, y) \leqslant 2\lambda_p$ for $q \geqslant p$, so $(u_p.y)$ converges to some $x \in \widehat{\mathscr{I}}$.

If $x \in \mathscr{I}$, write $x = u.y$ with $u \in U_D^+$. Then $d(u.y, u_p.y) = \lim_{q\to\infty} d(u_q.y, u_p.y) \leqslant 2\lambda_p$, so by (7.4.27) (i) $u \in u_p U_{x+c(v)^{-1}\lambda_p v+D}$ for every p. But for $q \geqslant p$ large enough $\lambda_p \geqslant c(v)^{-1}\lambda_q$, whence $u \in u_q U_{x+\lambda_p v+D}$, which with (2 bis) gives (4). Conversely, if (4) holds then (7.4.27) (ii) gives $d(u.y, u_p.y) \leqslant 2\lambda_p$, so $x = u.y \in \mathscr{I}$. □

(7.5.3) For $a \in \Phi$, call *ball with center $u \in U_a$ and radius $k \in \mathbf{R}$* in U_a the left coset $uU_{a,k}$.

Proposition **(7.5.4)**. — *In order that the building $\mathscr{I}$ be complete it is necessary and sufficient that the following condition hold for every $a \in \Phi^{\mathrm{red}}$:*

(MC) Every decreasing (for inclusion) sequence of balls of bounded radius of U_a has non-empty intersection.

Proof. Necessity. Let $a \in \Phi^{\mathrm{red}}$ and, for integers $p \geqslant 1$, let $u_p \in U_a$ and $k_p \in \mathbf{R}$ with $k_q \geqslant k_p$ and $u_q U_{a,k_q} \subset u_p U_{a,k_p}$ for $q \geqslant p$, and $k = \sup_p k_p < \infty$. Let D be a vector chamber with $a \in \Phi_D^+$ and v an element of length 1 of D. Let $y \in A$ with $a(y) = -k$ and put $\lambda_p = a(v)^{-1}(k - k_p)$, so that $U_{a,k_p} = U_a \cap U_{y+\lambda_p v+D}$. By (7.5.2) the sequence $(u_p.y)$ converges to a point $x \in \widehat{\mathscr{I}}$. If $\mathscr{I}$ is complete then $x \in \mathscr{I}$ and, by (7.5.2), the intersection of the $u_p U_{y+\lambda_p v+D}$ is non-empty. Let u be in it. By (6.1.6) there are $u_1 \in U_a$ and $u_2 \in \prod_{b\in\Phi_D^{+\mathrm{red}},\, b\neq a} U_b$ with $u = u_1 u_2$. Since $u_p^{-1} u_1 u_2 \in U_{y+\lambda_p v+D}$, we get $u_p^{-1} u_1 \in U_a \cap U_{y+\lambda_p v+D} = U_{a,k_p}$ (6.4.9), whence $\bigcap_p u_p U_{a,k_p} \neq \varnothing$.

Sufficiency. Suppose (MC) holds for every $a \in \Phi^{\mathrm{red}}$. By (7.5.2) it suffices to show that if $y \in A$, $u_p \in U_D^+$, $\lambda_p \in \mathbf{R}_+$ and $v \in D$ are such that $u_p U_{y+\lambda_p v+D} \supset u_q U_{y+\lambda_q v+D}$ for $q \geqslant p$, then $X = \bigcap_p u_p U_{y+\lambda_p v+D} \neq \varnothing$.

Arrange the roots of $\Phi_D^{+\mathrm{red}}$ in a nibbling order $(a_1, \dots, a_m)$ (1.3.15). For $i = 1, \dots, m$ put $U_i = U_{a_i}$ and let U'_i be the group generated by the U_j for $i < j \leqslant m$. By (DR 2) and (6.1.6), U'_i is normalized by U_i and U'_{i-1} is the semi-direct product of U_i by U'_i. Put moreover

$$U^p = U_{y+\lambda_p v+D}, \qquad U_i^p = U^p \cap U_i, \qquad U_i'^p = U^p \cap U'_i.$$

By (6.4.9),

$$U_{i-1}'^p = U_i^p\, U_i'^p \qquad \text{and} \qquad U_i^p = U_{a_i, -a_i(y)-a_i(v)\lambda_p} \supset U_{a_i,-a_i(y)}.$$

We show by induction on i that there exist $t_i \in U_D^+$ and $u_{p,i} \in U'_i$ (for $i = 0, \dots, m$ and $p \geqslant 1$) with

(1 ; i) $\qquad t_i u_p U^p \supset u_{p,i} U_i'^p \quad$ for all p;

(2 ; i) $\qquad u_{q,i} U_i'^q \subset u_{p,i} U_i'^p \quad$ for $q \geqslant p$.

This holds for $i = 0$ (with $U'_0 = U_D^+$, $U_0'^p = U^p$): take $t_0 = 1$, $u_{p,0} = u_p$. Let $i \geqslant 1$ and suppose t_{i-1} and the $u_{p,i-1}$ already constructed, satisfying (1 ; $i-1$) and (2 ; $i-1$). Let f be the homomorphism of U'_{i-1} onto U_i defined by $x \in f(x) U'_i$ for $x \in U'_{i-1}$. Then $f(u_{p,i-1} U_{i-1}'^p) = f(u_{p,i-1}) U_i^p$, and (2 ; $i-1$) together with (MC) and the relation $U_i^p \supset U_{a_i,-a_i(y)}$ shows that there exists $s \in \bigcap_p f(u_{p,i-1}) U_i^p$. Put $t_i = s^{-1} t_{i-1}$. Since $U_i^p U'_i = U'_i U_i^p$, there is exactly one $u_{p,i} \in U'_i$ with $s^{-1} u_{p,i-1} \in u_{p,i} U_i^p$. For every p,

$$t_i u_p U^p \supset s^{-1} u_{p,i-1} U_{i-1}'^p = u_{p,i} U_{i-1}'^p \supset \big(u_{p,i} U_{i-1}'^p\big) \cap U'_i = u_{p,i} U_i'^p$$

and, for $q \geqslant p$,

$$u_{q,i} U_i'^q = \big(s^{-1} u_{q,i-1} U_{i-1}'^q\big) \cap U'_i \subset \big(s^{-1} u_{p,i-1} U_{i-1}'^p\big) \cap U'_i = u_{p,i} U_i'^p,$$

whence (1 ; i) and (2 ; i).

Since $U'_m = \{1\}$ we have $u_{p,m} = 1$ for all p, and (1 ; m) gives $t_m^{-1} \in u_p U^p$ for all p, so $X \neq \varnothing$. □

Example **(7.5.5)**. — *Take up the notation of* (6.2.3) a) *(canonical valued root datum of $SL_2(K)$, K a field, commutative or not, endowed with a non-improper valuation ω) or of* (6.2.3) b) *(valued root datum of the group of rational points over K of a semi-simple algebraic group split over K, K a commutative field with a non-improper valuation ω). Each group U_a with the valuation φ_a is then isomorphic to the additive group of K with ω. Condition* (MC) *therefore says that the completion of K for ω is maximally complete* [34]. *Note that this always holds when ω is discrete, which matches the fact that the building of G is always complete when φ is discrete.*

Remark **(7.5.6)**. — *Condition* (MC) *may hold for every $a \in \Phi^{\mathrm{red}}$ without holding for every $a \in \Phi$. Take up for instance the notation of example* (6.2.3) d)*, taking for K the field of formal series with well-ordered exponents in a subgroup Γ of* $\mathbf{R}$ ([6], ch. VI, §3, exerc. 2)*, dense in* $\mathbf{R}$ *and with $\Gamma/2\Gamma$ infinite, with coefficients in a field k of characteristic* 2*, and taking for L the additive subgroup generated by the elements $u^2 X^\gamma$ with $u \in K$ and $\gamma \in \Gamma$. One sees easily that L is a subfield of K. On the other hand it is well known that K is maximally complete* (ibid., §5, exerc. 5)*, which shows that* (MC) *holds when a is a short root. But one sees easily that the completion of L is not maximally complete and that* (MC) *fails when a is a long root.*

(7.5.7) Let A be an apartment of $\mathscr{I}$ and C a chamber of A. Clearly the retraction $\rho_{A;C}$ extends by continuity to a map, still denoted $\rho_{A;C}$, from $\widehat{\mathscr{I}}$ onto A, and (7.4.20) (ii) remains valid with $\mathscr{I}$ replaced by $\widehat{\mathscr{I}}$. Similarly, a retraction with respect to a sector germ extends by continuity to $\widehat{\mathscr{I}}$.

(7.5.8) Suppose the valuation φ dense, and let σ, τ be the constants whose existence (7.4.34) guarantees. Let $\varepsilon > 0$ be small enough that $\sigma' = (1-2\varepsilon)\sigma - \varepsilon$ and $\tau' = (1-2\varepsilon)\tau - 2\varepsilon$ are strictly positive. Let x, y be two distinct points of $\widehat{\mathscr{I}}$ and choose $x', y' \in \mathscr{I}$ with $d(x, x')$ and $d(y, y')$ less than $\varepsilon d(x, y)$, so that $d(x', y') \geqslant (1-2\varepsilon)d(x, y)$. By (7.4.34) there is $g \in G$ fixing every point of $\mathscr{I}$ in the ball with center x' and radius $(1-2\varepsilon)\sigma d(x, y)$ and with $d(y', g.y') \geqslant (1-2\varepsilon)\tau d(x, y)$. By continuity g fixes every point of $\widehat{\mathscr{I}}$ in the ball with center x and radius $\sigma' d(x, y)$, and $d(y, g.y) \geqslant \tau' d(x, y)$. In other words, after decreasing the constants σ and τ, corollary (7.4.34) remains true with $\mathscr{I}$ replaced by $\widehat{\mathscr{I}}$.

7.6. Restriction to a subgroup. —

Definition **(7.6.1)**. — *A subset Φ_1 of Φ is said to be* quasi-closed (cf. [3], 3.8) *if, for all $a, b \in \Phi_1$, the commutator group (U_a, U_b) is contained in the group generated by the U_{pa+qb} for p, q strictly positive integers with $pa + qb \in \Phi_1$.*

Clearly a closed subset is quasi-closed.

(7.6.2) Throughout the rest of 7.6, Φ_1 denotes a quasi-closed root subsystem. We write G_1^0 for the subgroup of G generated by the U_a, $a \in \Phi_1$; N_1^0 for the subgroup generated by the M_a^0, $a \in \Phi_1$ ((6.1.2) (2)); and we put $T_1^0 = T \cap N_1^0$. We are moreover given a subgroup T_1 of T containing T_1^0, and we write G_1 for the subgroup generated by G_1^0 and T_1.

Proposition **(7.6.3)**. — $\big(T_1, (U_a, M_a^0 T_1)_{a\in\Phi_1}\big)$ *is a generating root datum of type Φ_1 in G_1, and $\varphi_1 = (\varphi_a)_{a\in\Phi_1}$ is a valuation of it.*

Proof. The only point that is not completely obvious is that φ_1 satisfies (V 3). Let $a, b \in \Phi_1$ and $h, k \in \mathbf{R}$ with $b \notin -\mathbf{R}_+^* a$. For an arbitrary order on the set E of elements of Φ of the form $pa + qb$ with $p, q \in \mathbf{N}^*$, p and q not both even, one has

$$(U_{a,k}, U_{b,h}) \subset \Big(\prod_{c=pa+qb\in E} U_{c,\,pk+qh}\Big) \cap \Big(\prod_{c\in E\cap\Phi_1} U_c\Big),$$

taking into account that if $2pa + 2qb \in \Phi_1$ $(p, q \in \mathbf{N}^*)$ then $pa + qb \in \Phi_1$, as a glance at the rank-2 root systems shows. Since the product map $\prod_{c\in E} U_c \to G$ is injective (6.1.6), (V 3) follows. □

The objects that §§6 and 7 attach to the valued root datum of G_1 will be denoted by the same letters as for G, but with the index or exponent 1. For example V_1^* is the subspace of V^* generated by Φ_1, and V_1 is the quotient of V by the intersection of the kernels of the $a \in \Phi_1$, an intersection we denote L_1. We write ${}^v\pi$ for the canonical map of V onto V_1 and π for the affine map of A onto A_1 defined by $\pi(\varphi + v) = \varphi_1 + {}^v\pi(v)$ for $v \in V$. One identifies ${}^vW_1$ with the subgroup of ${}^vW$ generated by the reflections r_a, $a \in \Phi_1$. Note that ${}^vW_1$ acts trivially on L_1.

It is immediate that $N_1 = N_1^0 T_1 \subset N$.

If f is a quasi-concave function on Φ_1, we denote by U_f^1 the corresponding subgroup of G_1 p. 185
(6.4.2); the notations U_Ω^1, P_Ω^1 and $\widehat{P}_\Omega^1$ (for $\Omega \subset A_1$) are self-explanatory.

Proposition **(7.6.4)**. — *(i) There exists one and only one map $\widetilde{\pi} : G_1.A \to \mathscr{I}_1$ from the set of the transforms of the points of A by the elements of G_1 into the building of G_1, extending $\pi : A \to A_1$ and commuting with the actions of G_1 on $G_1.A$ and on $\mathscr{I}_1$.*

(ii) One has $\widetilde{\pi}^{-1}(A_1) = A$, and the inverse image under $\widetilde{\pi}$ of any apartment (resp. half-apartment, wall) of $\mathscr{I}_1$ is an apartment (resp. half-apartment, wall) of $\mathscr{I}$.

(iii) There exists one and only one law of operation $(x, v) \mapsto x + v$ of L_1 on $G_1.A$, extending the action of L_1 on A and such that $g.(x + v) = g.x + v$ for $g \in G_1$, $x \in G_1.A$ and $v \in L_1$. The map $\widetilde{\pi}$ passes to the quotient by L_1 and defines a bijection of $(G_1.A)/L_1$ onto $\mathscr{I}_1$.

Let us first show that

(1) $$\pi \circ \nu(n) = \nu_1(n) \circ \pi \qquad \text{for all } n \in N_1.$$

This is immediate if there exist $a \in \Phi_1$ and $k \in \mathbf{R}$ such that $n \in M^1_{a,k}$: indeed, $\nu(n)$ (resp. $\nu_1(n)$) is then the orthogonal reflection with respect to the hyperplane of equation $a(x - \varphi) + k = 0$ (resp. $a(x - \varphi_1) + k = 0$). If $n \in T_1$, there exists $v \in V$ (resp. $v_1 \in V_1$) such that $n.y = y + v$ (resp. $n.y_1 = y_1 + v_1$) for all $y \in A$ (resp. $y_1 \in A_1$). One has, for $a \in \Phi_1$ and $k \in \mathbf{R}$:

$$nU_{a,k}n^{-1} = U_{a,\, k-a(v)} = U_{a,\, k-a(v_1)}$$

whence $a(v) = a(v_1)$ for all $a \in \Phi_1$. Consequently, $v_1 = {}^v\pi(v)$ and $\pi(n.y) = n.\pi(y)$ for all $y \in A$.

Formula (1) then follows from what precedes, since N_1 is generated by T_1 and by the union of the $M^1_{a,k}$.

Now let $x \in A$. It is immediate that $U^1_{\pi(x)} \subset U_x$. Since $G_1 = U^1_{\pi(x)}.N_1.U^1_{\pi(x)}$ (7.3.4), one has, in view of (1):

(2) $$G_1 \cap \widehat{P}_x = U^1_{\pi(x)} \cdot (N_1 \cap \widehat{P}_x) \cdot U^1_{\pi(x)} \subset \widehat{P}^1_{\pi(x)}.$$

Now let $g \in G_1$ and put $\Omega = A \cap g^{-1}.A$. Suppose Ω non-empty and let $x \in \Omega$. Put $\Omega' = \Omega \cap (x + L_1)$. In view of (7.4.8), there exists $n \in N$ such that $g.y = n.y$ for all $y \in \Omega$, whence

$$g \in n\widehat{P}_\Omega \cap G_1 = n\widehat{P}_\Omega \cap U^1_{\pi(x),D_1}.N_1.U^1_{\pi(x)} \subset n\widehat{P}_\Omega \cap U_{x,D}.N_1.\widehat{P}_{\Omega'}$$

denoting by D (resp. D_1) a vector chamber of Φ (resp. Φ_1) such that $D \subset \pi^{-1}(D_1)$. In view of (7.4.15), there therefore exists $n_1 \in N_1$ such that

$$n \in (N \cap U_{x,D})\, n_1\, (N \cap \widehat{P}_{\Omega'}) = Hn_1\widehat{N}_{\Omega'} = n_1\widehat{N}_{\Omega'},$$

whence

(3) $$g \in n_1(\widehat{P}_u \cap G_1) \subset n_1\widehat{P}^1_{\pi(u)} \qquad \text{for all } u \in A \cap g^{-1}.A \cap (x + L_1).$$

This being so, let us prove (i). The uniqueness of $\widetilde{\pi}$ is clear. To show its existence, it suffices to show that if $x \in A$ and $g \in G_1$ are such that $g.x \in A$, then $\pi(g.x) = g.\pi(x)$. Let then $n_1 \in N_1$ satisfy (3); in view of (1) and (2), one has

$$\pi(g.x) = \pi(n_1.x) = n_1.\pi(x) = g.\pi(x).$$

Let us prove (ii). Let $x \in A$ and $g \in G_1$ be such that $\pi(g.x) \in A_1$. In view of (7.4.8), there exists $n \in N_1$ such that $g \in n\widehat{P}^1_{\pi(x)}$. Since $\widehat{P}^1_{\pi(x)} = \widehat{N}^1_{\pi(x)}.U^1_{\pi(x)} \subset N_1.\widehat{P}_x$, one has $g.x \in N_1.x \subset A$, whence (ii). p. 186

Let us prove (iii). The uniqueness of the sought law of operation is clear. To show its existence, it suffices to show that if $z = g.y$, with $y, z \in A$, and if $g \in G_1$ and $v \in L_1$, then $g.(y + v) = z + v$. Now one has in that case

$$\widetilde{\pi}(g.(y + v)) = g.\widetilde{\pi}(y + v) = g.\widetilde{\pi}(y) = \widetilde{\pi}(g.y) = \pi(z) \in A_1$$

and $g.(y + v) \in A$ in view of (ii). Let then $n_1 \in N_1$ satisfy (3), with $x = y$. One has

$$g.(y + v) = n_1.(y + v) = n_1.y + {}^v\nu(n_1).v = z + v$$

since ${}^v\nu(n_1) \in {}^vW_1$ is the identity on L_1.

Corollary (**7.6.5**). — *Let p be the orthogonal projection of V onto L_1 and let S_1 be the subgroup formed by the $t \in T$ such that $p \circ \nu(t) = 0$. Let Ω be a non-empty subset of A.*

(i) One has $T_1^0 = T \cap G_1^0 \subset S_1$.

(ii) If $T_1 \subset S_1$, one has $G_1 \cap \widehat{P}_\Omega = \widehat{P}^1_{\pi(\Omega)}$.

(iii) One has $G_1 \cap \widehat{P}_\Omega = G_1 \cap \widehat{P}_{\Omega+L_1} = (G_1 \cap G_1^0.S_1) \cap \widehat{P}_\Omega \subset \widehat{P}^1_{\pi(\Omega)}$.

Assertion (i) follows at once from (6.1.2)(12), (6.1.13) and (6.2.10)(ii). Let us prove (ii). In view of (7.6.4)(i), it suffices to show that $\widehat{P}^1_{\pi(\Omega)} \subset \widehat{P}_\Omega$ as soon as $T_1 \subset S_1$, or again, in view of (7.1.8), that $\widehat{N}^1_{\pi(\Omega)} \subset \widehat{P}_\Omega$. Let then $n \in \widehat{N}^1_{\pi(\Omega)} \subset N_1 \subset N_1^0.S_1$. Every element of N_1^0 or of S_1 leaves invariant the affine subspaces of the form $y + \operatorname{Ker} p$ for $y \in A$; on the other hand, $\nu(n)$ leaves invariant the affine subspaces of the form $x + L_1$ for $x \in \Omega$ ((7.6.4)(i)). It follows that $n \in \widehat{P}_\Omega$, whence (ii).

Let us prove (iii). The first equality follows from (7.6.4)(iii) and the last inclusion from (7.6.4)(i). It remains to show that $G_1 \cap \widehat{P}_\Omega \subset G_1^0.S_1$. Let $g \in G_1 \cap \widehat{P}_\Omega$. Since the buildings of G_1 and of G_1^0 are the same, there exists $h \in G_1^0 \cap \widehat{P}^1_{\pi(\Omega)}$ such that $h^{-1}.A_1 = g.A_1$. In view of (ii), one has $h \in \widehat{P}_\Omega$ and $n = hg \in N_1 \cap \widehat{P}_\Omega$. Let us write $n = tn_1$ with $n_1 \in N_1^0$ and $t \in T_1$, and let $x \in \Omega$. One has on the one hand

$$\nu(n)(x + \operatorname{Ker} p) = \nu(t)(x + \operatorname{Ker} p) \tag{1}$$

and on the other hand

$$\nu(n)(x + \operatorname{Ker} p) = \nu(n).x + {}^v\nu(n)(\operatorname{Ker} p) = x + \operatorname{Ker} p. \tag{2}$$

Comparing (1) and (2), one sees that $t \in S_1$ and one has indeed

$$g = h^{-1}n \in G_1^0.N_1^0.S_1 = G_1^0.S_1.$$

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8. Bornology defined by a valued root datum

We keep the notations of §§6 and 7.

8.1. Bornologies compatible with a valued root datum. —

Definition (**8.1.1**). — *A bornology $\mathscr{B}$ on G is said to be* weakly compatible *with the valued root datum of G (or, more simply, with the valuation φ) if it is compatible with the group law of G, if the subgroups $U_{a,k}$ are bounded and the subgroups U_a unbounded for $\mathscr{B}$, whatever $a \in \Phi$ and $k \in \mathbf{R}$.*

One says that $\mathscr{B}$ is compatible *with φ if moreover the subgroup $H_{[0]}$ defined in (6.4.14) is bounded for $\mathscr{B}$.*

Proposition (**8.1.2**). — *Let $\mathscr{B}$ be a bornology weakly compatible with φ. Let x be a point of A and D a vector chamber.*

(i) For a subset X of G to be bounded for $\mathscr{B}$, it is necessary and sufficient that

$$(U_{x+D}.U_{x-D}.X.U_{x-D}.U_{x+D}) \cap N$$

be bounded for $\mathscr{B}$.

(ii) The set $\mathscr{B} \cap \mathfrak{P}(N)$ of bounded subsets of N is a bornology on N, compatible with the group law of N, and satisfying the two following conditions:

(BCN 1) *The image under ν of any bounded subset of N is a bounded subset of* Isom A *(endowed with its natural bornology)* ((3.1.2) *b*)).

(BCN 2) *Whatever* $a \in \Phi$ *and* $k \in \mathbf{R}$*, the intersection*

$$T \cap (U_{a,k} U_{-a,k} U_{a,k} U_{-a,k} U_{a,k})$$

is bounded.

(iii) Every bounded subset of U_a *is contained in some* $U_{a,k}$ $(a \in \Phi,\ k \in \mathbf{R})$.

By (7.3.4) and (7.2.6)(2), one has $G = U_{x-D} U_{x+D} N U_{x+D} U_{x-D}$, whence

$$X \subset U_{x-D}.U_{x+D}.\big((U_{x+D} U_{x-D} X U_{x-D} U_{x+D}) \cap N\big).U_{x+D}.U_{x-D}.$$

On the other hand, U_{x+D} and U_{x-D} are bounded for $\mathscr{B}$ since one has, for example (6.4.9),

$$U_{x+D} = \prod_{a \in \Phi_D^{+\,\mathrm{red}}} U_{a,\,-a(x)}.$$

One deduces (i).

Let us prove (BCN 1) by contradiction. Let X be a bounded subset of N whose image $\nu(X)$ is not bounded in Isom A. Up to multiplying X by a finite subset, one may suppose that $\nu(X)$ is stable under left multiplication by the stabilizer of a special point, and $\nu(X)$ then contains translations of arbitrarily large length. More precisely, X contains a sequence (x_n) such that $t_n = \nu(x_n)$ is a translation of length $|t_n|$ tending to infinity and of direction $t_n/|t_n|$ tending to an element $v \in V$ of length 1. Let then $a \in \Phi$ be such that $a(v) > 0$. One has $\lim_{n\to\infty} a(t_n) = +\infty$ and p. 188

$$U_a = \bigcup_n U_{a,\,-a(t_n)} = \bigcup_n x_n U_{a,0} x_n^{-1} \subset X.U_{a,0}.X^{-1}$$

is bounded, which is absurd.

This proves (ii), for (BCN 2) is obvious.

Let us prove (iii) by contradiction. Let (u_n) be a bounded sequence of points of U_a such that $\lim_{n\to\infty} \varphi_a(u_n) = -\infty$ and let $k = \sup_n \varphi_a(u_n)$. By (V 5), one has $m(u_n) \in U_{-a,-k} u_n U_{-a,-k}$ and the set of reflections $r_{a,\varphi_a(u_n)} = \nu(m(u_n))$ is bounded, which is absurd.

Proposition **(8.1.3)**. — *Let* D *be a vector chamber and let* $\mathscr{U}_D^+$ *be the smallest bornology on* U_D^+ *compatible with the group law and containing the* $U_{a,k}$ *for* $a \in \Phi_D^+$ *and* $k \in \mathbf{R}$*. Let* $X \subset U_D^+$*; the following conditions are equivalent:*

(i) X *is bounded for* $\mathscr{U}_D^+$*;*

(ii) whatever the order in which the roots $a \in \Phi_D^{+\,\mathrm{red}}$ *are arranged, there exists* $k \in \mathbf{R}$ *such that* $X \subset \prod_{a\in\Phi_D^{+\,\mathrm{red}}} U_{a,k}$*;*

(iii) there exists $x \in A$ *such that* $X \subset U_{x+D}$.

It is clear that (iii) implies (ii) (6.4.9) and that (ii) implies (i). Let $X \in \mathscr{U}_D^+$; then X is contained in a finite product of subgroups U_{a_i,k_i} with $a_i \in \Phi_D^+$ and $k_i \in \mathbf{R}$. For $a \in \Phi_D^+$, put $h(a) = \inf\{k_i \mid a_i = a\}$ and let $x \in A$ be such that $a(x) > -h(a)$ for all $a \in \Phi_D^+$. One then has $X \subset U_{x+D}$.

Proposition **(8.1.4)**. — *Let* $\mathscr{N}$ *be a bornology on* N*, compatible with the group law and satisfying the two conditions* (BCN 1) *and* (BCN 2) *of (8.1.2). Let* $\mathscr{G}$ *be the smallest bornology on* G *containing* $\mathscr{N}$ *and the* $U_{a,k}$ *for* $a \in \Phi$ *and* $k \in \mathbf{R}$ *and such that* $M, M' \in \mathscr{G}$ *implies* $MM' \in \mathscr{G}$.

(i) $\mathscr{G}$ *is a bornology weakly compatible with* φ *and in particular is compatible with the group law of* G.

(ii) $\mathscr{G}$ *induces on* N *the given bornology* $\mathscr{N}$ *and, for every vector chamber* D*, the bornology* $\mathscr{G}$ *induces on* U_D^+ *the bornology* $\mathscr{U}_D^+$ *(8.1.3).*

(iii) Let D *be a vector chamber. For a subset* X *of* G *to be bounded for* $\mathscr{G}$*, it is necessary and sufficient that there exist* $Y^+ \subset \mathscr{U}_D^+$*,* $Y^- \subset \mathscr{U}_D^- = \mathscr{U}_{-D}^+$ *and* $Y^0 \subset \mathscr{N}$ *such that*

$$X \subset Y^+.Y^-.Y^0.$$

Let us introduce a few notational conventions: if $\mathscr{X}$ and $\mathscr{Y}$ are two sets of subsets of G, we put $\mathscr{X}.\mathscr{Y} = \{XY \mid X \in \mathscr{X},\ Y \in \mathscr{Y}\}$ and $\mathscr{X}^{-1} = \{X^{-1} \mid X \in \mathscr{X}\}$, and we write $\mathscr{X} \prec \mathscr{Y}$ if every element of $\mathscr{X}$ is contained in a finite union of elements of $\mathscr{Y}$. Let $\mathscr{E} = \mathscr{N} \cap \mathfrak{P}(T)$ be the bornology induced by $\mathscr{N}$ on T and, for $a \in \Phi$, let us denote by $\mathscr{U}_a$ the set of the $U_{a,k}$ for $k \in \mathbf{R}$.

Let then $\mathscr{H}$ be the set of the subsets X of G satisfying the condition of (iii) for a given vector p. 189
chamber D. Since $G = U_D^+ U_D^- N$ (6.1.12), this is a bornology on G. Let us show that $\mathscr{H}$ is a bornology compatible with the group law of G, or again that

(1) $$\mathscr{H}^{-1}.\mathscr{H} \prec \mathscr{H}.$$

Every element of $\mathscr{H}^{-1}$ is contained in a finite union of finite products of sets belonging either to $\mathscr{E}$, or to one of the $\mathscr{U}_a$ for $a \in \Phi$, or to $\{\{n\} \mid n \in N\}$. Since $\mathscr{U}_{-a} = n\mathscr{U}_a n^{-1}$ for $n \in M_a$ and since N is generated by T and by the union of the M_b for b running through the basis Π_D of Φ corresponding to D (6.1.11), one sees that every element of $\mathscr{H}^{-1}$ is contained in a finite union of finite products of sets belonging either to $\mathscr{E}$, or to one of the $\mathscr{U}_a$ for $a \in \Phi_D^+$, or to the set of the $\{m\}$ for $b \in \Pi_D$ and $m \in M_b$.

By (8.1.3), one has $\mathscr{U}_a.\mathscr{H} \prec \mathscr{H}$, and condition (BCN 1) implies that $\mathscr{E}.\mathscr{H} \prec \mathscr{H}$. To prove (1), it therefore suffices to show that:

(2) *for $b \in \Pi_D$ and $m \in M_b$, one has $m\mathscr{H} \prec \mathscr{H}$.*

Let U'^+ (resp. U'^-) be the group generated by the U_c for $c \in \Phi_D^+$ (resp. Φ_D^-) not proportional to b. Put $\mathscr{U}^\pm = \mathscr{U}_D^\pm$, $\mathscr{U}'^+ = \mathscr{U}^+ \cap \mathfrak{P}(U'^+)$ and $\mathscr{U}'^- = \mathscr{U}^- \cap \mathfrak{P}(U'^-)$. The following relations are obvious:

(3) $$m\mathscr{U}_b m^{-1} = \mathscr{U}_{-b}, \qquad m\mathscr{U}_{-b}m^{-1} = \mathscr{U}_b;$$

(4) $$m\mathscr{U}'^+ m^{-1} = \mathscr{U}'^+, \qquad m\mathscr{U}'^- m^{-1} = \mathscr{U}'^-;$$

(5) $$\mathscr{E}.\mathscr{U}_b \prec \mathscr{U}_b.\mathscr{E}, \qquad \mathscr{E}.\mathscr{U}_{-b} \prec \mathscr{U}_{-b}.\mathscr{E};$$

(6) $$\mathscr{E}.\mathscr{U}'^+ \prec \mathscr{U}'^+.\mathscr{E}, \qquad \mathscr{E}.\mathscr{U}'^- \prec \mathscr{U}'^-.\mathscr{E}.$$

We shall show that:

(7) $$\mathscr{U}_{-b}.\mathscr{U}_b \prec \mathscr{U}_b.\mathscr{U}_{-b}.\mathscr{E} \ \cup\ \mathscr{U}_b.\mathscr{U}_{-b}.m\mathscr{E}.$$

For this, it suffices to show that, whatever $h, k \in \mathbf{R}$, one has:

(8) $$\{U_{-b,h}.U_{b,k}\} \prec \mathscr{U}_b.\mathscr{U}_{-b}.\mathscr{E} \ \cup\ \mathscr{U}_b.\mathscr{U}_{-b}.m\mathscr{E}$$

and one may suppose $h = k < 0$. For $u \in U_{-b}$ with $r = \varphi_b(u) \leqslant -2k$, one has:

$$u \in U_{b,-r} mTU_{b,-r} \subset U_{b,2k} mTU_{b,2k}.$$

Consequently, there exists a subset Y of T such that:

(9) $$U_{-b,k} - U_{-b,-2k} \subset U_{b,2k} mYU_{b,2k}$$

and one may choose Y in such a way that:

(10) $$Y \subset m^{-1}U_{b,2k}U_{-b,k}U_{b,2k}.$$

Let $s \in \Gamma_b$, $s > 2k$. There exists $t \in T$ such that $m^{-1} \in tM_{b,s}^0 \subset tU_{b,s}U_{-b,-s}U_{b,s}$, and (10) implies that $Y \in \mathscr{E}$ by (BCN 2).

On the other hand, one has by (6.3.2) $U_{-b,-2k}U_{b,k} \subset U_{b,k}U_{-b,-2k}H$, and one sees as above p. 190
that there exists $Y' \in \mathscr{E}$ such that:

(11) $$U_{-b,-2k}.U_{b,k} \subset U_{b,k}.U_{-b,-2k}.Y'.$$

It then follows from (9) and (11) that:

$$U_{-b,k}.U_{b,k} \subset (U_{b,2k}.mY.U_{b,2k}) \cup (U_{b,k}.U_{-b,-2k}.Y').$$

Taking (3) and (5) into account, one deduces (8), whence (7). Finally, using (3), (4), (6) and (7), one sees that:

$$
\begin{aligned}
m\mathscr{H} &\prec m.\mathscr{U}'^{+}.\mathscr{U}_b.\mathscr{U}_{-b}.\mathscr{U}'^{-}.\mathscr{N} = \mathscr{U}'^{+}.\mathscr{U}_{-b}.\mathscr{U}_b.\mathscr{U}'^{-}.m\mathscr{N}\\
&\prec \mathscr{U}'^{+}.\mathscr{U}_b.\mathscr{U}_{-b}.\{1,m\}.\mathscr{E}.\mathscr{U}'^{-}.\mathscr{N} \prec \mathscr{U}^{+}.\mathscr{U}_b.\mathscr{U}_{-b}.\mathscr{U}'^{-}.\{1,m\}.\mathscr{E}.\mathscr{N}\\
&\prec \mathscr{U}^{+}.\mathscr{U}^{-}.\mathscr{N} = \mathscr{H}.
\end{aligned}
$$

It is now clear that $\mathscr{H} = \mathscr{G}$.

On the other hand, for $Y^+ \subset U_D^+$, $Y^- \subset U_D^-$ and $Y^0 \subset N$, one has ((6.1.15) c))

$$
\begin{aligned}
U_D^+ \cap Y^+Y^-Y^0 &\subset Y^+\\
N \cap Y^+Y^-Y^0 &\subset Y^0.
\end{aligned}
$$

Consequently, $\mathscr{G} = \mathscr{H}$ induces on N (resp. U_D^+) the bornology $\mathscr{N}$ (resp. $\mathscr{U}_D^+$). In view of the invariance of $\mathscr{G}$ under inner automorphisms, one deduces that $\mathscr{G}$ is weakly compatible with φ and satisfies (ii), which completes the proof.

Corollary **(8.1.5)**. — *The map which, to a bornology on G, associates the induced bornology on N, is a bijection from the set of the bornologies weakly compatible with φ onto the set of the bornologies on N compatible with the group law of N and satisfying conditions* (BCN 1) *and* (BCN 2).

This follows from (8.1.2) and (8.1.4).

The inverse bijection of the one described in the corollary will be denoted $\mathscr{N} \mapsto \mathscr{B}(\mathscr{N})$.

Remarks **(8.1.6)**. — *a)* A bornology $\mathscr{B}$ on G which contains the $U_{a,k}$ for $a \in \Phi$ and $k \in \mathbf{R}$, which induces on N a bornology $\mathscr{N}$ compatible with the group law of N satisfying condition (BCN 1), and which is such that $M, M' \in \mathscr{B}$ implies $MM' \in \mathscr{B}$, is weakly compatible with φ: it is indeed clear that $\mathscr{N}$ satisfies (BCN 2) and that $\mathscr{B}$ satisfies (8.1.2)(i); consequently, one has $\mathscr{B} = \mathscr{B}(\mathscr{N})$.

b) Let us take up the notations of (8.1.4). It is not true that every bounded subset of G (for $\mathscr{G}$) is contained in a set of the form

(1) $$Y^+.Y^0.Y^- \qquad \text{with } Y^+ \in \mathscr{U}^+,\ Y^- \in \mathscr{U}^- \text{ and } Y^0 \in \mathscr{N}$$

nor even that every bounded subset of $U^+.T.U^-$ is contained in a product of the form (1) with moreover $Y^0 \in \mathscr{E}$. A counterexample is furnished by the set

$$m.(U_{b,k} - \{1\})$$

for $b \in \Phi^+$, $m \in M_b$ and $k \in \mathbf{R}$.

Definition **(8.1.7)**. — *One calls* bornology defined by φ, *and denotes $\mathscr{B}(\varphi)$, the bornology on G which is the inverse image of the natural bornology of* Isom $\mathscr{I}$ *under the canonical map from G into* Isom $\mathscr{I}$ *(7.4.20).* p. 191

One will note that, in the case where φ is discrete, this bornology coincides with the one defined by the Tits system defined by φ (3.1.8).

Proposition **(8.1.8)**. — *The bornology $\mathscr{B}(\varphi)$ is compatible with φ and coincides with the bornology $\mathscr{B}(\mathscr{N})$, where $\mathscr{N}$ is the bornology on N which is the inverse image under ν of the natural bornology of* Isom A.

Let $a \in \Phi$ and $k \in \mathbf{R}$ and let $x \in A$ be such that $a(x) \geqslant -k$. One then has $U_{a,k} \subset \widehat{P}_x$, which shows that $U_{a,k}$ is bounded for $\mathscr{B}(\varphi)$. On the other hand, corollary (7.4.27) shows that U_a is not bounded. Moreover, all of H, hence *a fortiori* $H_{[0]}$, is bounded, since H fixes every point of A. Finally, the image of a subset X of N is bounded in Isom $\mathscr{I}$ if and only if $\nu(X)$ is bounded in Isom A, whence the last assertion.

Corollary **(8.1.9)**. — *Let $\mathscr{B}$ be a bornology weakly compatible with φ. Suppose that, for every subset X of $\nu(T)$ bounded in* $\operatorname{Isom} A$, *there exists a subset Y of T belonging to $\mathscr{B}$ such that $X = \nu(Y)$. Then the bornology defined by φ is generated by $\mathscr{B}.H$.*

Indeed, the bornology which is the inverse image under ν of the natural bornology of $\operatorname{Isom} A$ is generated by $\mathscr{N}.H$, where $\mathscr{N}$ is the bornology induced by $\mathscr{B}$ on N.

Theorem **(8.1.10)**. — *Let ψ be another valuation of the root datum of G. The following conditions are equivalent:*

a) *the valuations φ and ψ are equivalent (6.2.5);*

b) *the bornologies $\mathscr{B}(\varphi)$ and $\mathscr{B}(\psi)$ are equal;*

c) *the bornologies $\mathscr{B}(\varphi)$ and $\mathscr{B}(\psi)$ induce the same bornology on U_a for every $a \in \Phi$;*

d) *the bornologies $\mathscr{B}(\varphi)$ and $\mathscr{B}(\psi)$ induce the same bornology on N;*

e) *for every $a \in \Phi$, there exists a bijection $f_a : {}^{\varphi}\Gamma_a \to {}^{\psi}\Gamma_a$ such that $\psi_a = f_a \circ \varphi_a$.*

Suppose φ and ψ equivalent and let i be the canonical bijection of the building ${}^{\varphi}\mathscr{I}$ onto ${}^{\psi}\mathscr{I}$ (7.4.3). In view of (7.4.3) b), the restriction of i to ${}^{\varphi}A$ is bounded, hence so is i itself ((7.4.3) a), (7.4.18) and (7.4.20)). One deduces at once that $\mathscr{B}(\varphi) \subset \mathscr{B}(\psi)$. One sees in the same way that $\mathscr{B}(\psi) \subset \mathscr{B}(\varphi)$, whence the implication a) $\Rightarrow$ b).

It is clear that b) implies c) and d). Let us show that c) implies d). For this it suffices to remark that a subset Y of T is bounded for $\mathscr{B}(\varphi)$ if and only if, for every $a \in \Phi$ and every subset $X \subset U_a$ bounded for $\mathscr{B}(\varphi)$, the set of the txt^{-1} for $t \in Y$ and $x \in X$ is bounded for the bornology induced by $\mathscr{B}(\varphi)$ on U_a, and that a subset of N is bounded if and only if it is contained in the union of a finite number of translates of bounded subsets of T.

Let us show that d) implies e). For this, let us consider the equivalence relation ${}^{\varphi}\mathscr{R}_a$: $\varphi_a(u) = \varphi_a(v)$ on U_a (for $a \in \Phi$). It is equivalent to the relation

$$m(u) \in Hm(v)$$

or again to the fact that the two reflections $\nu(m(u))$ and $\nu(m(v))$ are equal. But, whatever u and v in U_a, $\nu(m(u))$ and $\nu(m(v))$ are two reflections with respect to parallel hyperplanes, and to say that they are equal amounts to saying that they generate a bounded subgroup of $\operatorname{Isom} A$. In view of (8.1.8), this means that $m(u)$ and $m(v)$ generate a *bounded* subgroup of N. Consequently, d) implies that the equivalence relations ${}^{\varphi}\mathscr{R}_a$ and ${}^{\psi}\mathscr{R}_a$ are the same for every a, whence e). p. 192

Let us show finally that e) implies a). Up to replacing φ and ψ by equipollent valuations, one may suppose that, for every a belonging to a basis Π of Φ, there exists an element $u_a \in U_a$ such that $\varphi_a(u_a) = \psi_a(u_a) = 0$. In view of (6.2.7)(1), one has, for $u \in U_a$:

$$\varphi_{-a}(m(u_a)um(u_a)^{-1}) = \varphi_a(u), \qquad \psi_{-a}(m(u_a)um(u_a)^{-1}) = \psi_a(u)$$

whence $f_{-a} = f_a$.

Now let $u, v \in U_a$, $k = \varphi_a(u)$ and $\ell = \varphi_a(v)$. Again by (6.2.8)(1), one has

$$\varphi_{-a}(m(u)vm(u)^{-1}) = \ell - 2k$$
$$\psi_{-a}(m(u)vm(u)^{-1}) = f_a(\ell) - 2f_a(k)$$

whence $f_a(\ell - 2k) = f_a(\ell) - 2f_a(k)$. Since $0, k \in \Gamma_a$ and $\Gamma_a \supset \Gamma_a + 2\widetilde{\Gamma}_a$ (6.2.16), one sees that $pk \in \Gamma_a$ and that:

$$f_a(pk) = pf_a(k) \qquad \text{for all } k \in \Gamma_a \text{ and all } p \in \mathbf{Z}.$$

For $u \in U_a$, the subgroup ${}^{\varphi}U_{a,\varphi_a(u)}$ is generated by the $v \in U_a$ such that $\varphi_a(v) = \varphi_a(u)$. It therefore coincides with the subgroup ${}^{\psi}U_{a,\psi_a(u)}$, whence one deduces easily that f_a is decreasing. Since Γ_a is dense in $\mathbf{R}$, or else is a subgroup of $\mathbf{R}$ isomorphic to $\mathbf{Z}$, one deduces easily that *there exists a strictly positive constant $\lambda(a)$ such that f_a is multiplication by $\lambda(a)$*, and this for every $a \in \Pi$.

Now let $a \in \Pi$ and $b \in \Phi$ be such that f_b is also multiplication by a constant $\lambda(b)$. Using once again (6.2.7)(1), one has, for $v \in U_b$:

$$\varphi_{r_a(b)}(m(u_a)vm(u_a)^{-1}) = \varphi_b(v), \qquad \psi_{r_a(b)}(m(u_a)vm(u_a)^{-1}) = \psi_b(v)$$

whence one deduces that $f_{r_a(b)}$ is multiplication by $\lambda(b)$.

Since the r_a for $a \in \Pi$ generate the Weyl group ${}^vW$ of Φ, and since every $b \in \Phi^{\mathrm{red}}$ is the transform of an element of Π by an element of ${}^vW$, one sees that f_b *is multiplication by a constant* $\lambda(b)$ *for every* $b \in \Phi^{\mathrm{red}}$, hence also for every $b \in \Phi$ by (V 4).

Finally, for $u \in U_a$, $v \in U_b$, $k = \varphi_a(u)$ and $\ell = \varphi_b(v)$, one has

$$\begin{aligned}\varphi_{r_a(b)}(m(u)vm(u)^{-1}) &= \ell - kb(a^\vee)\\ \psi_{r_a(b)}(m(u)vm(u)^{-1}) &= \psi_b(v) - \psi_a(u)b(a^\vee) = \lambda(b)\ell - \lambda(a)kb(a^\vee)\\ &= \lambda(b)\big(\ell - kb(a^\vee)\big)\end{aligned}$$

whence one deduces that if $b(a^\vee) \neq 0$, that is to say *if a and b are not orthogonal, one has* $\lambda(a) = \lambda(b)$. Consequently, the function λ is indeed constant on the irreducible components of Φ, which completes the proof.

Corollary **(8.1.11)**. — *Let $\gamma : G \to G$ be an automorphism of bornological group preserving T and the set of the U_a. There exists one and only one map $\gamma_* : \mathscr{I} \to \mathscr{I}$ possessing the following properties:* p. 193

a) $\gamma_*(g.x) = \gamma(g).\gamma_*(x)$ *for all $g \in G$ and $x \in \mathscr{I}$;*

b) $\gamma_*(A) = A$ *and the restriction of γ_* to A is an affinity.*

The map γ_* is bijective and equicontinuous: more precisely, there exist constants $m, M \in \mathbf{R}_+^*$ such that

$$m\, d(x,y) \leqslant d(\gamma_*(x), \gamma_*(y)) \leqslant M\, d(x,y)$$

whatever $x, y \in \mathscr{I}$.

The existence and the uniqueness of γ_* are immediate consequences of (8.1.10) and (7.4.3), applied to the valuations φ and $\varphi \circ \gamma$. The rest of the statement follows at once from a), b) and (7.4.20), taking into account that an affinity of a Euclidean space onto itself possesses the properties in question.

Remark **(8.1.12)**. — One can show that, if $\gamma : G \to G$ is an automorphism of bornological group, there exists one and only one map $\gamma_* : \mathscr{I} \to \mathscr{I}$ possessing property (8.1.11) a) and such that:

b′) for every bounded convex subset Ω of A, the restriction of γ_* to Ω is an isomorphism of affine sets onto a bounded convex subset of an apartment.

If φ is dense, a) already implies the uniqueness of γ_* (this follows from (8.2.1)). If φ is discrete, γ_* is characterized by a) and:

b″) the restriction of γ_* to every facet F of $\mathscr{I}$ is an affine map from F onto another facet.

In the discrete case, these assertions follow from §§2 and 3, notably from 3.3 and 3.5.

Proposition **(8.1.13)**. — *Let us take up the hypotheses and notations of 7.6. Let p be the orthogonal projection of V onto L_1 and let S_1 be the subgroup of T formed by the elements $t \in T$ such that $p \circ \nu(t) = 0$.*

(i) $\mathscr{B}(\varphi)$ induces on G_1 a bornology compatible with φ_1.

(ii) For $\mathscr{B}(\varphi)$ to induce on G_1 the bornology $\mathscr{B}(\varphi_1)$, it is necessary and sufficient that G_1 be contained in $G_1^0 S_1$.

(iii) For a subset M of G_1 to be bounded for $\mathscr{B}(\varphi)$, it is necessary and sufficient that it be bounded for $\mathscr{B}(\varphi_1)$ and contained in a subset of the form $G_1^0.(p\circ\nu)^{-1}(X)$, where X is a bounded subset of L_1.

(iv) For a subgroup of G_1 to be bounded for $\mathscr{B}(\varphi)$, it is necessary and sufficient that it be bounded for $\mathscr{B}(\varphi_1)$ and contained in $G_1^0.S_1$.

Assertion (i) is immediate. Let us prove (iii). Let $M \subset G_1$, bounded for $\mathscr{B}(\varphi_1)$ (resp. $\mathscr{B}(\varphi)$). There exist subsets $Y^+ \subset U_1^+$, $Y^- \subset U_1^-$ and $Z \subset T_1$ bounded for $\mathscr{B}(\varphi_1)$ (resp. $\mathscr{B}(\varphi)$), and a finite subset $F \subset N_1 \cap G_1^0$, such that $M \subset Y^+.Y^-.F.Z$. Since Y^+, Y^- and F are bounded for $\mathscr{B}(\varphi)$ (resp. $\mathscr{B}(\varphi_1)$), one sees easily that one is reduced to proving that the image under ν of a subset Z of T_1 in Isom A is bounded if and only if $\nu_1(Z)$ is bounded in Isom A_1 and $p \circ \nu(Z)$ is bounded in L_1, which is immediate.

Finally, in view of (7.6.5)(i), the homomorphism $p \circ \nu : T \to L_1$ extends in a unique way to a homomorphism from $G_1^0.T$ into L_1 with kernel $G_1^0.S_1$. Assertions (ii) and (iv) then follow from (iii). p. 194

8.2. Maximal bounded subgroups. —

In this subsection, we suppose that *the valuation φ of the root datum of G is dense*: for the discrete case, see §3. We endow G with the bornology $\mathscr{B}(\varphi)$ defined by φ (8.1.7).

Proposition **(8.2.1)**. — *(i) The map $x \mapsto \widehat{P}_x$ is a bijection from the completion $\widehat{\mathscr{I}}$ of the building of G onto the set of the maximal bounded subgroups of G.*

(ii) Let x and y be two points of $\widehat{\mathscr{I}}$. The maximal bounded subgroups $\widehat{P}_x$ and $\widehat{P}_y$ are conjugate if and only if x and y belong to the same orbit of G in $\widehat{\mathscr{I}}$.

It follows at once from the definition of the bornology of G (8.1.7) that the subgroups $\widehat{P}_x$ are bounded. On the other hand, the complete metric space $\widehat{\mathscr{I}}$ satisfies condition (CN) of (3.2.3): indeed, one shows exactly as in (3.2.1) (taking (7.4.20) into account) that relation (3.2.1)(1) is exact whatever $x, y, z \in \mathscr{I}$, on taking $m = (x/2) + (y/2)$ (cf. (7.4.20)(v)), and this remains true by continuity for $x, y, z \in \widehat{\mathscr{I}}$ (cf. (7.5.1)). Consequently, *every bounded subgroup of* Isom $\widehat{\mathscr{I}}$*, hence also every bounded subgroup of G, possesses a fixed point in $\widehat{\mathscr{I}}$* (3.2.3), hence is contained in a subgroup $\widehat{P}_x$ with $x \in \widehat{\mathscr{I}}$. To complete the proof of (i), it suffices to show that if x and y are two distinct points of $\mathscr{I}$, one has $\widehat{P}_x \not\subset \widehat{P}_y$; now this follows from (7.4.34) and (7.5.8). Assertion (ii) is then obvious.

(8.2.2) One will compare (8.2.1) with the results of §3; when φ is dense, all the stabilizers of the various points of $\mathscr{I}$ are maximal bounded subgroups (and there are still others when $\mathscr{I}$ is not complete), whereas, when φ is discrete, only certain among them are: those which correspond to the vertices of the polysimplicial complex $\mathscr{I}$ when G is generated by H and the U_a, and, in the general case, those which correspond to the barycenters of certain facets. Moreover, when φ is discrete, there is only a finite number, in general greater than one, of conjugacy classes of maximal bounded subgroups. When φ is dense, there are in general infinitely many conjugacy classes; however, when $\mathscr{I}$ is complete (condition (MC) of (7.5.4)) and $\Gamma_a = \mathbf{R}$ for every $a \in \Phi$, there is only one conjugacy class!

Lemma **(8.2.3)**. — *Let x be a point of A, D a vector chamber and L a subgroup of G containing $B_{x,D}$ and not contained in $\widehat{P}_x$. There exists a root $a \in \Phi$ such that $L \cap U_a \not\subset U_{a,-a(x)} = U_a \cap \widehat{P}_x$.*

In view of (7.3.4), one has $L = B_{x,D}.(N \cap L).B_{x,D}$ and there exists $n \in N \cap L$ with $n \notin \widehat{N}_x$. Put $v = \nu(n).x - x \in V$; since $v \neq 0$, there exists $a \in \Phi^{\mathrm{red}}$ such that $a(v) > 0$. Put $b = {}^v\nu(n^{-1}).a$. p. 195
Since $U_{b,-b(x)+} \subset B_{x,D} \subset L$, one has:

$$U_{a,-a(x+v)+} = n.U_{b,-b(x)+}.n^{-1} \subset L$$

and since φ is dense, there exists $h \in \Gamma_a$ such that $-a(x+v) < h < -a(x)$, whence the lemma, since $U_{a,h} \subset L$ and $U_{a,h} \not\subset U_{a,-a(x)}$.

Proposition (**8.2.4**). — *Let $x \in A$ and let D be a vector chamber.*

(i) Every bounded subgroup of G containing $B_{x,D}$ is contained in $\widehat{P}_x$.

(ii) If Φ is irreducible, every subgroup of G containing $B_{x,D}$ either is contained in $\widehat{P}_x$, or contains the subgroup G' of G generated by H and the U_a for $a \in \Phi$.

Let L be a subgroup containing $B_{x,D}$ and not contained in $\widehat{P}_x$. In view of lemma (8.2.3), there exists a root $a \in \Phi^{\mathrm{red}}$ such that $L \cap U_a \not\subset U_{a,-a(x)}$. Taking into account that φ is dense, one deduces that there exist $h, k \in \Gamma_a$ such that $h < k < -a(x)$ and $U_{a,h} \subset L$. The subgroup L then contains on the one hand the two subgroups $U_{a,h}$ and $U_{-a,-h}$, on the other hand the two subgroups $U_{a,k}$ and $U_{-a,-k}$. Consequently, $\nu(N \cap L)$ contains the two reflections $r_{a,k}$ and $r_{a,h}$, hence the translation $t = r_{a,h} \circ r_{a,k}$ of non-zero vector $(k-h)a^\vee$. It follows that L is not bounded, whence (i).

Now let c be a root not orthogonal to a, and put $m = \langle c, a^\vee \rangle$; one has $m \neq 0$. For $q \in \mathbf{Z}$, one has

$$L \supset t^q U_{c,-c(x)+} t^{-q} = U_{c,\,-c(x)-qm(k-h)+}$$

which shows that L contains all of U_c, and the same reasoning shows that L contains U_{-c}. Consequently, $\nu(N \cap L)$ contains a translation of the form $\ell.c^\vee$, with $\ell \neq 0$. One shows thus step by step that L contains all the groups U_c for the roots $c \in \Phi$ which can be joined to a by a sequence $(c_0 = a, c_1, \dots, c_s = c)$ of roots such that c_i and c_{i+1} are not orthogonal for $0 \leqslant i < s$, in other words for the roots c of the irreducible component of Φ which contains a. This proves (ii).

Let us recall (6.2.20) that the quotient group G/G' is a torsion group every element of which is annihilated by a power of the connection index of Φ.

Remark (**8.2.5**). — Proposition (8.2.4), together with the results recalled in (7.2.7) and in (1.2.19), allows one to determine all the subgroups of G containing $B_{x,D}$.

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9. Descent of a valued root datum

We keep the notations of §§6, 7 and 8. The aim of this section is to show that, under certain conditions, the valued root datum of G "descends" to a subgroup and provides a valued root datum of that subgroup.

9.1. Induced valuations. —

(9.1.1) Throughout §9, we denote by $V^\natural$ a vector subspace of V. For $a \in V^*$, we denote by $a^\natural$ the restriction of a to $V^\natural$. For every $b \in (V^\natural)^*$, we denote by Φ_b (resp. Φ_b^+) the set of the $a \in \Phi$ such that $a^\natural \in \mathbf{R}.b$ (resp. $a^\natural \in \mathbf{R}_+^*.b$). It is clear that Φ_b is a rationally closed root subsystem of Φ. We denote by Z the subgroup generated by T and the U_a for $a \in \Phi_0$ (i.e. for $a \in \Phi$ with $a^\natural = 0$). For $b \in (V^\natural)^*$, we denote:

- by U_b the subgroup generated by the U_a for $a \in \Phi_b^+$;
- by L_b the subgroup generated by Z, U_b and U_{-b};
- by M_b the set of the $x \in L_b$ such that $xZx^{-1} = Z$, $xU_bx^{-1} = U_{-b}$ and $xU_{-b}x^{-1} = U_b$.

Lemma (**9.1.2**). — *For every $b \in (V^\natural)^*$, the pair $(T, (U_a, M_a)_{a \in \Phi_b})$ is a generating root datum of type Φ_b in L_b. Moreover, Z normalizes U_b and U_{-b}; the subgroups $Z.U_b$ and $Z.U_{-b}$ are parabolic subgroups of L_b and one has $ZU_b \cap ZU_{-b} = Z$.*

The first assertion is immediate (cf. (7.6.3)). The others are so as well when $b = 0$. Let us therefore suppose $b \neq 0$. That Z normalizes U_b and U_{-b} follows at once from condition (DR 2) of (6.1.1) and from (6.1.2)(3). Moreover, if one chooses the vector chamber D_0 in such a way that $\Phi_b^+ \subset \Phi^+$, which is permissible, and if one puts $U_Z^+ = Z \cap U^+$ and $U_Z^- = Z \cap U^-$, one sees that $TU_Z^+U_b$ and $TU_Z^-U_{-b}$ are minimal parabolic subgroups of L_b, which implies that ZU_b and ZU_{-b} are parabolic subgroups of L_b. Finally, let $u \in U_b$, $v \in U_{-b}$ and $z \in Z$ be such that $v = zu$. In view of (6.1.15), there exists one and only one element $n \in Z \cap N$ such that $z = v'nu'$, with $v' \in U_Z^-$ and $u' \in U_Z^+$. One then has $v = v'nu'u \in U^-nU^+ \cap U^-$, whence $n = 1$ by (6.1.15) and $v = v'$ by (DR 6). One therefore has $v \in U_{-b} \cap U_Z^- = \{1\}$ (6.1.6), which proves the last assertion.

Lemma **(9.1.3)**. — *Let $b \in (V^\natural)^*$; if $M_b \neq \emptyset$, there exists an element $n \in N \cap L_b$ such that $M_b = Zn$.*

This is obvious if $b = 0$. Suppose $b \neq 0$ and let us take up the notations of (9.1.2). In view of (6.1.15) c), there exists $n \in N \cap L_b$ such that

$$n^{-1}TU_Z^+U_bn = TU_Z^-U_{-b} \qquad \text{and} \qquad n^{-1}TU_Z^-U_{-b}n = TU_Z^+U_b.$$

If $m \in M_b$, the subgroups $n^{-1}ZU_bn$ and $m^{-1}ZU_bm = ZU_{-b}$ are two conjugate parabolic subgroups both containing the same minimal parabolic subgroup $TU_Z^-U_{-b}$. They are therefore equal, and mn^{-1} normalizes ZU_b, hence belongs to ZU_b. One sees in the same way that mn^{-1} belongs to ZU_{-b}, hence finally to Z by (9.1.2); whence the lemma. p. 197

(9.1.4) For $b \in (V^\natural)^*$, $b \neq 0$, and $k \in \mathbf{R}$, we denote by $U_{b,k}$ the subgroup of G generated by the $U_{a,rk}$ for $a \in \Phi_b^+$, $r \in \mathbf{R}_+^*$ and $a^\natural = rb$. With the notations of 6.4, $U_{b,k}$ is the subgroup U_f associated with the function $f = f(b,k)$ on Φ defined by

$$\text{(1)} \qquad \begin{cases} f(b,k)(a) = rk & \text{if } a = rb \text{ with } r \in \mathbf{R}_+^* \\ f(b,k)(a) = +\infty & \text{if } a \notin \Phi_b^+. \end{cases}$$

One verifies at once that $f(b,k)$ is a *concave* function.

Lemma **(9.1.5)**. — *Let $b \in (V^\natural)^*$, $b \neq 0$, and $k \in \mathbf{R}$. For $r \in \mathbf{R}_+^*$, let $\beta(r)$ be the set of the $a \in \Phi^{\mathrm{red}}$ such that $a^\natural = rb$. The product map (for an arbitrary fixed order) is a bijection from $\prod_{r \in \mathbf{R}_+^*}\left(\prod_{a\in\beta(r)} U_a\right)$ (resp. $\prod_{r \in \mathbf{R}_+^*}\left(\prod_{a\in\beta(r)} U_{a,rk}\right)$) onto U_b (resp. $U_{b,k}$).*

This follows at once from (6.1.6) and (6.4.9).

Lemma **(9.1.6)**. — *Let $b \in (V^\natural)^*$, $b \neq 0$. For $u \in U_b$, put*

$$\varphi_b(u) = \sup\{k \in \mathbf{R} \cup \{\infty\} \mid u \in U_{b,k}\}.$$

The inverse image $\varphi_b^{-1}([k, +\infty])$ is the subgroup $U_{b,k}$. For every $s \in \mathbf{R}_+^*$, one has $U_{sb} = U_b$ and $\varphi_{sb} = s\varphi_b$.

This is evident.

(9.1.7) Let us take up the notations of §7: let $\mathscr{I}$ be the building of G, A the canonical apartment, identified with the set $\varphi + V$ of the valuations equipollent to φ. Put $A_\natural = \varphi + V^\natural$.

Lemma **(9.1.8)**. — *Let $b \in (V^\natural)^*$, $b \neq 0$, $k \in \mathbf{R}$ and $u \in U_b$. The two following conditions are equivalent:*

(i) $\varphi_b(u) = k$;

(ii) the set of the fixed points of u in $A_\natural$ is the half-space $\alpha_{b,k}$ of $A_\natural$ formed by the $x \in A_\natural$ such that $b(x - \varphi) + k \geqslant 0$.

Moreover, if $k \in \varphi_b(U_b - \{1\})$, the half-space $\alpha_{b,k}$ is the intersection with $A_\natural$ of at least one affine root of A.

One may suppose that Φ_b^+ is not empty.

Let us denote by D_b the intersection of the closed half-spaces of V of the form $\{x \in V \mid a(x) \geqslant 0\}$ for $a \in \Phi_b^+$. This is a convex cone with non-empty interior, containing a Weyl chamber D of Φ, and whose intersection with $V^\natural$ is the closed half-space $R_b = \{x \in V \mid b(x) \geqslant 0\}$.

Let $a \in \Phi$ and $h \in \mathbf{R}$. Every element of $U_{a,h}$ fixes the points of the half-space $\alpha_{a,h} = \{x \in A \mid a(x - \varphi) + h \geqslant 0\}$ of A (7.4.5); if $a^\natural = rb$, with $r \in \mathbf{R}_+^*$, one has $\alpha_{a,rk} \cap A_\natural = \alpha_{b,k}$. Consequently, *every element of $U_{b,k}$ fixes all the points of $\alpha_{b,k}$*. p. 198

On the other hand, if $a \in \Phi_b^+$ and $h \in \mathbf{R}$, the half-space $\alpha_{a,h}$ contains a translate of D_b. Since the intersection of a finite number of translates of the convex cone D_b still contains a translate of D_b, and since the set $F(u)$ of the fixed points of u in A is convex, one sees that $F(u) + D_b = F(u)$. Let $x \in A_\natural$ be such that $u.x = x$; put $h = -b(x - \varphi)$. Then u fixes all the points of $\Omega = x + D_b$ and belongs to the subgroup $\widehat{P}_\Omega \cap U_D^+$, which is generated by the U_α for $\alpha \supset \Omega$ and $\alpha \in \Sigma$ (7.1.8); now, the only affine roots containing Ω are the $\alpha_{a,\ell}$ for $a \in \Phi_b^+$, $s \in \mathbf{R}_+^*$, $a^\natural = sb$ and $\ell \geqslant sh$. One therefore has $u \in U_{b,h}$ and $\varphi_b(u) \geqslant h$. This, together with the preceding result, shows the equivalence of (i) and (ii).

Finally, if $\varphi_b(u) = k$, there exist $s \in \mathbf{R}_+^*$ and $a \in \Phi_b^+$ such that $a^\natural = sb$ and $sk \in \varphi_a(U_a)$, whence the last assertion.

Definition **(9.1.9)**. — *Let $G^\natural$ be a subgroup of G, $\Phi^\natural$ a root system in $(V^\natural)^*$*[4] *and $(T^\natural, (U_b^\natural)_{b\in\Phi^\natural})$ a generating root datum of type $\Phi^\natural$ in $G^\natural$. One says that this root datum is* compatible *with that of G if one has*

$$\text{(DDR 1)} \qquad U_b^\natural \subset U_b \quad \textit{for every } b \in \Phi^\natural.$$

Throughout the rest of 9.1, we are given the vector subspace $V^\natural$ of V, a subgroup $G^\natural$ of G, a root system $\Phi^\natural$ in $(V^\natural)^*$ and a generating root datum $(T^\natural, (U_b^\natural))$ of type $\Phi^\natural$ in $G^\natural$, compatible with that of G. The objects which §6 allows one to associate with this root datum will be denoted by the same letter as in §6, but affected with the exponent $\natural$: for example, one denotes by ${}^vW^\natural$ the Weyl group of $\Phi^\natural$ and one denotes by $D^\natural$ a vector chamber of $\Phi^\natural$ in $V^\natural$. We assume, which is permissible, that the vector chamber D is such that $\overline{D} \cap V^\natural \subset \overline{D}^\natural$. The notations $M_b^\natural$, $N^\natural$, $\Phi^{\natural+}$, $U^{\natural+}$, etc. then explain themselves.

For $b \in \Phi^\natural$, we denote by $\varphi_b^\natural$ the restriction of φ_b to $U_b^\natural$ and we put $\varphi^\natural = (\varphi_b^\natural)_{b\in\Phi^\natural}$.

Proposition **(9.1.10)**. — *The family $\varphi^\natural$ satisfies conditions* (V 1) *and* (V 4) *of (6.2.1). It also satisfies* (V 3) *except perhaps when the two roots a and b of $\Phi^\natural$ are equal and $2a \in \Phi^\natural$. However, this last restriction can be lifted if one supposes that the following condition is fulfilled:*

(DDR 2) *for every $a \in \Phi^\natural$ such that $2a \in \Phi^\natural$, one has* $\mathrm{Card}\{c^\natural \mid c \in \Phi_a^+\} \leqslant 2$.

That (V 1) and (V 4) are satisfied follows from lemma (9.1.6). Now let $a, b \in \Phi^\natural$. Let us put

$$\Psi^\natural = \{pa + qb \mid p, q \in \mathbf{N}^*\} \cap \Phi^\natural$$

$$\Psi = \{c \in \Phi \mid c^\natural \in \mathbf{R}_+^* a + \mathbf{R}_+^* b\}$$

Let Y be the group generated by the U_c for $c \in \Psi$ and let $Y^\natural$ be the group generated by the $U_d^\natural$ for $d \in \Psi^\natural$. In view of (6.1.6), the following product maps are bijections: p. 199

$$\theta : \Big(\prod_{d\in\Psi^{\natural\,\mathrm{red}}} U_d\Big) \times \prod_{c\in\Psi^{\mathrm{red}},\ c^\natural\notin\mathbf{R}_+^*.\Psi^\natural} U_c \longrightarrow Y$$

$$\theta : \Big(\prod_{d\in\Psi^{\natural\,\mathrm{red}}} U_d^\natural\Big) \times \{1\} \longrightarrow Y^\natural.$$

[4] One will take care that, contrary to what the notation might lead one to believe, one does *not* assume that $\Phi^\natural = \{a^\natural \mid a \in \Phi\}$.

Let us then apply (6.4.29) to the two concave functions $f = f(a,k)$ and $g = f(b,\ell)$ (with $k,\ell \in \mathbf{R}$): one has $(U_f, U_g) \subset U_h$, where the function $h : \Phi \to \widetilde{\mathbf{R}}$ is defined by

$$h(c) = \inf\Big\{\sum_i f(a_i) + \sum_j g(b_j)\Big\} \tag{1}$$

the infimum being taken over the set of the decompositions $c = \sum_{i\in I} a_i + \sum_{j\in J} b_j$, where $a_i, b_j \in \Phi$ and $\operatorname{Card} I \geqslant 1$, $\operatorname{Card} J \geqslant 1$.

Let us suppose a and b non-proportional. One then verifies immediately that $h(c) = +\infty$ if $c \notin \Psi$ and that $h(c) = rk + s\ell$ if $c^\natural = ra + sb$. In view of (6.4.9), the restriction of θ to

$$\Big(\prod_{d\in\Psi^{\natural\,\mathrm{red}}} U_{d,h(d)}\Big) \times \prod_{c\in\Psi^{\mathrm{red}},\ c^\natural\notin\mathbf{R}_+^*.\Psi^\natural} U_{c,h(c)}$$

(one has put $h(ra+sb) = rk+s\ell$) is a bijection onto U_h. Now, by (DR 2), one has $(U_a^\natural, U_b^\natural) \subset Y^\natural$. Consequently, $(U^\natural_{a,k}, U^\natural_{b,\ell}) \subset Y^\natural \cap U_h$ and one deduces condition (V 3) for the roots a and b by taking the inverse image of $Y^\natural \cap U_h$ under θ.

If a and b are proportional, but $a + b \notin \Phi^\natural$, there is nothing to prove since $U_a^\natural$ and $U_b^\natural$ commute.

There remains to examine the case where $b = a$, with $2a \in \Phi^\natural$, supposing that $\operatorname{Card}\{c^\natural \mid c \in \Psi\} \leqslant 2$. One then sees that U_a is commutative, and that there is nothing to prove, except when there exists $s \in \mathbf{R}_+^*$ such that $\{c^\natural \mid c \in \Psi\} = \{sa, 2sa\}$. But in this last case one verifies easily that the function h given by (1) is infinite except on the roots $c \in \Psi$ of the form $c = c_1 + c_2$ with $c_1, c_2 \in \Psi$ and $c_1^\natural = c_2^\natural = sa$, in which case $h(c) = s(k+\ell)$. One deduces that $(U_{a,k}, U_{a,\ell}) \subset U_{a,(1/2)(k+\ell)}$, whence

$$(U_{a,k}, U_{a,\ell}) \subset U_{a,(1/2)(k+\ell)} \cap U_{2a} = U_{2a,k+\ell}$$

which completes the proof.

Definition **(9.1.11)**. — *If the family $\varphi^\natural$ is a valuation of the root datum $(T^\natural, (U_b^\natural))$ of $G^\natural$, one says that φ* descends *to $G^\natural$ and that $\varphi^\natural$ is the valuation* induced *by φ.*

If φ descends to $G^\natural$, we extend to the objects associated with the valued root datum of $G^\natural$ the notational conventions of (9.1.9).

Remarks **(9.1.12)**. — *a)* Under the hypotheses of (9.1.10) (including (DDR 2)), the valuation φ descends if and only if the family $\varphi^\natural$ satisfies conditions (V 0), (V 2) and (V 5) of (6.2.1). One will note that these are all relative "to rank 1", in the sense that they involve simultaneously only the integer multiples of one and the same root. We shall give later (9.2) conditions for them to be satisfied. p. 200

b) Let $V^\sharp$ be a vector subspace of $V^\natural$ and let $G^\sharp$ be a subgroup of $G^\natural$, endowed with a root datum compatible with that of $G^\natural$. Then the root datum of $G^\sharp$ is compatible with that of G. If φ descends to $G^\natural$, one verifies easily that the valuation $\varphi^\natural$ descends to $G^\sharp$ if and only if φ descends to $G^\sharp$.

(9.1.13) For $b \in (V^\natural)^*$, $b \neq 0$, and $k \in \mathbf{R}$, let us denote by U_b^e the subgroup of G generated by the U_a for $a \in \Phi$ and $a^\natural \in \mathbf{N}^*.b$, and by $U^e_{b,k}$ the one generated by the $U_{a,nk}$ for $a \in \Phi$, $n \in \mathbf{N}^*$ and $a^\natural = nb$. Using (6.1.6) and (6.4.9), one sees easily that

$$U^e_{b,k} = U_{b,k} \cap U^e_b.$$

In practice, the root datum of $G^\natural$ will most of the time satisfy the following condition, stronger than (DDR 1):

$$\text{(DDR 1 bis)}\qquad U_b^\natural \subset U_b^e \quad \text{for every } b \in \Phi^\natural.$$

(9.1.14) Let us suppose φ extendable and let $\varphi_0 : H \longrightarrow \mathbf{R}_+ \cup \{\infty\}$ be an extension of φ (6.4.38). For $u \in Z$ (9.1.1), let $\varphi_Z(u)$ be the supremum, finite or $+\infty$, of the set of the $k \in \mathbf{R}$ such that u belongs to the subgroup generated by $U_{0,k} = \varphi_0^{-1}([k, +\infty])$ and the $U_{a,k}$ for $a \in \Phi_0$.

Let us suppose also that φ descends to $G^\natural$. One may then consider the subgroup $H^\natural = \nu^{\natural -1}(1)$ and put, for $u \in H^\natural$:

$$\varphi_0^\natural(u) = \sup\{0, \varphi_Z(u)\}.$$

Proposition **(9.1.15)**. — *Let us keep the hypotheses and notations of (9.1.14). Suppose moreover that $\Phi^\natural = \{a^\natural \mid a \in \Phi\}$ and that there exists a constant $t \geqslant 0$ such that, if b and $2b$ belong to $\Phi^\natural$, one has*

$$\text{(DP)}\qquad (U_{b,k+t}.U^e_{2b,2k-h}) \cap U_b^\natural = U^\natural_{b,k}.U^\natural_{2b,2k-h}$$

whatever $k \in \mathbf{R}$ and $h \in \mathbf{R}_+$. Then $\varphi_0^\natural$ is an extension of $\varphi^\natural$.

For $\ell \geqslant 0$, put $U^\natural_{0,\ell} = \varphi_0^{\natural -1}([\ell, +\infty])$. We must show (6.4.38) the following assertions:

a) One has $(U^\natural_{0,k}, U^\natural_{0,\ell}) \subset U^\natural_{0,k+\ell}$ for $k, \ell \geqslant 0$. For this, it suffices to apply (6.4.44) to the functions $f, g : \Phi \cup \{0\} \longrightarrow \mathbf{R} \cup \{\infty\}$ such that $f(a) = k$ and $g(a) = \ell$ if $a \in \Phi_0 \cup \{0\}$, and $f(a) = g(a) = \infty$ otherwise.

b) For $b \in \Phi^\natural$ and $k, \ell \in \mathbf{R}$ such that $k + \ell > 0$, the $H^\natural$-components (6.3.8) of the commutators (u, u') with $u \in U^\natural_{b,k}$ and $u' \in U^\natural_{-b,\ell}$ belong to $U^\natural_{0,k+\ell}$. For this one applies (6.4.44) to the functions $f, g : \Phi \cup \{0\} \longrightarrow \mathbf{R} \cup \{\infty\}$ defined by $f(a) = k$ if $a^\natural = b$, $f(a) = 2k$ if $a^\natural = 2b$, $f(a) = \infty$ otherwise, and $g(a) = \ell$ if $a^\natural = -b$, $g(a) = 2\ell$ if $a^\natural = -2b$, $g(a) = \infty$ otherwise. One draws from this that

$$(U_{b,k}, U_{-b,\ell}) \subset U^e_{2b,3k+\ell}.U_{b,2k+\ell}.Y.U_{-b,2\ell+k}.U^e_{-2b,3\ell+k}$$

where Y is the subgroup generated by $U_{0,k+\ell}$ and the $U_{a,k+\ell}$ for $a \in \Phi_0$, whence the result. p. 201
One sees in the same way that the $H^\natural$-components of the commutators (u, u') and (u', u) with $u \in U^\natural_{b,k/2}$ and $u' \in U^\natural_{-2b,\ell}$ belong to $U^\natural_{0,k+\ell}$. This shows that $H^\natural_{[k+\ell]} \subset U^\natural_{0,k+\ell}$.

c) For $b \in \Phi$, $k \in \mathbf{R}$ and $\ell \in \mathbf{R}^*_+$, one has $(U^\natural_{b,k}, U^\natural_{0,\ell}) \subset U^\natural_{b,k+\ell}.U^\natural_{2b,2k+\ell}$; in other words $U^\natural_{0,\ell} \subset H^\natural_{(\ell)}$. A reasoning of the same type as the preceding ones shows that

$$(U_{b,k}, U^\natural_{0,\ell}) \subset U_{b,k+\ell}.U^e_{2b,2k+\ell} \qquad \text{for all } k \in \mathbf{R}$$

whence the result if $2b \notin \Phi^\natural$. If $2b \in \Phi^\natural$, one uses (DP) and what precedes to show, taking into account that $(U_b, U^e_{2b}) = \{1\}$, that

$$\begin{aligned}(U^\natural_{b,k}, U^\natural_{0,\ell}) &\subset (U_{b,k+t}, U^\natural_{0,\ell}).U^e_{2b,2k+\ell} \cap U^\natural_b \\ &\subset (U_{b,k+\ell+t}.U^e_{2b,2k+\ell}) \cap U^\natural_b = U^\natural_{b,k+\ell}.U^\natural_{2b,2k+\ell}\end{aligned}$$

which completes the proof.

Remark **(9.1.16)**. — One cannot always take $t = 0$ in (DP). Let for example K be a commutative field, endowed with an involution σ and with a non-improper valuation ω invariant under σ. Suppose K of residue characteristic 2 and let $\lambda \in K$ be such that $\lambda + \lambda^\sigma = 1$. With the notations that we shall employ in §10 (cf. (10.1.1)), let us consider the hermitian form on K^3 defined by $q(xe_1 + ze_0 + ye_{-1}) = \lambda^\sigma zz + x^\sigma y + K_{\sigma,1}$ and let us endow $G^\natural = SU_3(q)$ with the root datum defined in (10.1.6). Let us endow $G = SL_3(K)$ with the root datum defined in (10.2.2) and with the valuation φ defined in (10.2.3) (relative to the basis (e_1, e_0, e_{-1}) of K^3). One verifies easily that these root data are compatible and that φ descends to $G^\natural$ and induces there the valuation defined in (10.1.13). Moreover, we shall see that φ and $\varphi^\natural$ are extendable ((10.1.24) and (10.2.4)). But it is easy to see that condition (DP) is satisfied if and only if one has $t = -(1/2)\sup\{\omega(\mu) \mid \mu \in K,\ \mu + \mu^\sigma = 1\}$. If K and the field of invariants of σ have the same residue field, or if the residue extension is inseparable, one has $t > 0$.

Proposition **(9.1.17)**. — *Suppose that φ descends to $G^\natural$ and that there exists a subgroup $S^\natural$ of $T^\natural$, normal in $N^\natural$, such that $G^\natural$ is generated by $S^\natural$ and the $U^\natural_a$ for $a \in \Phi^\natural$, that $\nu^\natural(S^\natural)$ generates the vector space $V^\natural$ and that $S^\natural.A_\natural = A_\natural$. Then the scalar product induced on $V^\natural$ by that of V is invariant under the Weyl group of $\Phi^\natural$. If one endows $\mathscr{I}^\natural$ with the corresponding distance, there exists one and only one isometric map $j : \mathscr{I}^\natural \to \mathscr{I}$ such that $j(\varphi^\natural) = \varphi$ and such that $j(g.x) = g.j(x)$ for all $g \in G^\natural$ and all $x \in \mathscr{I}^\natural$. Moreover, $j(\mathscr{I}^\natural)$ is convex, $j(A^\natural) = A_\natural$ and $A^\natural = j^{-1}(A)$.*

Let us denote by j the bijection of $A^\natural = \varphi^\natural + V^\natural$ onto $A_\natural$ defined by $j(\varphi^\natural + v) = \varphi + v$. Let $s \in S^\natural$, $\psi \in A_\natural$ and $v \in V^\natural$ be such that $s.\psi = \psi + v$. It follows at once from the definitions that $(\psi + v)^\natural = \psi^\natural + v$. On the other hand, the inner automorphism defined by s preserves the root datum $(T^\natural, (U^\natural_b))$ and transforms the root datum $(T, (U_a))$ and the apartment A into $(sTs^{-1}, (sU_as^{-1}))$ and $s.A$. Since $s.A_\natural = A_\natural$, one deduces from (9.1.8) that the valuation $\psi^\natural + v$ of $(T^\natural, (U^\natural_b))$ is equal to the valuation obtained by descent from the valuation $s.\psi$ of $(sTs^{-1}, (sU_as^{-1}))$, which is none other than $s.\psi^\natural$ (transport of structure!). One therefore has:

$$j(s.x) = s.j(x) \qquad \text{for } x \in A^\natural \text{ and } s \in S^\natural. \tag{1}$$

Let $b \in \Phi^\natural$, $k \in \mathbf{R}$, $m \in M^\natural_{b,k}$ (6.2.2) and $y \in A^\natural$ be such that $m.y = y$. One has: p. 202

$$b(j(y) - \varphi) + k = b(y - \varphi^\natural) + k = 0.$$

Since m belongs to the group generated by $U_{b,k}$ and $U_{-b,-k}$, this implies that $m.j(y) = j(y)$. For $s \in S^\natural$, one then has, in view of (1):

$$j(ms.y) = j(msm^{-1}.y) = msm^{-1}.j(y) = ms.j(y) = m.j(s.y)$$

whence one deduces at once that $j(m.x) = m.j(x)$ for all $x \in A^\natural$. The group $N^\natural$ being generated by $S^\natural$ and the $M^\natural_{b,k}$, one concludes that $j(n.x) = n.j(x)$ for $n \in N^\natural$ and $x \in A^\natural$. This implies that the scalar product induced on $V^\natural$ by that of V is invariant under ${}^vW^\natural$. If one endows $A^\natural$ with the corresponding distance, j is isometric.

Let still $x \in A^\natural$. One knows that the stabilizer of x in $G^\natural$ is generated by its stabilizer $\widehat{N}^\natural_x$ in $N^\natural$ and the $U^\natural_{a,-a(x-\varphi^\natural)}$ for $a \in \Phi^\natural$. But the very definition of $\varphi^\natural$ shows that these last are contained in the stabilizer of $j(x)$, and what precedes shows that $\widehat{N}^\natural_x$ is too. One therefore has $\widehat{P}^\natural_x \subset \widehat{P}_{j(x)}$ and j extends in one and only one way to a map, still denoted j, from $\mathscr{I}^\natural$ into $\mathscr{I}$ such that $j(g.x) = g.j(x)$ for $g \in G^\natural$ and $x \in \mathscr{I}^\natural$. Since the restriction of j to $A^\natural$ is isometric and since any two points of $\mathscr{I}^\natural$ are transformed into two points of $A^\natural$ by an operation of $G^\natural$, one sees that j is isometric and that $j(\mathscr{I}^\natural)$ is convex. On the other hand, the uniqueness of j is immediate.

There remains finally to show that $j^{-1}(A) = A^\natural$. Let $x \in \mathscr{I}^\natural$, $x \notin A^\natural$, be such that $j(x) \in A$. Let C be a chamber of $A^\natural$ and let A' be an apartment of $\mathscr{I}^\natural$ containing C and x. Since $j(C) \subset A_\natural$, there exists a non-empty open subset Ω of A' such that $j(\Omega)$ is a non-empty open subset of $A_\natural$. The restriction of j to the convex hull Ω' of $\Omega \cup \{x\}$ is then an isometry onto a subset with non-empty interior of the affine subspace L of A generated by $A_\natural \cup \{j(x)\}$, whence one draws $\dim A' \geqslant \dim \Omega' = \dim L > \dim A_\natural$, which is absurd.

Remark **(9.1.18)**. — Conversely, if φ_1 is a valuation of the root datum of $G^\natural$ and j an isometry of the corresponding building $\mathscr{I}_1$ into $\mathscr{I}$, such that $j(g.x) = g.j(x)$ for $x \in \mathscr{I}_1$ and $g \in G^\natural$, that $j(\varphi_1) = \varphi$ and that $j(A_1) \subset A$ (where $A_1 = \varphi_1 + V^\natural$), then one can show that φ descends, that $\varphi^\natural = \varphi_1$ and that the hypotheses of (9.1.17) are satisfied with $S^\natural = T^\natural$.

Examples **(9.1.19)**. — *a)* If $V^\natural = V$, $T^\natural \subset T$ and $N^\natural \subset N$, one verifies easily that (DDR 2) is satisfied and that $\varphi^\natural$ satisfies conditions (V 2) and (V 5) of (6.2.1). Suppose moreover that $\operatorname{Card} \varphi^\natural_b(U^\natural_b) \geqslant 3$ for every $b \in \Phi^\natural$: then *φ descends to $G^\natural$ and the conditions of (9.1.17) are satisfied* (with $S^\natural = T^\natural$).

For example, let K be a commutative field endowed with a valuation ω, and let L be a subfield of K such that the restriction of ω to L is non-improper. Take for G (resp. $G^\natural$) the group of the rational points over K (resp. L) of a connected semi-simple algebraic group $\mathscr{G}$ defined and split over L, endowed with the root datum defined in (6.1.3) b). Then (DDR 1 bis) and (DDR 2) are satisfied. If φ (resp. φ_1) is the valuation of G (resp. $G^\natural$) defined in (6.2.3) b), one sees that φ descends to $G^\natural$ and that $\varphi^\natural = \varphi_1$. p. 203

Another example is that where K is a commutative field complete for a non-improper discrete valuation, with perfect residue field, where G is the group of the rational points over K of a connected semi-simple algebraic group $\mathscr{G}$ defined over K, endowed with the root datum defined in (6.1.3) c) and with the valuation φ that we shall construct later, and where $G^\natural$ is the group of the rational points over K of a semi-simple subgroup $\mathscr{G}^\natural$ of $\mathscr{G}$, defined over K and of relative rank over K equal to that of $\mathscr{G}$ (for example one of the maximal split subgroups constructed in [3], §7). Then (DDR 1 bis) and (DDR 2) are satisfied ([3], 3.12) and $\operatorname{Card} \varphi^\natural_b(U^\natural_b) = \infty$ for every $b \in \Phi^\natural$. Hence φ descends to $G^\natural$ and one may apply (9.1.17).

b) Under the conditions of the first paragraph of *a)*, and with the notations of (9.1.17), one has $A_\natural = A$ and one may identify $A^\natural$ with A thanks to the isometry j. The set of the walls (resp. affine roots) of $A^\natural$ is then contained in that of A (9.1.8), but is not necessarily equal to it. For example, under the hypotheses of the second paragraph of *a)*, the affine roots of A are the closed half-spaces $\alpha_{a,k}$ for $a \in \Phi$ and $k \in \omega(K^*)$, whereas those of $A^\natural$ are the $\alpha_{a,k}$ for $a \in \Phi$ and $k \in \omega(L^*)$.

One can also give examples where *no special point of A is a special point of $A^\natural$*.

c) Let $\mathscr{G}$ be a Chevalley scheme (over $\mathbf{Z}$), $\mathscr{T}$ a maximal split torus of $\mathscr{G}$ and $\mathscr{G}^\natural$ a closed group subscheme of $\mathscr{G}$ which is also a Chevalley scheme and such that $\mathscr{T}^\natural = \mathscr{T} \cap \mathscr{G}^\natural$ is a maximal split torus of $\mathscr{G}^\natural$. Put $V = \operatorname{Hom}(\mathrm{Mult}, \mathscr{T}) \otimes \mathbf{R}$, $V^\natural = \operatorname{Hom}(\mathrm{Mult}, \mathscr{T}^\natural) \otimes \mathbf{R}$ and let $\Phi \subset V^*$ and $\Phi^\natural \subset (V^\natural)^*$ be the root systems of $\mathscr{G}$ and $\mathscr{G}^\natural$. Let K be a field endowed with a non-improper valuation ω. Let $G = \mathscr{G}(K)$ and $G^\natural = \mathscr{G}^\natural(K)$, endowed with the root data of types Φ and $\Phi^\natural$ defined as in (6.1.3) b) (with $T = \mathscr{T}(K)$ and $T^\natural = \mathscr{T}^\natural(K)$); it is easy to see that the compatibility condition (DDR 1) of (9.1.9) is satisfied. Finally, let φ and φ_1 be the valuations of the root data in question defined as in (6.2.3) b). Let us denote by $\mathscr{I}$ (resp. $\mathscr{I}_1$) the building of G (resp. $G^\natural$) relative to φ (resp. φ_1). Then:

The valuation φ descends to $G^\natural$ and induces there a valuation $\varphi^\natural$ equivalent to φ_1 (6.2.5). Moreover, the hypotheses of (9.1.17) are satisfied and there exists (7.4.3) one and only one map $\gamma : \mathscr{I}_1 \to \mathscr{I}$, compatible with the actions of $G^\natural$ on $\mathscr{I}_1$ and $\mathscr{I}$, and such that $\gamma(\varphi_1) = \varphi$ and such that the restriction of γ to $A_1 = \varphi_1 + V^\natural$ is an affine injection of A_1 into A. If G is almost simple, γ multiplies distances by a constant.

We shall only sketch the proof of these assertions.

It is first of all easy to reduce to the case where $G^\natural$ is almost simple, which we shall suppose from now on.

Let $\overline{K}$ be a field containing K, endowed with a valuation $\overline{\omega}$ extending ω and whose value group is $\mathbf{R}$. Applying remark (9.1.12) b) to the inclusions

$$\mathscr{G}(\overline{K}) \supset \mathscr{G}(K) \supset \mathscr{G}^\natural(K)$$
$$\mathscr{G}(\overline{K}) \supset \mathscr{G}^\natural(\overline{K}) \supset \mathscr{G}^\natural(K)$$

and taking into account example *a)* above, one sees that one may, without loss of generality, substitute $\overline{K}$ for K. We shall therefore suppose from now on that $\omega(K^*) = \mathbf{R}$. p. 204

Let O be the ring of integers of K. The stabilizer $\widehat{P}_\varphi$ of φ (considered as a point of $\mathscr{I}$) in G is none other than $\mathscr{G}(O)$. Similarly, $\widehat{P}_{\varphi_1} = \mathscr{G}^\natural(O)$. Consequently $\widehat{P}_{\varphi_1} = \widehat{P}_\varphi \cap G^\natural$ and, since $G^\natural$ is transitive on $\mathscr{I}_1$, there exists one and only one map $\gamma : \mathscr{I}_1 \to \mathscr{I}$ compatible with the action of $G^\natural$, that is to say such that

$$\text{(1)} \qquad \gamma(gx) = g.\gamma(x) \qquad (x \in \mathscr{I}_1;\ g \in G^\natural).$$

Let us take g in $T^\natural$. In view of (6.1.3) b), one has $\nu(g) = \nu^\natural(g) \in V^\natural$ and $\nu^\natural(T^\natural) = V^\natural$, whence

$$\gamma(\varphi_1 + v) = \varphi + v \qquad (v \in V^\natural).$$

In particular, γ induces an affine injection of A_1 into A. Let $N^\natural$ be the normalizer of $T^\natural$ in $G^\natural$. The metric induced on A_1 (resp. $\gamma(A_1)$) by the metric of $\mathscr{I}_1$ (resp. of $\mathscr{I}$) is, up to a factor, the only Euclidean metric invariant under $N^\natural$. In view of (1), it follows that $\gamma|_{A_1}$ multiplies distances by a constant. Again by virtue of (1), and taking (7.4.18) into account, the same holds for γ. A reference to remark (9.1.18) completes the proof.

9.2. A descent theorem. —

(9.2.1) Let us take up for a moment the hypotheses and notations of (9.1.17). Put $\mathscr{I}_\natural = j(\mathscr{I}^\natural)$, $A_\natural = j(A^\natural)$ and let F be a facet of $\mathscr{I}$ meeting $\mathscr{I}_\natural$ of the largest possible dimension. Since $\mathscr{I}_\natural = G^\natural.A_\natural$, one may suppose that F meets $A_\natural$, hence is contained in A. One then sees immediately that the following conditions are satisfied:

(DI 1) *$\mathscr{I}_\natural$ is a convex subset of $\mathscr{I}$ stable under $G^\natural$;*
(DI 2) *$A_\natural = \mathscr{I}_\natural \cap A$ is an affine subspace of A;*
(DI 3) *there exists a facet F of $\mathscr{I}$ meeting $A_\natural$ (hence contained in A) which is contained in the closure of no other facet $F' \neq F$ of $\mathscr{I}$ meeting $\mathscr{I}_\natural$.*

(DV 1) $\varphi \in A_\natural$.

(9.2.2) Throughout 9.2, we are given a subgroup $G^\natural$ of G and a subset $\mathscr{I}_\natural$ of the building $\mathscr{I}$ of G satisfying conditions (DI 1), (DI 2) and (DI 3) above. The aim of this subsection is to show that these conditions, joined to conditions (DDR 1) (9.1.9), (DDR 2) (9.1.10), (DV 1) (9.2.1) and to two other conditions (DDR 3) (9.2.8) and (DV 2) (9.2.8), imply that the valuation φ descends to $G^\natural$ and that the hypotheses of (9.1.17) are satisfied.

Example **(9.2.3)**. — Let Γ be a group of automorphisms of G preserving the root datum and the bornology of G. We have seen (8.1.11) that Γ then operates on the building $\mathscr{I}$ of G in such a way that $\gamma(g.x) = \gamma(g).\gamma(x)$ whatever $\gamma \in \Gamma$, $g \in G$ ...

The setting at the top of p. 205: a group Γ acts on the building $\mathscr{I}$, preserving the apartment A and acting on it by affine automorphisms. More generally $\gamma \in \Gamma$ carries an apartment A' onto an apartment A'' by an affine isomorphism. Taking

$$\mathscr{I}^\natural = \mathscr{I}^\Gamma, \qquad G^\natural = G^\Gamma$$

(the Γ-fixed points) yields data satisfying (DI 1) and (DI 2). If in addition Γ preserves a chamber C of A, then $A^\natural$ contains the center of C (7.2.5) and (DI 3) holds with $F = C$ — the situation that occurs for quasi-split algebraic groups over a valued field.

(9.2.4) Dimension of a filter base. The dimension of a facet was defined in (7.2.4). More generally, let $\mathscr{F}$ be a filter base in a finite-dimensional affine space. The affine subspaces spanned by the members of $\mathscr{F}$ form a filter base $\mathscr{A}\mathscr{F}$; the *affine subspace generated by* $\mathscr{F}$ is the intersection L of the members of $\mathscr{A}\mathscr{F}$ — equivalently, the smallest member of $\mathscr{A}\mathscr{F}$ — and one sets $\dim \mathscr{F} = \dim L$. Note that $\mathscr{F}$ is contained in L in the sense of (7.2.1).

Proposition (9.2.5). — *Let M be a convex subset of $\mathscr{I}$, and let F (resp. F') be a facet of $\mathscr{I}$ that meets M and that lies in the closure of no other facet of $\mathscr{I}$ meeting M. Then:*

(i) $\dim F = \dim F'$;

(ii) $\dim F \cap M = \dim F' \cap M$;

(iii) *if M lies in an apartment A' of $\mathscr{I}$, then* $\dim F \cap M = \dim M$, *M is contained in the affine subspace L of A' generated by $F \cap M$, and $F \cap M$ has non-empty interior relative to L (cf. (7.2.1));*

(iv) *let A' be an apartment containing F and let F'' be a facet of A' meeting M. Suppose that at least one of the following holds:* $\dim F'' = \dim F$; *or* $\dim F'' \cap M = \dim F \cap M$; *or $F'' \cap M$ has non-empty interior relative to the affine subspace L generated by $F \cap M$ in A'. Then F'' lies in the closure of no other facet of $\mathscr{I}$ meeting M.*

Argument (outline). For (iii) one reduces to $A' = A$ and writes $F = F_{x,E}$ with $x \in \overline{M}$ and E the direction of F at x (7.2.4). Let $\widetilde{E}$ be the span of the convex cone E, and choose a convex $X \in F_{x,E}$ with $X \subset x + E$ and $X \cap M \subset L$ (when φ is discrete one may take for X the facet of § 2 corresponding to F). Picking y interior to $X \cap M$ relative to L, the open segment $]xy[$ consists of interior points of $X \cap M$. Suppose some $z \in M$ lies outside L. The interior of the triangle xyz lies in M. If $z \in x + \widetilde{E}$, a point $u \in \,]yz[\, \cap (x + E)$ produces an open segment inside $X \cap M$, forcing $u \in L$ and $z \in L$ — contradiction. Hence $z \notin x + \widetilde{E}$ and $[yz] \cap (x + \widetilde{E}) = \{y\}$, so there is a facet direction E' at x with $E \subset \overline{E}'$ and $]yz[\, \cap (x + E') \neq \emptyset$; this gives $F_{x,E'} \cap M \neq \emptyset$, contradicting the maximality hypothesis on F since $F \subset \overline{F}_{x,E'}$. Therefore $M \subset L$, and (iii) follows.

For (i) and (ii): translating by an element of G one may assume $F, F' \subset A$, and replacing M by $M \cap A$ one may assume $M \subset A$. Then (ii) is immediate from (iii). For (i), write $F' = F_{x',E'}$ with $x' \in \overline{M}$. By (iii), $M \subset L \subset x + \widetilde{E} = x' + \widetilde{E}$. Since E is a facet direction at x, the affine subspace $x + \widetilde{E}$ is an intersection of walls of A, and every facet direction at x' meeting $\widetilde{E}$ lies in $\widetilde{E}$. If $\dim E' > \dim E$ one would get $E' \cap \widetilde{E} = \emptyset$ and hence $M \cap F_{x',E'} \subset L \cap F_{x',E'} = \emptyset$, which is false. So $\dim E' \leq \dim E$.

For (iv): choose a facet F''' meeting M, maximal in the above sense, with $F'' \subset \overline{F'''}$; reduce to $F''' = F'$, $A' = A$, $M \subset A$, so $M \subset L$. Each of the three hypotheses implies the next in the chain

$$\dim F'' \cap M = \dim F \cap M \implies \mathrm{int}_L(F'' \cap M) \neq \emptyset \implies \dim F'' = \dim F,$$

the middle implication because all facets meeting L in a germ of non-empty interior have the same dimension, namely that of the intersection of the affine roots containing L. Finally $\dim F'' = \dim F = \dim F'$ gives $F'' = F'$.

Corollary (9.2.6). — *(i) Let F be a facet satisfying (DI 3), A' an apartment containing F, and L the affine subspace of A' generated by $F \cap A^\natural$. Then $A' \cap \mathscr{I}^\natural \subset L$.*

(ii) If A' is an apartment containing $A^\natural$, then $A' \cap \mathscr{I}^\natural = A^\natural$.

Argument (outline). (i) is (9.2.5)(iii) applied to $M = A' \cap \mathscr{I}^\natural$; (ii) follows from (i).

Proposition (9.2.7). — *Let F be a facet satisfying (DI 3) and C a chamber of A with $F \subset \overline{C}$. The retraction $\rho_{A;C}$ of $\mathscr{I}$ onto A maps $\mathscr{I}^\natural$ onto $A^\natural$; moreover its restriction to $\mathscr{I}^\natural$ is independent of the choice of such a chamber $C \subset A$.*

Argument (outline). For $x \in \mathscr{I}^\natural$ take an apartment A' containing x and F. Then $F \cap \mathscr{I}^\natural = F \cap \mathscr{I}^\natural \cap A \cap A'$ consists of fixed points of $\rho_{A;C}$ (7.2.1), so $\rho_{A;C}$ carries the affine span L of $F \cap \mathscr{I}^\natural$ in A' onto its span in A, i.e. onto $A^\natural$ by (9.2.5)(iii); with (9.2.6) this gives $\rho_{A;C}(x) \in A^\natural$. Since $F \cap \mathscr{I}^\natural$ has non-empty interior in L (resp. $A^\natural$), the restriction of $\rho_{A;C}$ to L is the *unique* affine map $L \to A^\natural$ restricting to the identity on $F \cap \mathscr{I}^\natural$ — whence the independence of C.

(9.2.8) Let $V^\natural$ be the direction of $A^\natural$ in V; notation as in (9.1.1).

Lemma. — $Z.A \cap \mathscr{I}^\natural = A^\natural$.

Argument (outline). Apply 7.6 with $\Phi_1 = \Phi_0$ (a closed subsystem) and $T_1 = T$, so that $G_1 = Z$. Let $\pi : Z.A \to \mathscr{I}_1$ be the canonical projection onto the building of Z ((7.6.4)(i)). On A, π is the quotient by the subspace $L_1 \subset V$ cut out by the kernels of the $a \in \Phi_0$; since $V^\natural \subset L_1$, the image $\pi(A^\natural)$ is a single point. Given $x \in A$, $z \in Z$, choose an apartment A_1 of $\mathscr{I}_1$ containing $\pi(A^\natural)$ and $\pi(z.x)$; then $A' = \pi^{-1}(A_1)$ is an apartment of $\mathscr{I}$ ((7.6.4)(ii)) containing $A^\natural$ and $z.x$. So if $z.x \in \mathscr{I}^\natural$ then $z.x \in A^\natural$ by (9.2.6).

(9.2.9) Standing hypotheses for the rest of § 9.2. In addition to $G^\natural$ and $\mathscr{I}^\natural$ satisfying (DI 1), (DI 2), (DI 3), we are given a root system $\Phi^\natural$ in $(V^\natural)^*$ and a generating root datum $(T^\natural, (U_b^\natural, M_b^\natural)_{b \in \Phi^\natural})$ in $G^\natural$ satisfying (DDR 1) (9.1.9) and (DDR 2) (9.1.10). Keeping the notation of (9.1.9), we assume:

(DDR 3) $\qquad T^\natural \subset Z \quad$ and $\quad M_b^\natural \subset M_b \quad$ for all $b \in \Phi^\natural$;

(DV 1) $\qquad \varphi \in A^\natural$;

(DV 2) $\qquad \mathrm{Card}\big(\varphi_b^\natural(U_b^\natural)\big) \geq 3 \quad$ for all $b \in \Phi^\natural$.

Theorem (9.2.10). — *Under the hypotheses of (9.2.9) the valuation φ descends to $G^\natural$: that is, the family $\varphi^\natural$ is a valuation of the root datum of $G^\natural$. If φ is discrete, so is $\varphi^\natural$.*

Argument (outline). One must verify (V 0)–(V 5) of (6.2.1). Condition (V 0) is exactly (DV 2), and (V 1), (V 3), (V 4) are already known from (9.1.10). Discreteness is immediate from the definition of $\varphi_b^\natural$ (9.1.6), which gives

$$\varphi_b^\natural(U_b^\natural) \;\subset \bigcup_{a \in \Phi,\ r \in \mathbf{R}_+^*,\ a^\natural = rb} r^{-1}\varphi_a(U_a) \qquad (b \in \Phi^\natural).$$

There remain (V 2) and (V 5), proved in (9.2.12) and (9.2.13).

Lemma (9.2.11). — *Let $N^\natural$ be the group generated by $T^\natural$ and the $M_b^\natural$, $b \in \Phi^\natural$ (cf. (6.1.2)(10)). For every $n \in N^\natural$:*

(1) $$n.A^\natural = A^\natural.$$

If $\nu^\natural(n) \in \mathrm{Aut}\,A^\natural$ denotes the restriction of n to $A^\natural$, then

(2) $$\lambda \circ \nu^\natural(n) = {}^v\nu^\natural(n)$$

where $\lambda : \mathrm{Aut}\,A^\natural \to \mathrm{Aut}\,V^\natural$ is the canonical homomorphism and ${}^v\nu^\natural : N^\natural \to {}^vW^\natural$ is the homomorphism given by the root datum of $G^\natural$ ((6.1.2)(10)).

Argument (outline). For (1) it is enough to treat $n \in T^\natural$ or $n \in M_b^\natural$; in both cases $n = zn'$ with $n' \in N$, $z \in Z$ (by (DDR 3) and (9.1.3)), and (9.2.8) gives

$$n.A^\natural \subset (n.A) \cap \mathscr{I}^\natural = (z.A) \cap \mathscr{I}^\natural \subset A^\natural.$$

For (2): if $\Phi^\natural = \emptyset$ then $V^\natural = \{0\}$ and there is nothing to prove. Otherwise $N^\natural$ is generated by the $M_b^\natural$, so it suffices to see that $\lambda \circ \nu^\natural(m) = r_b$ for $m \in M_b^\natural$. By (DR 5), $mU_c^\natural m^{-1} = U^\natural_{r_b(c)}$ for $c \in \Phi^\natural$; by (9.1.8), $\nu^\natural(m)$ carries a closed half-space $\{x \in A^\natural \mid c(x-\varphi) + k \geq 0\}$, $k \in \Gamma_c^\natural$, onto one of the form $\{x \in A^\natural \mid r_b(c)(x-\varphi) + h \geq 0\}$, $h \in \mathbf{R}$. As $\Phi^\natural$ spans $(V^\natural)^*$ and $\Gamma_c^\natural \neq \emptyset$, this forces $\lambda \circ \nu^\natural(m) = r_b$.

(9.2.12) Proof of (V 2). Let $b \in \Phi^\natural$, $m \in M_b^\natural$. By (9.2.11)(2), $\nu^\natural(m)$ is a translation $t \in V^\natural$ followed by the orthogonal reflection in the hyperplane $\{x \in A^\natural \mid b(x-\varphi) = 0\}$; hence it carries $\{x \in A^\natural \mid b(x-\varphi) + k \geq 0\}$ $(k \in \mathbf{R})$ onto $\{x \in A^\natural \mid -b(x-\varphi) - b(t) + k \geq 0\}$. With (9.1.8) this yields

$$\varphi^\natural_{-b}(mum^{-1}) = \varphi^\natural_b(u) - b(t) \qquad \text{for all } u \in U_b^\natural,$$

which is (V 2).

(9.2.13) Proof of (V 5). Let $b \in \Phi^\natural$, $u \in U_b^\natural$, $u \neq 1$, and put $k = \varphi_b^\natural(u)$. Choose $u', u'' \in U^\natural_{-b}$ and $m \in M_b^\natural$ with $u = u'mu''$, and set $k' = \varphi^\natural_{-b}(u')$, $k'' = \varphi^\natural_{-b}(u'')$. Let $\alpha, \alpha', \alpha''$ be the fixed-point sets in $A^\natural$ of u, u', u'' respectively; by (9.1.8),

$$\alpha = \{x \in A^\natural \mid b(x-\varphi) + k \ \geq 0\},$$
$$\alpha' = \{x \in A^\natural \mid -b(x-\varphi) + k' \geq 0\},$$
$$\alpha'' = \{x \in A^\natural \mid -b(x-\varphi) + k'' \geq 0\}.$$

It suffices to show that $\partial\alpha = \partial\alpha'$, equivalently that $\alpha \cap \alpha'$ is neither empty nor of non-empty interior.

Case 1: $\alpha \cap \alpha'$ has interior. Let x be an interior point. Then $u''.x = m^{-1}u'^{-1}u.x = m^{-1}.x \in A^\natural$, and for every $y \in \alpha''$, $d(u''.x, y) = d(u''.x, u''.y) = d(x,y)$. Since α'' contains an open half-space of $A^\natural$, this forces $u''.x = x$. Hence m fixes the non-empty open set $\operatorname{int}(\alpha \cap \alpha')$ pointwise and induces the identity on $A^\natural$, contradicting (9.2.11).

Case 2: $\alpha \cap \alpha' = \emptyset$. Take $x \in \alpha''$ and put $x' = u.x = u'm.x$, so $x' \in u(A^\natural) \cap u'(A^\natural) \subset \mathscr{I}^\natural$. The set $X = A^\natural \smallsetminus (\alpha \cup \alpha')$ is open and non-empty in $A^\natural$, so some facet F of A has $F \cap X$ of non-empty interior relative to $A^\natural$; by (9.2.5)(iv) this F satisfies (DI 3). Let C be a chamber of A with $F \subset \overline{C}$ and ρ the retraction of $\mathscr{I}$ onto A centred at C; by (9.2.7), $x'' = \rho(x') \in A^\natural$. Choose an apartment A' containing C and x', and a point $y \in A' \cap X$ with $y \neq x''$ and $[yx'']$ not parallel to $\partial\alpha$ or $\partial\alpha'$. The line through this segment meets both α and α', which cut out on it two half-lines of opposite directions; hence either there is $z \in \alpha$ with $y \in [zx'']$, or there is $z' \in \alpha'$ with $y \in [z'x'']$.

In the first case, since ρ does not increase distances and restricts to an isometry $A' \to A$ (7.4.19),

$$d(z,x') \ \geq \ d(z,x'') = d(z,y) + d(y,x'') = d(z,y) + d(y,x') \ \geq \ d(z,x'),$$

so $d(z,x') = d(z,y) + d(y,x')$ and $y \in [zx']$ (7.4.20). As z and x' both lie in the convex set $u(A^\natural)$, we get $y \in u(A^\natural)$. But $d(u.y, t) = d(u.y, u.t) = d(y,t)$ for every t in the closed half-space $\alpha = u(\alpha)$ of the affine space $u(A^\natural)$, which forces $u.y = y$; i.e. $y \in \alpha$, contradicting $y \notin \alpha \cup \alpha'$. The second case gives $y \in u'(A^\natural)$ and $y = u'.y$, the same contradiction. Hence $\alpha \cap \alpha' \neq \emptyset$, and the proofs of (V 5) and of Theorem (9.2.10) are complete.

(9.2.14) The canonical embedding j. The hypotheses of (9.1.17) hold with $S^\natural = T^\natural$: indeed $T^\natural.\varphi \subset A^\natural = \varphi + V^\natural$ by (9.2.11)(1). Consequently:

there exists exactly one isometry j of the building $\mathscr{I}^\natural$ of $G^\natural$ (relative to $\varphi^\natural$) into $\mathscr{I}$ such that $j(g.x) = g.j(x)$ for $x \in \mathscr{I}^\natural$, $g \in G^\natural$, and $j(\varphi^\natural) = \varphi$. If $A^\natural = \varphi^\natural + V^\natural$ is the model apartment of $\mathscr{I}^\natural$, then $j(A^\natural) = A^\natural$ and $A^\natural = A \cap j(\mathscr{I}^\natural)$. Moreover $j(\mathscr{I}^\natural) \subset \mathscr{I}^\natural$, *but in general $j(\mathscr{I}^\natural) \neq \mathscr{I}^\natural$.*

Furthermore, *the building $\mathscr{I}^\natural$ of $G^\natural$ does not depend (up to isomorphism) on the choice of $\mathscr{I}^\natural$ nor on the choice of φ within its equivalence class* (subject to the conditions of (9.2.9)): this follows from (8.1.10), since the bornology of $G^\natural$ defined by $\varphi^\natural$ is, by (8.1.7), the one induced by the bornology of G. In fact one can show that $\mathscr{I}^\natural$ depends only on $\mathscr{I}$ and $G^\natural$ (cf. (8.1.12)). The next proposition shows that j depends only on $\mathscr{I}^\natural$.

Proposition (9.2.15). — *(i) $A^\natural$ is the smallest non-empty convex subset of $\mathscr{I}^\natural$ stable under $T^\natural$.*

(ii) $j(\mathscr{I}^\natural)$ is the smallest non-empty convex subset of $\mathscr{I}^\natural$ stable under $G^\natural$.

(iii) Let i be an isometry of $\mathscr{I}^\natural$ into $\mathscr{I}^\natural$ with $i(g.x) = g.i(x)$ for $g \in G^\natural$, $x \in \mathscr{I}^\natural$. Then $i = j$.

Argument (outline). For (i), argue as in (2.8.11), taking for g an element of $T^\natural$ such that $v = {}^v\nu^\natural(g)$ lies in the kernel of no vector root outside Φ_0, and applying (7.4.30) in place of (2.8.10), after noting that (9.2.7) gives $\mathscr{I}^\natural \subset \widehat{P}_x.A^\natural$ for every $x \in A^\natural$. Assertion (ii) is an immediate consequence of (i). For (iii): $i(\mathscr{I}^\natural)$ is a non-empty convex subset of $\mathscr{I}^\natural$, so $A^\natural \subset i(\mathscr{I}^\natural)$; since the stabilizers in $G^\natural$ of two distinct chambers of $\mathscr{I}^\natural$ are distinct, one deduces $i(A^\natural) = A^\natural$ and $i|A^\natural = j|A^\natural$, whence (iii).

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10. Valuations of the classical groups

The aim of § 10 is to show that the “natural” root datum of the classical groups over a complete valued field — possibly of infinite rank over its center — can be equipped with a valuation, making explicit along the way several notions introduced in the earlier sections. Contrary to custom, the authors begin with the orthogonal, unitary and symplectic groups; the (much easier) linear group is treated more briefly in 10.2.

10.1. Orthogonal, unitary and symplectic groups. —

(10.1.1) Sesquilinear and pseudo-quadratic forms. (For the notions and properties used here, cf. [38], [39].)

Let K be a field, not necessarily commutative, σ an involution of K, and $\varepsilon \in K$ with $\varepsilon = \pm 1$. Put $\mathrm{K}_{\sigma,\varepsilon} = \{t - \varepsilon t^\sigma \mid t \in K\}$. Let X be a right vector space over K, let $f : X \times X \to K$ be a sesquilinear form (relative to σ) such that

(1) $$f(y,x) = \varepsilon f(x,y)^\sigma \qquad (x, y \in X)$$

(2) $$f(x,x) = 0 \quad \text{if } (\sigma,\varepsilon) = (\mathrm{id.}, -1) \qquad (x \in X),$$

and let $q : X \to K/\mathrm{K}_{\sigma,\varepsilon}$ be a *pseudo-quadratic form* associated with f, i.e. a function such that

(3) $$q(xk) = k^\sigma q(x) k \qquad (k \in K;\ x \in X)$$

(4) $$q(x+y) = q(x) + q(y) + \big(f(x,y) + \mathrm{K}_{\sigma,\varepsilon}\big) \qquad (x, y \in X).$$

If $(\sigma,\varepsilon) = (\mathrm{id.}, 1)$, then q is a quadratic form in the usual sense and f is the associated symmetric bilinear form.

Since a non-zero sesquilinear form has all of K as image, q determines f except when $\mathrm{K}_{\sigma,\varepsilon} = K$, which is equivalent to $\sigma = \mathrm{id.}$ and $\varepsilon \neq 1$; in that last case q is zero and f is alternating. In all cases f is *trace-valued*:

(5) $$f(x,x) = k + \varepsilon k^\sigma \qquad \text{for } k \in q(x).$$

Indeed, (4) gives

(6) $$q(x(1+t)) = q(x) + q(xt) + f(x,xt) + \mathrm{K}_{\sigma,\varepsilon}$$

whence $(f(x,x) - k - \varepsilon k^\sigma)t \in \mathrm{K}_{\sigma,\varepsilon}$ for every $t \in K$.

If $\operatorname{char} K \neq 2$, then f determines q. The same holds if $\operatorname{char} K = 2$ and the set of σ-invariant elements of K equals $\mathrm{K}_{\sigma,\varepsilon}$: for if $f = 0$ then $k = k^\sigma$ for every $k \in q(x)$ by (5), whence $q = 0$.

In particular, suppose there is a central element λ of K with $\lambda + \lambda^\sigma = 1$. This happens if $\operatorname{char} K \neq 2$ (take $\lambda = 1/2$), or if $\operatorname{char} K = 2$ and σ restricted to the center of K is not the identity. Taking $t = -\lambda$ in (6) gives

$$q(x) = \lambda f(x,x) + \mathrm{K}_{\sigma,\varepsilon}. \tag{7}$$

Moreover

$$\mathrm{K}_{\sigma,\varepsilon} = \{t \in K \mid t^\sigma = -\varepsilon t\}, \tag{8}$$

since $t^\sigma = -\varepsilon t$ implies $t = (\lambda t) - \varepsilon(\lambda t)^\sigma$.

Extension of scalars. Let $K' \supset K$ be a field and σ' an involution of K' extending σ. Then f and q extend naturally to $X \otimes_K K'$. For f this is the familiar operation; the extension q' of q is obtained as follows. Define $\mathrm{K}'_{\sigma',\varepsilon}$ as $\mathrm{K}_{\sigma,\varepsilon}$ was defined, and for

$$x = \sum_{i=1}^{m} x_i \otimes k_i \qquad (x_i \in X;\ k_i \in K') \tag{9}$$

put

$$q'(x) = \sum_{i=1}^{m} k_i^{\sigma'}\big(q(x_i) + \mathrm{K}'_{\sigma',\varepsilon}\big)k_i \ + \sum_{\substack{i,j=1\\ i<j}}^{m} k_i^{\sigma'}\big(f(x_i,x_j) + \mathrm{K}'_{\sigma',\varepsilon}\big)k_j. \tag{10}$$

This is well defined, since one checks easily that the right-hand side of (10) does not depend on the decomposition (9) chosen for x. Note that when $\mathrm{K}'_{\sigma',\varepsilon} \cap K \neq \mathrm{K}_{\sigma,\varepsilon}$ — which can happen if (8) fails — the extension q' does not determine q.

Standing assumptions. From now on the pair (f,q) is assumed *non-degenerate*, i.e.:

$$\text{for } x \in X, \quad f(x,X) = \{0\} \ \text{and} \ q(x) = 0 \implies x = 0. \tag{11}$$

We also assume (f,q) has *finite Witt index* r (recall the Witt index is the maximal dimension of a totally singular subspace $X' \subset X$, i.e. one with $f(X',X') = \{0\}$ and $q(X') = \{0\}$).

Put $I = \{\pm 1, \dots, \pm r\}$ and set $\varepsilon(i) = 1$ (resp. $\varepsilon(i) = \varepsilon$) for $i \in I$ with $i > 0$ (resp. $i < 0$). Then X admits a decomposition

$$X = \coprod_{i\in I} e_i.K \ + \ X_0$$

where the e_i are elements of X and X_0 a vector subspace, such that for $i,j \in I$:

$$\begin{cases} f(e_i,e_j) = q(e_i) = 0 & \text{if } i \neq -j, \\ f(e_i,e_{-i}) = \varepsilon(i), \\ f(e_i,X_0) = \{0\}, \\ \qquad 0 \notin q(X_0 - \{0\}). \end{cases} \tag{12}$$

Such a decomposition being fixed once and for all, put $X_i = e_i.K$; for an endomorphism g of X let $c_{ij}(g)$ $(i,j \in I \cup \{0\})$ denote the element of $\operatorname{Hom}(X_j,X_i)$ obtained by composing the canonical injection $X_j \to X$, g, and the canonical projection $X \to X_i$ — i.e. the (i,j) entry of the matrix of g for this decomposition. Since a basis element e_i has been singled out in X_i for $i \neq 0$, we may, where convenient, identify $\operatorname{Hom}(X_j,X_i)$ with K when $i \neq 0 \neq j$, with X_0 when $i = 0 \neq j$, and with the dual X_0^* of X_0 when $i \neq 0 = j$.

In the remainder of 10.1, Z denotes the set $\{(z,k) \mid z \in X_0,\ k \in K,\ q(z) = k + \mathrm{K}_{\sigma,\varepsilon}\}$. *Unless explicitly stated otherwise, r is always assumed non-zero, and* $\neq 1$ *if* $(\sigma, X_0) = (\mathrm{id.}, \{0\})$.

(10.1.2) For $i \in I$ and $(z,k) \in Z$ (resp. $Z - \{(0,0)\}$), let $u_i(z,k)$ (resp. $m_i(z,k)$) be the linear transformation of X defined by

$$u_i(z,k) : \begin{cases} x \mapsto x - e_{-i}.f(z,x).\varepsilon(i) & (x \in X_0) \\ e_i \mapsto e_i + z - e_{-i}.k.\varepsilon(i) & \\ e_j \mapsto e_j & (j \in I;\ j \neq i) \end{cases}$$

$$\left(\text{resp.}\quad m_i(z,k) : \begin{cases} x \mapsto x - z.k^{-1}.f(z,x) & (x \in X_0) \\ e_i \mapsto -e_{-i}.k.\varepsilon(i) & \\ e_{-i} \mapsto -e_i.(k^\sigma)^{-1}.\varepsilon(-i) & \\ e_j \mapsto e_j & (j \in I;\ j \neq \pm i) \end{cases}\right)$$

For $i, j \in I$ with $j \neq \pm i$, and $k \in K$ (resp. $K - \{0\}$), let $u_{ij}(k)$ (resp. $m_{ij}(k)$) be the linear transformation of X defined by

$$u_{ij}(k) : \begin{cases} x \mapsto x & (x \in X_0) \\ e_i \mapsto e_i + e_{-j}.k^\sigma.\varepsilon(-j) & \\ e_j \mapsto e_j - e_{-i}.k.\varepsilon(i) & \\ e_h \mapsto e_h & (h \in I - \{i,j\}) \end{cases}$$

$$\left(\text{resp.}\quad m_{ij}(k) : \begin{cases} x \mapsto x & (x \in X_0) \\ e_i \mapsto e_{-j}.k^\sigma.\varepsilon(-j) & \\ e_j \mapsto -e_{-i}.k.\varepsilon(i) & \\ e_{-i} \mapsto e_j.k^{-1}.\varepsilon(i) & \\ e_{-j} \mapsto -e_i.(k^\sigma)^{-1}.\varepsilon(-j) & \\ e_h \mapsto e_h & (h \in I - \{\pm i, \pm j\}) \end{cases}\right)$$

Consider the space $V^* = \mathbf{R}^r$ with its usual Euclidean norm, let $(a_h)_{h=1,\ldots,r}$ be its canonical basis, and set $a_{-h} = -a_h$ and $a_{ij} = a_i + a_j$ for $i, j \in I$, $j \neq \pm i$. For $i \in I$ put

$$\begin{aligned} U_{a_i} &= \{u_i(z,k) \mid (z,k) \in Z\}, \\ U_{2a_i} &= \{u_i(0,k) \mid k \in \mathrm{K}_{\sigma,\varepsilon}\}, \\ M^0_{a_i} &= \{m_i(z,k) \mid (z,k) \in Z - \{(0,0)\}\}, \\ M^0_{2a_i} &= \{m_i(0,k) \mid k \in \mathrm{K}_{\sigma,\varepsilon} - \{0\}\}, \end{aligned}$$

for $i, j \in I$ with $j \neq \pm i$ put

$$U_{a_{ij}} = \{u_{ij}(k) \mid k \in K\}, \qquad M^0_{a_{ij}} = \{m_{ij}(k) \mid k \in K - \{0\}\},$$

and for $a \in V^* - \{a_i, 2a_i, a_{ij} \mid i \in I,\ j \in I - \{\pm i\}\}$ put $U_a = \{1\}$.

The set Φ of those $a \in V^*$ with $U_a - U_{2a} \neq \emptyset$ is a root system. Precisely, with i, j ranging over I and $j \neq \pm i$:

(B)	$\Phi = \{a_i, a_{ij}\}$	when $X_0 \neq \{0\}$ and $(\sigma,\varepsilon) = (\text{id.}, 1)$;
(BC)	$\Phi = \{a_i, 2a_i, a_{ij}\}$	when $X_0 \neq \{0\}$ and $\sigma \neq \text{id.}$;
(C)	$\Phi = \{2a_i, a_{ij}\}$	when $X_0 = \{0\}$ and $(\sigma,\varepsilon) \neq (\text{id.}, 1)$;
(D)	$\Phi = \{a_{ij}\}$	when $X_0 = \{0\}$ and $(\sigma,\varepsilon) = (\text{id.}, 1)$.

Remark (10.1.3) ("Change of coordinates"). — Let $\eta \in K$ be equal to 1 if $\sigma = \text{id.}$ and to ± 1 if $\sigma \neq \text{id.}$, and let k be an invertible element of K with $k^\sigma = \eta k$. The conditions imposed on

$\varepsilon, \sigma, f, q$ are preserved, and the groups $U_{a_i}, U_{a_{ij}}$ are unchanged, if one replaces $\sigma, \varepsilon, f, q, e_i, X_0$ by $\sigma', \varepsilon', f'$, etc., where

$$\varepsilon' = \varepsilon\eta,$$
$$t^{\sigma'} = kt^{\sigma}k^{-1} \quad \text{for } t \in K \quad (\text{whence } \mathrm{K}_{\sigma',\varepsilon'} = k.\mathrm{K}_{\sigma,\varepsilon}),$$
$$f' = kf, \qquad q' = kq,$$
$$e_i' = e_i.k^{-1} \ \text{ or } \ e_i \quad \text{according as } i < 0 \text{ or } i > 0,$$
$$X_0' = X_0.$$

If $\sigma \neq$ id. and $\varepsilon \neq 1$, one may take for k a non-zero anti-invariant element, e.g. one of the form $h - h^{\sigma}$; then $\varepsilon' = 1$.

If $(\sigma, \varepsilon) \neq$ (id., 1) and one takes for k an invertible element of $\mathrm{K}_{\sigma,\varepsilon}$ (whose inverse then also lies in $\mathrm{K}_{\sigma,\varepsilon}$), one gets $\varepsilon' = -1$ and $1 \in \mathrm{K}_{\sigma',\varepsilon'}$. On the other hand, if $(\sigma, \varepsilon) =$ (id., 1) and $X_0 \neq \{0\}$, it is possible to choose k so that q' takes the value 1 at an arbitrarily prescribed point of X_0. Hence, for the study of the system of groups $(U_a)_{a\in\Phi}$, one may always reduce to one of the following cases:

(1) $\varepsilon = -1$ and $1 \in \mathrm{K}_{\sigma,\varepsilon}$ (whence $\Phi = C_r$ or BC_r);

(2) $(\sigma, \varepsilon) = (\text{id.}, 1)$ and $1 \in q(X_0)$ (whence $\Phi = B_r$);

(3) $(\sigma, \varepsilon, X_0) = (\text{id.}, 1, \{0\})$ (whence $\Phi = D_r$).

(10.1.4) Let $g \in \mathrm{Hom}(X, X)$. One says g is a *similitude* if there is an element c of the center of K, invariant under σ, such that for $x, y \in X$

$$f(gx, gy) = cf(x, y), \qquad q(gx) = cq(x).$$

This element c is then unique; it is called the *ratio* (multiplier) of the similitude g and is written $c(g)$. An *isometry* is by definition a similitude of ratio 1.

We write G^0 for the group generated by the U_a ($a \in \Phi$), $\mathrm{Is}(f, q)$ for the group of isometries, $\mathrm{Sim}(f, q)$ for the group of similitudes, and T^0 (resp. $T(f, q)$; resp. $\widetilde{T}(f, q)$) for the group of those elements t of G^0 (resp. $\mathrm{Is}(f, q)$; resp. $\mathrm{Sim}(f, q)$) stabilizing each of the X_i ($i \in [-r, +r]$), i.e. represented by a diagonal matrix $((c_{ij}(t)))$. If Φ is of type D, it is known that the linear subspaces of X of dimension r on which q vanishes (the maximal totally singular subspaces) split into two families, two such subspaces belonging to the same family if and only if their intersection has even codimension in each of them; we then write $\mathrm{Is}^+(f, q)$ (resp. $\mathrm{Sim}^+(f, q)$) for the subgroup of index 2 of $\mathrm{Is}(f, q)$ (resp. $\mathrm{Sim}(f, q)$) consisting of those elements preserving each of these two families.

Proposition (10.1.5). — *(i) For a linear transformation t of X stabilizing each X_i ($i \in I \cup \{0\}$) to be a similitude of ratio c, it is necessary and sufficient that*

(1) $$c_{ii}(t)^{\sigma}.c_{-i,-i}(t) = c \qquad \text{for } i \in I$$

and

(2) $$q\big(c_{00}(t)(z)\big) = c.q(z) \qquad \text{for } z \in X_0.$$

(ii) The group $\widetilde{T}(f, q)$ normalizes each of the groups U_a ($a \in \Phi$) and G^0.

(iii) If Φ is not of type D, then $\mathrm{Is}(f, q) = T(f, q).G^0$ *and* $\mathrm{Sim}(f, q) = \widetilde{T}(f, q).G^0$. *If Φ is of type D, then* $\mathrm{Is}^+(f, q) = T(f, q).G^0$ *and* $\mathrm{Sim}^+(f, q) = \widetilde{T}(f, q).G^0$.

Proposition (10.1.6). — *Let T be a subgroup of $\widetilde{T}(f, q)$ containing T^0, let G be the group $T.G^0$, and for $a \in \Phi$ put $M_a = T.M_a^0$. Then the pair $(T, (U_a, M_a)_{a\in\Phi})$ is a generating root datum of type Φ in G. With the notation of (6.1.2)(2) one has, for this datum, $m(u_i(z, k)) = m_i(z, k)$*

and $m(u_{ij}(k)) = m_{ij}(k)$; in particular the set M_a^0 $(a \in \Phi)$ coincides with the set so denoted in (6.1.2)(2).

These two propositions are proved in (10.1.10) and (10.1.12) respectively.

Proposition (10.1.7). — *(i) The group N^0 generated by the M_a^0 $(a \in \Phi)$ stabilizes X_0 and permutes the X_i $(i \in I)$. The group ${}^vW$ of permutations of the index set I that it induces is the group W_C of all permutations commuting with multiplication by -1, or the group W_D of all even permutations with that property, according as the root system Φ is not, or is, of type D.*

(ii) The group N_1' generated by the $m_{ij}(1)$ $(i, j \in I;\ j \neq \pm i)$ is a finite subgroup of N^0 preserving the set $\{\pm e_i \mid i \in I\}$, fixing X_0, and whose image in ${}^vW$ is the group W_D.

(iii) Suppose one of the conditions (1), (2), (3) of (10.1.3) holds, and let e_0 be an element of X_0 equal to 0 in case (1) and such that $q(e_0) = 1$ in case (2). Then the group N' generated by N_1' and, if Φ is not of type D, by the $m_i(e_0, 1)$ $(i \in I)$, is a finite group preserving the set $\{\pm e_i \mid i \in I\}$, whose image in ${}^vW$ is all of ${}^vW$.

Argument (outline). It is clear that N_1' preserves $\{\pm e_i \mid i \in I\}$ and fixes X_0; hence it is a finite group. Finiteness of N' follows because every element of the given generating system — hence every element of the group itself — preserves $\{\pm e_i \mid i \in I\}$ and induces on X_0 either the identity or the transformation $x \mapsto x - e_0.f(e_0, x)$, which has order 1 or 2. The remaining assertions are obvious.

Lemma (10.1.8). — *(i) For $i \in I$, U_{a_i} (resp. U_{2a_i}) is the group of all isometries $1 + n$ of X such that*

$$\begin{aligned} &n(X_i) \subset X_0 + X_{-i} && \text{(resp. } X_{-i}),\\ &n(X_0) \subset X_{-i} && \text{(resp. } = \{0\}),\\ &n(X_j) = \{0\} && \text{for } j \neq i, 0. \end{aligned}$$

This group acts simply transitively on the set of singular lines of q (i.e. one-dimensional subspaces on which q vanishes) contained in $X_{-i} + X_0 + X_i$ (resp. $X_{-i} + X_i$) and distinct from X_{-i}.

(ii) For $i, j \in I$ with $j \neq \pm i$, $U_{a_{ij}}$ is the group of all isometries $1 + n$ of X such that

$$\begin{aligned} &n(X_i) \subset X_{-j},\\ &n(X_j) \subset X_{-i},\\ &n(X_h) = \{0\} && \text{for } h \in [-r, r] - \{i, j\}. \end{aligned}$$

This group acts simply transitively on the set of lines of $X_i + X_{-j}$ distinct from X_{-j}. It is contained in $\mathrm{Is}^+(f, q)$ if Φ is of type D.

The verification is easy.

Lemma (10.1.9). — *(i) For every $i \in I$, the stabilizer of e_i in G^0 acts transitively on the lines Y of X with $f(e_i, Y) \neq \{0\}$ and $q(Y) = \{0\}$.*

(ii) The group G^0 acts transitively on the singular lines of q.

Argument (outline). (i) is proved by induction on r; one may assume $i = 1$. For $r = 1$ the assertion is (10.1.8)(i). Let $r \geq 2$ and let $x \in X$ satisfy $f(e_1, x) \neq 0$ and $q(x) = 0$; it suffices to find $g \in G^0$ with $g(e_1) = e_1$ and $g(x) \in X_{-1}$. The set of $y \in X_1 + X_r$ with $f(y, x) = 0$ is a line distinct from X_1; by (10.1.8)(ii), after transforming x by a suitable element of $U_{a_{r,-1}}$, one may assume this line is X_r, i.e. $f(e_r, x) = 0$. Then $x \in \sum_{i=-r+1}^{r} X_i$; write $x = x' + e_r.k$ with $x' \in X' = \sum_{i=-r+1}^{r-1} X_i$ and $k \in K$. By the inductive hypothesis applied to X' and the restrictions of f and q, there is an element of G^0 fixing e_1, e_r, e_{-r} and carrying x' into a point of X_{-1}. One may therefore assume directly that $x \in X_{-1} + X_r$; but then $U_{a_{-1,-r}}$ contains an element g with the required properties, by (10.1.8)(ii).

For (ii) it suffices to show that if $x \in q^{-1}(0)$ there is $g \in G^0$ with $g(x) \in X_1$. Assume $x \neq 0$; there is $i \in I$ with $f(e_i, x) \neq 0$. If $i \neq -1$, put $y = e_{-1} + e_{-i}$; if $i = -1$ and $r \geq 2$, put $y = e_1 - e_2 + e_{-2} - e_{-1}$; finally, if $i = -1$ and $r = 1$ — in which case $\dim X_0 > 0$ by hypothesis — take $y = u_1(k, z)(e_1) = e_1 + z - e_{-1}k\varepsilon(1)$ with $(z, k) \in Z - \{(0, 0)\}$. In each case $q(y) = 0$, $f(e_1, y) \neq 0$ and $f(e_i, y) \neq 0$. By (i) there is an element g_1 (resp. g_i) of the stabilizer of e_1 (resp. e_i) in G^0 with $g_1(y) \in X_{-1}$ (resp. $g_i(y) \in xK$), whence the assertion.

(10.1.10) Proof of Proposition (10.1.5). Assertion (i) results from a straightforward computation, and (ii) is an immediate consequence of (10.1.8).

For (iii) we restrict to the case where Φ is not of type D, the case $\Phi = D_r$ being handled analogously. By (10.1.8) one has $G^0 \subset \mathrm{Is}(f, q) \subset \mathrm{Sim}(f, q)$, so it suffices to prove $\mathrm{Sim}(f, q) \subset \widetilde{T}(f, q).G^0$. The proof is by induction on r, the assertion being obvious for $r = 0$ (when $\Phi = D_r$ one starts the induction at $r = 1$). Let $g \in \mathrm{Sim}(f, q)$. Applying (10.1.9)(ii) and then (10.1.9)(i), one finds $g^0 \in G^0$ with $g^0(g(e_1)) = e_1$ and $g^0(g(e_{-1})) = e_{-1}$, whence $g^0(g(X')) = X'$, where

$$X' = \{x \mid x \in X,\ f(e_1, x) = f(e_{-1}, x) = 0\} = \sum_{i \neq \pm 1} X_i.$$

By the inductive hypothesis applied to the restriction of $g^0 g$ to X', one gets $g^0 g \in G^0.\widetilde{T}(f, q)$, whence $g \in G^0.\widetilde{T}(f, q)$. q.e.d.

(10.1.11) Relations. In the relations below, h, i, j, i', j' denote elements of I, k, k' elements of K, and z, z' elements of X_0. These elements are in each case assumed to satisfy the conditions needed for the relations written to make sense (thus in relation (3) one must have $(z, k) \in Z - \{(0, 0)\}$; in relation (15), $j \neq \pm i$ and $j' \neq \pm i'$; etc.).

(1) $u_i(z, k).u_i(z', k') = u_i(z + z',\ k + k' + f(z, z'))$,

(2) $u_i(z, k)^{-1} = u_i(-z,\ \varepsilon k^\sigma)$,

(3) $u_i(z, k) = u_{-i}\big(-z.k^{-1}.\varepsilon(i),\ k^{-1}.\varepsilon\big).m_i(z, k).u_{-i}\big(-z.(k^\sigma)^{-1}.\varepsilon(-i),\ k^{-1}.\varepsilon\big)$,

(4) $u_{ij}(k).u_{ij}(k') = u_{ij}(k + k')$,

(5) $u_{ij}(k)^{-1} = u_{ij}(-k)$,

(6) $u_{ij}(k) = u_{-i,-j}\big(-(k^\sigma)^{-1}.\varepsilon(i).\varepsilon(j)\big).m_{ij}(k).u_{-i,-j}\big(-(k^\sigma)^{-1}.\varepsilon(i).\varepsilon(j)\big)$,

(7) $u_{ji}(k) = u_{ij}(-\varepsilon k^\sigma)$,

(8) $\big(u_i(z, k),\ u_i(z', k')\big) = u_i\big(0,\ f(z, z') - f(z', z)\big)$,

(9) $\big(u_i(z, k),\ u_{i'}(z', k')\big) = u_{ii'}(f(z, z'))$ for $i' \neq \pm i$,

(10) $\big(u_{-i}(z, k),\ u_{ij}(k')\big) = u_j\big(-z.k'.\varepsilon(i),\ \varepsilon k'^\sigma.k^\sigma.k'\big).u_{-i,j}\big(-k.k'.\varepsilon(i)\big)$,

(11) $\big(u_i(z, k),\ u_{i',j'}(k')\big) = 1$ for $i \notin \{-i', -j'\}$,

(12) $\big(u_{ih}(k),\ u_{-h,j}(k')\big) = u_{ij}\big(-k.k'.\varepsilon(-h)\big)$ for $j \neq \pm i$,

(13) $\big(u_{ih}(k),\ u_{-h,i}(k')\big) = u_i\big(0,\ (k'^\sigma.k^\sigma - k.k'.\varepsilon).\varepsilon(h)\big)$,

(14) $\big(u_{ij}(k),\ u_{ij'}(k')\big) = 1$ for $j' \neq -j$,

(15) $\big(u_{ij}(k),\ u_{i',j'}(k')\big) = 1$ for $i', j' \notin \{\pm i, \pm j\}$.

The verifications are easy.

(10.1.12) Proof of Proposition (10.1.6). For $a \in \Phi$, the quotient of two elements of M_a^0 stabilizes each X_i; hence it lies in T^0, and $M_a = T.M_a^0$ is a right coset of T. Granting this, the fact that the system $(T, (U_a, M_a)_{a \in \Phi})$ satisfies axioms (DR 1), (DR 2) and (DR 4) of root data (6.1.1) follows from the relations of (10.1.11). Axiom (DR 3) is obvious. To check (DR 5) it clearly suffices to treat the case where n_a belongs to M_a^0; the required relation then follows from the shape of the m_i and m_{ij}, from the fact that they lie in $\mathrm{Is}(f, q)$ (cf. (10.1.5)), and from Lemma (10.1.8). Axiom (DR 6) follows because the matrix $((c_{ij}(g)))$ (cf. (10.1.1)) representing an element g of TU^+ (resp. of U^-) is lower triangular (resp. upper triangular with only 1's on the diagonal). Finally, the assertion of the statement concerning the $m(u_i)$ and $m(u_{ij})$ is a consequence of relations (3) and (6) of (10.1.11).

(10.1.13) For any function $\omega : K \to \mathbf{R} \cup \{\infty\}$, we write $\varphi^\omega = (\varphi^\omega_a)_{a\in\Phi}$ for the system of functions $\varphi^\omega_a : U_a \to \mathbf{R} \cup \{\infty\}$ defined by

(1) $$\varphi^\omega_{a_i}\big(u_i(z,k)\big) = \tfrac{1}{2}\omega(k) \qquad \text{if } a_i \in \Phi$$

(2) $$\varphi^\omega_{2a_i}\big(u_i(0,k)\big) = \omega(k) \qquad \text{if } 2a_i \in \Phi$$

(3) $$\varphi^\omega_{a_{ij}}\big(u_{ij}(k)\big) = \omega(k).$$

(10.1.14) Let $\omega : K \to \mathbf{R} \cup \{\infty\}$ be a non-improper valuation of K, invariant under σ, and let $\widehat{K}$ be the completion of K for ω. The continuous extensions of ω and σ to $\widehat{K}$ are again written ω and σ. Put $\widehat{X} = X \otimes_K \widehat{K}$, $\widehat{X}_i = X_i \otimes_K \widehat{K}$, write $\widehat{f} : \widehat{X} \times \widehat{X} \to \widehat{K}$ and $\widehat{q} : \widehat{X} \to \widehat{K}/\widehat{\mathrm{K}}_{\sigma,\varepsilon}$ for the natural extensions of f and q (10.1.1), and set

$$\widehat{Z} = \{(z,k) \mid z \in \widehat{X}_0,\ k \in \widehat{K},\ \widehat{q}(z) = k + \widehat{\mathrm{K}}_{\sigma,\varepsilon}\}.$$

We equip K and $\widehat{K}$ with the topology defined by ω, and every finite-dimensional vector space over K or $\widehat{K}$ with the topology naturally associated with it (the topology transported from the product topology on K^m or $\widehat{K}^m$). Closure for these topologies is denoted by a bar. One has

(1) $$\widehat{\mathrm{K}}_{\sigma,\varepsilon} \subset \overline{\mathrm{K}_{\sigma,\varepsilon}}.$$

Put

$$X^\perp = \{x \in X \mid f(x,X) = \{0\}\}, \qquad \widehat{X}^\perp = \{x \in \widehat{X} \mid f(x,\widehat{X}) = \{0\}\}.$$

One also has $X^\perp = \{z \in X_0 \mid f(z, X_0) = \{0\}\}$ and

(2) $$\widehat{X}^\perp = \{z \in \widehat{X}_0 \mid f(z, X_0) = \{0\}\} = X^\perp \otimes_K \widehat{K}.$$

Note further that, for $(z,k) \in Z$,

(3) $$\omega(f(z,z)) = \omega(k + \varepsilon k^\sigma) \ \geq\ \omega(k).$$

Theorem (10.1.15). — *With the notation and conventions of (10.1.14), the following conditions are equivalent.*

(i) *For* $(z,k), (z',k') \in Z$, $\omega(f(z,z')) \ \geq\ \frac{1}{2}\big(\omega(k) + \omega(k')\big)$.

(ii) *For* $(z,k), (z',k') \in Z$, $\omega(f(z,z')) \ \geq\ \inf\{\omega(k), \omega(k')\}$.

(iii) $\widehat{q}(\widehat{X}_0 - \widehat{X}^\perp) \cap \overline{\mathrm{K}_{\sigma,\varepsilon}} = \emptyset$.

(iv) φ^ω *is a valuation of the root datum* $(T, (U_a))$.

Remarks (10.1.16). — *a)* If $\widehat{\mathrm{K}}_{\sigma,\varepsilon}$ is closed in $\widehat{K}$ (for instance if K is finite-dimensional over its center, or if $\varepsilon = 1$ and there is a central element λ with $\lambda + \lambda^\sigma = 1$, cf. (10.1.1)(8)), then condition (iii) means that the restriction of q to $\widehat{X}_0$ does not vanish outside $\widehat{X}^\perp$.

b) If $\mathrm{char}\, K = 2$, it may happen that q has non-trivial zeros in $\widehat{X}^\perp$. For example, suppose K is a non-perfect commutative field whose completion is perfect, and let σ be the identity, X_0 the field $K^{1/2}$ regarded as a vector space over K, $q|_{X_0}$ the quadratic form $z \mapsto z^2$, and $f|_{X_0\times X_0}$ the zero bilinear form. Then $\widehat{q}$ is the square of a linear form and hence has a kernel of codimension 1 in $\widehat{X}_0$.

Lemma (10.1.17). — *Let Y be a finite-dimensional subspace of X_0 and $\widehat{Y} = Y \otimes_K \widehat{K}$; identify Y with $Y \otimes 1$. Then:*

(i) There is a continuous function $\overline{q} : \widehat{Y} \to \widehat{K}$ such that $\overline{q}(Y) \subset K$ and

$$q(z) = \overline{q}(z) + \mathrm{K}_{\sigma,\varepsilon} \qquad \text{for } z \in Y,$$

$$\widehat{q}(z) = \overline{q}(z) + \widehat{\mathrm{K}}_{\sigma,\varepsilon} \qquad \text{for } z \in \widehat{Y}.$$

(ii) Let $(z,k) \in \widehat{Y}$ and $z_n \in Y$ $(n \in \mathbf{N})$ with $\lim_{n\to\infty} z_n = z$. Then there exist $k_n \in K$ with $\lim_{n\to\infty} k_n = k$ and $q(z_n) = k_n + \mathrm{K}_{\sigma,\varepsilon}$ for every n.

Argument (outline). In what follows s, s' denote integers running through $[1, \dim Y]$. Let (x_s) be a basis of Y and, for each s, let d_s be a representative of $q(x_s)$ in K. Then the function $\overline{q}$ defined by

$$\overline{q}\Big(\sum_s x_s\xi_s\Big) = \sum_s \xi_s^\sigma d_s \xi_s + \sum_{s<s'} \xi_s^\sigma f(x_s, x_{s'})\xi_{s'}$$

satisfies the conditions of (i).

For (ii), let (z,k) and z_n be as in the statement. By (10.1.14)(1) there are $k'_n \in \mathrm{K}_{\sigma,\varepsilon}$ with $\lim_{n\to\infty} k'_n = k - \overline{q}(z)$. The sequence $k_n = \overline{q}(z_n) + k'_n$ then has the required property.

Lemma (10.1.18). — *Let $(z,k), (z_0,k_0) \in Z - \{(0,0)\}$ and*

$$\eta = \omega(k) + \omega(k_0) - 2\omega(f(z,z_0)).$$

Suppose η is strictly positive. For $n \in \mathbf{N}^*$, define $z_n \in X_0$, $k_n \in K$ and $\lambda_n \in K^*$ inductively by

$$\lambda_n = -f(z_{n-1},z)^{-1}.k_{n-1}, \qquad z_n = z_{n-1} + z.\lambda_n, \qquad k_n = \lambda_n^\sigma.k.\lambda_n.$$

Then:

(i) For every $n \in \mathbf{N}$,

(1) $$(z_n, k_n) \in Z$$

(2) $$\omega(\lambda_{n+1}) = (2^n - 1)\eta + \omega(k_0) - \omega(f(z_0,z));$$

(3) $$\omega(f(z_n,z)) = \omega(f(z_0,z));$$

(4) $$\omega(k_n) = (2^n-1)\eta + \omega(k_0).$$

(ii) The sequence (z_n) converges in $z.\widehat{K} + z_0.\widehat{K}$ to a point $\widehat{z}$ such that

$$\omega(f(\widehat{z},z)) = \omega(f(z_0,z)) \qquad \text{and} \qquad \widehat{q}(\widehat{z}) \subset \overline{\mathrm{K}_{\sigma,\varepsilon}}/\widehat{\mathrm{K}}_{\sigma,\varepsilon}.$$

(iii) For every real number A there exist $(z',k'), (z'_0,k'_0) \in Z$ with $z', z'_0 \in z.K + z_0.K$, $\omega(f(z'_0,z')) = \omega(f(z_0,z))$, $\omega(k') \geq A$ and $\omega(k'_0) \geq A$.

Argument (outline). The verification of (i) by induction on n is immediate, taking (10.1.14)(3) into account. Assertion (ii) follows at once from (i). For (iii), note first that by (i) there is an integer n with $\omega(k_n) \geq A$; put $(z',k') = (z_n,k_n)$ and apply (i) again, substituting (z',k') and (z,k) for (z,k) and (z_0,k_0) respectively.

(10.1.19) Proof of Theorem (10.1.15). It is clear that (i) implies (ii). The implications (ii)⇒(i) and (iii)⇒(i) are immediate consequences of (10.1.18)(iii) and (10.1.18)(ii) respectively.

Suppose there is $\widehat{z} \in \widehat{X}_0 - \widehat{X}^\perp$ with $\widehat{q}(\widehat{z}) \in \overline{\mathrm{K}_{\sigma,\varepsilon}}/\mathrm{K}_{\sigma,\varepsilon}$, and let $z' \in X_0$ and $\widehat{k} \in \overline{\mathrm{K}_{\sigma,\varepsilon}}$ be such that $f(\widehat{z},z') \neq 0$ and $\widehat{q}(\widehat{z}) = \widehat{k} + \widehat{\mathrm{K}}_{\sigma,\varepsilon}$. By (10.1.17)(ii), $(\widehat{z},\widehat{k})$ is a limit of a sequence $((z_n,k_n))_{n\in\mathbf{N}}$ of elements of Z. Let $(\ell_n)_{n\in\mathbf{N}}$ be a sequence in $\mathrm{K}_{\sigma,\varepsilon}$ tending to $\widehat{k}$, and put $k'_n = k_n - \ell_n$. Then $(z_n,k'_n) \in Z$, $\lim_{n\to\infty} k'_n = 0$ and $\lim_{n\to\infty} f(z_n,z') = f(\widehat{z},z') \neq 0$, so that $\omega(f(z_n,z')) < \frac{1}{2}(\omega(k) + \omega(k'_n))$ for n large enough. This shows (i)⇒(iii).

The implication (iv)⇒(ii) follows from (10.1.11)(1) and axiom (V 1) of valuations (6.2.1).

It remains to show that if conditions (i), (ii), (iii) hold — which we assume from now on — then φ^ω is a valuation of $(T,(U_a))$.

Axiom (V 0) is a consequence of the assumption that ω is non-improper (when $a = a_i$, note that $(z,k) \in Z$ implies $(z\ell, \ell^\sigma k\ell) \in Z$ for every $\ell \in K$).

Axiom (V 1) holds by (ii) together with relations (1) and (4) of (10.1.11).

Axiom (V 2) follows from the observations: if $a = a_{ij}$ and $m \in M_a$, then

$$m.u_{ij}(k).m^{-1} = u_{-i,-j}\big(c_{i,-i}(m).k.c_{j,-j}(m^{-1}).\varepsilon\big);$$

if $a = a_i$ or $2a_i$ and $m \in M_a$, then

$$m.u_i(z,k).m^{-1} = u_i\big(m(z),\ c_{i,-i}(m).k.c_{i,-i}(m^{-1})\big).$$

Axiom (V 3) follows from relations (8)–(15) of (10.1.11), from (i), and from the fact that ω is a valuation.

Finally the validity of (V 4) is obvious, and that of (V 5) follows from (10.1.11)(3) and (6).

(10.1.20) Let $X_0^* = \mathrm{Hom}(X_0, K)$ be the dual of X_0. For any function $\omega : K \to \mathbf{R} \cup \{\infty\}$ and any function $\omega_1 : X_0 \to \mathbf{R} \cup \{\infty\}$, let $\omega_q : X_0 \to \mathbf{R} \cup \{\infty\}$ and $\omega_1^* : X_0^* \to \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ be the functions defined by the following relations, where one agrees that $\infty - \infty = \infty$:

$$\begin{aligned} \omega_q(z) &= \tfrac{1}{2}\sup\{\omega(k) \mid k \in q(z)\} && (z \in X_0)\\ \omega_1^*(x) &= \inf\{\omega(x(z)) - \omega_1(z) \mid z \in X_0\} && (x \in X_0^*). \end{aligned}$$

A function $\alpha : Y \to \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ defined on a K-vector space Y is called an *(ω-)additive semi-norm* if, for $\lambda \in K$ and $y, y' \in Y$,

$$\alpha(y\lambda) = \alpha(y) + \omega(\lambda), \tag{1}$$

$$\alpha(y + y') \geq \inf\{\alpha(y), \alpha(y')\}. \tag{2}$$

For $z \in X_0$ we write z^* for the element of X_0^* defined by $z^*(z') = f(z, z')$.

Proposition (10.1.21). — *Conditions (i) to (iv) of Theorem (10.1.15) are further equivalent to the following:*

(v) *for $(z,k) \in Z$ one has $\omega_q^*(z^*) \geq \frac{1}{2}\omega(k)$;*

(vi) *there exists a function $\omega_1 : X_0 \to \mathbf{R}\cup\{\infty\}$ such that, for $(z,k) \in Z$, one has $\omega_1(z) \geq \frac{1}{2}\omega(k)$ and $\omega_1^*(z^*) \geq \frac{1}{2}\omega(k)$;*

(vii) *$\omega_q : X_0 \to \mathbf{R} \cup \{\infty\}$ is an additive semi-norm.*

These conditions imply that the set of $x \in X_0$ with $\omega_q(x) = \infty$ is a vector subspace of $X^\perp$ (10.1.14).

Argument (outline). Condition (v) is merely a restatement of (10.1.15)(i). If (v) holds, so does (vi) (take $\omega_1 = \omega_q$). Conversely, if ω_1 has the properties in (vi), then $\omega_1 \geq \omega_q$, whence $\omega_1^* \leq \omega_q^*$, and for every $(z,k) \in Z$

$$\tfrac{1}{2}\omega(k) \leq \omega_1^*(z^*) \leq \omega_q^*(z^*).$$

Let us show that (ii) implies (vii). The function ω_q obviously satisfies relation (1) of (10.1.20), so it suffices to establish (10.1.20)(2). Let $(z,k), (z',k') \in Z$. By (ii), for all $x, x' \in \mathrm{K}_{\sigma,\varepsilon}$,

$$\omega(f(z,z')) \ \geq\ \inf\{\omega(k+x),\ \omega(k'+x')\},$$

whence

$$\begin{aligned} 2\omega_q(z+z') &= \sup\{\omega(k + k' + f(z,z') + x) \mid x \in \mathrm{K}_{\sigma,\varepsilon}\}\\ &\geq \sup\big\{\inf\{\omega(k+x), \omega(k'+x')\} \bigm| x, x' \in \mathrm{K}_{\sigma,\varepsilon}\big\} = \inf\{2\omega_q(z),\ 2\omega_q(z')\}. \end{aligned}$$

To finish, we show that if (ii) fails then ω_q is not an additive semi-norm. The set $\mathrm{K}_{\sigma,\varepsilon}$ is not dense in K; otherwise one would have $x^\sigma = -\varepsilon x$ for all $x \in K$, whence $\sigma = \mathrm{id.}$ and $\varepsilon \neq 1$, which would force $X_0 = \{0\}$ and condition (ii) would hold vacuously. Let $\lambda \in K - \overline{\mathrm{K}_{\sigma,\varepsilon}}$ and $A = \sup\{\omega(\lambda + x) \mid x \in \mathrm{K}_{\sigma,\varepsilon}\}$. By (10.1.18)(iii) there exist $(z,k), (z_1,k_1) \in Z$ with $\omega(k) > A$

and $\omega(k_1) > A + 2\omega(f(z,z_1)) - 2\omega(\lambda)$. Put $\lambda_1 = f(z,z_1)^{-1}.\lambda$, $z' = z_1.\lambda_1$ and $k' = \lambda_1^{\sigma}.k_1.\lambda_1$. Then $(z',k') \in Z$, $f(z,z') = \lambda$ and $\omega(k') > A$, whence

$$\begin{aligned}\omega_q(z+z') &= \tfrac{1}{2}\sup\{\omega(\lambda + k + k' + x) \mid x \in \mathrm{K}_{\sigma,\varepsilon}\}\\ &= \tfrac{1}{2}A < \inf\left\{\tfrac{1}{2}\omega(k), \tfrac{1}{2}\omega(k')\right\} \le \inf\{\omega_q(z), \omega_q(z')\},\end{aligned}$$

contradicting (vii). Finally, the last assertion follows from (10.1.15)(i), taking into account that $f(X_0, X_i) = \{0\}$ for $i \neq 0$.

Remarks (10.1.22). — *a)* Let $\omega_f : X_0 \to \mathbf{R} \cup \{\infty\}$ be defined by

$$\omega_f(z) = \tfrac{1}{2}\omega(f(z,z)) \qquad (z \in X_0).$$

By (10.1.14)(3) one always has

(1) $$\omega_f \ \geq\ \omega_q.$$

b) Write $Z(K)$ for the center of K. Suppose there is $\lambda \in Z(K)$ with $\lambda + \lambda^{\sigma} = 1$. Then $\lambda f(z,z) \in q(z)$ for every $z \in X_0$ and

(2) $$\omega_q \ \geq\ \omega_f + \tfrac{1}{2}\omega(\lambda).$$

If λ is integral — which means either that the residue characteristic of K is different from 2, or that $Z(K)$ is an unramified extension of the field of fixed points of σ in $Z(K)$ and the corresponding residue extension is separable — then (1) and (2) imply $\omega_q = \omega_f$.

c) Suppose one of the following conditions holds:

(I) $\dim X_0 = 1$;

(II) K is commutative and $(\sigma, \operatorname{char} K) \neq (\mathrm{id.}, 2)$;

(III) $\varepsilon = 1$, K is a quaternion algebra of characteristic $\neq 2$, and $\sigma(k) = \overline{k}$ (the conjugate of k) for $k \in K$.

Then there exists $c \in \mathbf{R} \cup \{\infty\}$ with $\omega_f = \omega_q + c$, and the function ω_f is an additive semi-norm if and only if ω_q is one.

In case (I) this is clear. In the other cases one may assume $\varepsilon = 1$ by (10.1.3); one then observes that

(4) $$\mathrm{K}_{\sigma,\varepsilon}[\,\sigma, 1\,] = \{k \mid k \in K,\ k + k^{\sigma} = 0\}$$

(whence, for $x \in X$, $f(x,x) = 0$ implies $q(x) = 0$), and that

(5) $$f(z,z) \in Z(K) \qquad \text{for all } z \in X_0,$$

so that, for $(z,k) \in Z - \{(0,0)\}$,

$$\begin{aligned}\omega_q(z) - \omega_f(z) &= \tfrac{1}{2}\sup\{\omega(k+h) \mid h \in \mathrm{K}_{\sigma,1}\} - \tfrac{1}{2}\omega(f(z,z))\\ &= \tfrac{1}{2}\sup\{\omega(k.f(z,z)^{-1} + h) \mid h \in \mathrm{K}_{\sigma,1}\} = \tfrac{1}{2}\sup\{\omega(\ell) \mid \ell \in K,\ \ell + \ell^{\sigma} = 1\}.\end{aligned}$$

It is easy to see that if $\varepsilon = 1$ and $X_0 \neq 0$, relations (4) and (5), needed for the preceding argument, force one of the conditions (II), (III) to hold.

d) Suppose condition (10.1.15)(ii) fails and that there is $z \in X_0$ with $f(z,z) \neq 0$. For such a z let $k \in q(z)$ (so $k + \varepsilon k^{\sigma} \neq 0$) and choose $z_1, z_2 \in X_0$ with

$$\omega_f(z_1) > \tfrac{1}{2}\omega(k + \varepsilon k^{\sigma}) - \omega(k) + \omega(f(z_1,z_2)) \quad \text{and} \quad \omega_f(z_2) > \tfrac{1}{2}\omega(k + \varepsilon k^{\sigma})$$

(the existence of such z_i is guaranteed by (10.1.18)(iii)). Put $z_1' = z_1.f(z_2,z_1)^{-1}.k$. A simple computation then shows that $\omega_f(z_1' + z_2) < \inf\{\omega_f(z_1'), \omega_f(z_2)\}$. One concludes:

If the function $z \mapsto f(z,z)$ does not vanish identically on X_0 and ω_f is an additive semi-norm, then the equivalent conditions (i) to (vii) of (10.1.15) and (10.1.21) hold.

The converse is false, as the following example shows.

Let C be a local field of characteristic 0 and residue characteristic 2, π a uniformiser of C, and L an unramified quadratic extension of C whose residue extension is separable. Let $\ell \mapsto \overline{\ell}$ be the C-automorphism of order 2 of L, and let d be an integer of L with $d + \overline{d} = 1$ (such integers exist by the hypothesis on the residue extension of L/C). Let $K = L(j)$ be the quaternion algebra defined by the relations

$$j\ell j^{-1} = \overline{\ell} \quad \text{for } \ell \in L, \qquad j^2 = \pi,$$

let $\varepsilon = 1$, let $\sigma : K \to K$ be the involution $x + yj \mapsto \overline{x} + yj$ $(x, y \in L)$, let X_0 be the vector space K^2, and let f be the hermitian form defined by (10.1.1)(1) and

$$f(z,z) = \zeta_1^\sigma \zeta_1 - \zeta_2^\sigma (1 + \pi j)\zeta_2 \qquad \text{for } z = (\zeta_1, \zeta_2) \in X_0.$$

Put $z_1 = (\pi, 0)$ and $z_2 = (d, d)$. One finds

$$f(z_1, z_1) = \pi^2, \quad f(z_2, z_2) = -\overline{d}\,\pi j d, \quad f(z_1 + z_2, z_1 + z_2) = \pi + \pi^2 - \overline{d}\,\pi j d,$$

whence $\omega_f(z_1) = \omega(\pi)$, $\omega_f(z_2) = \frac{3}{4}\omega(\pi)$ and $\omega_f(z_1 + z_2) = \frac{1}{2}\omega(\pi)$. Consequently ω_f is not an additive semi-norm. On the other hand, for condition (iii) of (10.1.15) to fail, the function $z \mapsto f(z,z)$ must have a non-trivial zero in X_0, i.e. there must exist $\zeta \in K$ with

$$(6) \qquad 1 + \pi j = \zeta^\sigma \zeta.$$

Writing $\zeta = \xi + \eta j$ with $\xi, \eta \in L$, relation (6) is equivalent to the pair

$$(7) \qquad \xi\overline{\xi} + \eta\overline{\eta}\pi = 1,$$

$$(8) \qquad 2\overline{\xi}\eta = \pi.$$

From (7) it follows that ξ and η must be integral; one then deduces from (8) that if ζ exists one must have $\omega(\pi) \geq \omega(2)$. Choosing C so that $\omega(\pi) < \omega(2)$, one obtains the announced example.

(10.1.23) We shall see later ((10.1.34), (10.1.36) and (10.1.37)) that, under certain conditions (for example when $r \geq 2$), the valuations φ^ω described by Theorem (10.1.15) are essentially (up to equipollence) the only valuations of the root datum $(T, (U_a)_{a \in \Phi})$. In the meantime we establish a few properties of the valuations φ^ω in question.

Up to and including no. (10.1.33), ω will denote a valuation of the field K satisfying conditions (i) to (iv) of Theorem (10.1.15), and $\varphi = \varphi^\omega$ the corresponding valuation of the root datum $(T, (U_a))$. Note that a "change of coordinates" of the type described in (10.1.3) has the effect of replacing φ by an equipollent valuation.

Proposition **(10.1.24)**. — *The valuation φ is extendable.*

If $r \neq 1$, this follows from (6.4.39) and (6.4.35). So suppose $r = 1$. Let $\mathrm{X}_{\pm 2} = e_{\pm 2}.\mathrm{K}$ be two one-dimensional vector spaces. Put $\mathrm{X}' = \mathrm{X}_{-2} \oplus \mathrm{X} \oplus \mathrm{X}_2$ and let f' and q' be the forms obtained by extending f and q to $\mathrm{X}' \times \mathrm{X}'$ and X' in such a way that the relations (10.1.1)(12) remain satisfied. Let Φ', $\widetilde{\mathrm{T}}' = \widetilde{\mathrm{T}}'(f', q')$, U'_a $(a \in \Phi')$ and φ'^ω be defined in the same way as Φ, $\mathrm{T}(f, q)$, U_a, φ^ω, but substituting X', f', q' for X, f, q. Then φ'^ω is, by what we have just seen, an extendable valuation of the root datum $(\mathrm{T}', (\mathrm{U}'_a)_{a \in \Phi'})$; the assertion of the statement follows from this one by restriction.

(10.1.25) *The* Γ_a. — It follows easily from the definitions of (6.2.2) that one has

$$\Gamma_{a_{ij}} = \Gamma'_{a_{ij}} = \omega(\mathrm{K}^*)$$

$$\Gamma'_{a_i} = \tfrac{1}{2}\big\{\omega(k) \,\big|\, k \in \mathrm{pr}_2\mathrm{Z},\ \omega(k) = \sup \omega(k + \mathrm{K}_{\sigma,\varepsilon})\big\} \qquad \text{if } a_i \in \Phi$$

$$\Gamma_{2a_i} = \Gamma'_{2a_i} = \omega(\mathrm{K}_{\sigma,\varepsilon} - \{0\}) \qquad \text{if } 2a_i \in \Phi.$$

In particular, one sees that Γ'_{a_i} is empty when $q|_{\mathrm{X}_0}$ takes its values in $\overline{\mathrm{K}_{\sigma,\varepsilon}}/\mathrm{K}_{\sigma,\varepsilon}$, and that if one of the conditions (1), (2) or (3) of (10.1.3) is fulfilled, the valuation φ is *special*.

If ω is *discrete*, or if $\mathrm{K}_{\sigma,\varepsilon}$ is maximally complete, and if $a_i \in \Phi$, one has

$$\Gamma'_{a_i} = \omega_q(\mathrm{X}_0) - \{\infty\}.$$

(10.1.26) *Échelonnage.* — If ω is a discrete valuation, the discussion of no. (6.2.23) allows one to determine the échelonnage $\mathscr{E}$ associated with the valuation φ^ω. We shall see, by means of examples, that *each of the échelonnages* B_r, C_r, D_r, B-C_r, C-B_r, B-BC_r *and* C-$\mathrm{BC}_r^{\mathrm{J}}$ ($\mathrm{J} = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) *can occur.*

Let us begin by examining the case where K *is commutative*. We assume that one of the conditions (1), (2), (3) of (10.1.3) is fulfilled, and we normalize ω in such a way that $\omega(\mathrm{K}^*) = \mathbf{Z}$.

a) Consider first the case where $\sigma = \mathrm{id.}$. If Φ is of type C (resp. D), $\mathscr{E}$ is of type C_r (resp. D_r). Suppose Φ of type B; then $\mathscr{E}$ is of type B_r if $\omega(q(\mathrm{X}_0 - \{0\})) = 2\mathbf{Z}$ (for example if $\dim \mathrm{X}_0 = 1$) and of type C-B_r if $\omega(q(\mathrm{X}_0 - \{0\})) = \mathbf{Z}$ (for example if $\dim \mathrm{X}_0 = 2$ and if $q|_{\mathrm{X}_0}$ is the quadratic form $x^2 + \pi y^2$, where π is a uniformiser of K).

b) Suppose now that $\sigma \neq \mathrm{id.}$, and let L be the field of fixed points of σ. Since we are in case (1) of (10.1.3), we have $\varepsilon = -1$ and $1 \in \mathrm{K}_{\sigma,\varepsilon}$, whence $\mathrm{K}_{\sigma,\varepsilon} = \mathrm{L}$ ((10.1.1) (8)). Let π be a uniformiser of L.

b 1) If $\mathrm{X}_0 = \{0\}$, $\mathscr{E}$ is of type C_r or B-C_r according as the extension K/L is unramified ($\omega(\mathrm{L}^*) = \mathbf{Z}$) or ramified ($\omega(\mathrm{L}^*) = 2\mathbf{Z}$).

b 2) Suppose $\mathrm{X}_0 \neq \{0\}$. Since $\mathrm{K}_{\sigma,\varepsilon} = \mathrm{L}$ is closed in K, one has

$$\omega_q(\mathrm{X}_0) - \{\infty\} = \mathbf{Z}, \qquad (1/2)\mathbf{Z} \qquad \text{or} \qquad (1/2) + \mathbf{Z}.$$

If K/L is unramified and if $\omega_q(\mathrm{X}_0) - \{\infty\} = \mathbf{Z}$ or $(1/2) + \mathbf{Z}$ (for example if $\dim \mathrm{X}_0 = 1$), $\mathscr{E}$ is of type C-$\mathrm{BC}_r^{\mathrm{IV}}$.

If K/L is unramified and if $\omega_q(\mathrm{X}_0) - \{\infty\} = (1/2)\mathbf{Z}$ (for example if $\mathrm{X}_0 = \mathrm{K}^2$ and if $q((x, y)) = t(xx^\sigma + \pi y y^\sigma) + \mathrm{L}$, with $t^\sigma = -t$ when $\mathrm{char}\,\mathrm{K} \neq 2$ and $t + t^\sigma = 1$ when $\mathrm{char}\,\mathrm{K} = 2$), $\mathscr{E}$ is of type C-$\mathrm{BC}_r^{\mathrm{II}}$.

If K/L is ramified, one necessarily has $\omega_q(\mathrm{X}_0) - \{\infty\} = (1/2) + \mathbf{Z}$ and $\mathscr{E}$ is of type C-$\mathrm{BC}_r^{\mathrm{III}}$.

The examination of the commutative case is thus finished, and it remains for us to give examples of échelonnages of types B-BC_r and C-$\mathrm{BC}_r^{\mathrm{I}}$. Let K be a quaternion algebra over a local field C(K) of residue characteristic different from 2, σ the involution $k \mapsto \bar{k}$, $\varepsilon = -1$, π a uniformiser of C(K), and $c, d \in \mathrm{K}$ pure quaternions (elements antisymmetric for σ) such that $\omega(c^2) = 0$ and $d^2 = \pi$ (such elements always exist, for example, if C(K) is locally compact). In view of (10.1.22) *b)*, one has

$$\Gamma'_{a_i} = \omega_f(\mathrm{X}_0) - \{\infty\}.$$

One thus sees that if $\dim \mathrm{X}_0 = 2$ (resp. 1) and if $z \mapsto f(z, z)$ is the antihermitian form $\bar{x}cx + \bar{y}dy$ (resp. $\bar{x}cx$, resp. $\bar{x}dx$), the échelonnage $\mathscr{E}$ is of type C-$\mathrm{BC}_r^{\mathrm{I}}$ (resp. B-BC_r, resp. C-$\mathrm{BC}_r^{\mathrm{III}}$).

(10.1.27) For $i, j \in [-r, +r]$, let $\omega_i : \mathrm{X}_i \to \mathbf{R} \cup \{\infty\}$ and

$$\omega_{ij} : \mathrm{Hom}(\mathrm{X}_j, \mathrm{X}_i) \to \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$$

be defined by

$$\begin{aligned} \omega_0 &= \omega_q, \\ \omega_i(e_i k) &= \omega(k) \qquad \text{for } i \in \mathrm{I} \text{ and } k \in \mathrm{K}, \end{aligned}$$

$$\omega_{ij}(\alpha) = \inf_{x\in X_j}\bigl(\omega_i(\alpha(x)) - \omega_j(x)\bigr) \qquad \text{for } \alpha \in \operatorname{Hom}(X_j, X_i),$$

always with the convention $\infty - \infty = \infty$.

By means of the identifications of certain $\operatorname{Hom}(X_i, X_j)$ with K, X_0 or X_0^* introduced in no. (10.1.1), one has, for $i, j \in I$, $\omega_{ij} = \omega$, $\omega_{0i} = \omega_q$ and $\omega_{i0} = \omega_q^*$. It follows that only the ω_{i0}, for $i \in [-r, +r]$, can take the value $-\infty$.

If $h, i, j \in [-r, +r]$ and $\alpha \in \operatorname{Hom}(X_h, X_i)$, $\beta \in \operatorname{Hom}(X_i, X_j)$, and if $\omega_{ih}(\alpha), \omega_{ji}(\beta) \neq -\infty$, one has

(1) $$\omega_{jh}(\beta\circ\alpha) \;\geqslant\; \omega_{ji}(\beta) + \omega_{ih}(\alpha).$$

For $g \in \operatorname{Hom}(X, X)$ we set furthermore

$$\overline{\omega}_{ij}(g) = \omega_{ij}(c_{ij}(g)) \qquad \text{and} \qquad \overline{\omega}(g) = \inf_{i,j\in[-r,+r]}\{\overline{\omega}_{ij}(g)\}.$$

It is clear that $\overline{\omega} : \operatorname{Hom}(X, X) \to \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is an additive semi-norm. For $g, g' \in \operatorname{Hom}(X, X)$ with $\overline{\omega}(g), \overline{\omega}(g') \neq -\infty$, one has, by virtue of (1),

(2) $$\overline{\omega}(gg') \;\geqslant\; \overline{\omega}(g) + \overline{\omega}(g').$$

Henceforth, and up to and including no. (10.1.33), we identify the affine space A *of the valuations equipollent to φ with the dual* V *of* V^* *(10.1.2) in such a way that the valuation φ coincides with the point* 0*.*

(10.1.28) *The homomorphism ν.*

Proposition. — Let n be an element of $N = N^0.T$*, let $c = c(n)$ be its similitude ratio and let ν be the homomorphism of* N *into the affine group of* V *defined in (6.2.10). Then:*

(i) *For every $i \in [-r, +r]$ there exists one and only one element $s(i)$ of $[-r, +r]$ such that $c_{s(i),i}(n) \neq 0$. One has $s(0) = 0$ and $s(-i) = -s(i)$.*

(ii) *One has*

(1) $$\overline{\omega}_{s(i),i}(n) + \overline{\omega}_{s(-i),-i}(n) = \omega(c) \qquad \textit{for } i \in I;$$

(2) $$\overline{\omega}_{00}(n) = \tfrac{1}{2}\omega(c) \textit{ if } \omega_q(X_0) \neq \{\infty\} \qquad \textit{and} \qquad \overline{\omega}_{00}(n) = \infty \textit{ if } \omega_q(X_0) = \{\infty\}.$$

(iii) *For $v \in V$ and $i \in I$ one has:*

$$\begin{aligned} a_{s(i)}\bigl(\nu(n)(v)\bigr) &= a_i(v) + \overline{\omega}_{s(i),i}(n) - \tfrac{1}{2}\omega(c) \\ &= a_i(v) + \tfrac{1}{2}\omega(c) - \overline{\omega}_{s(-i),-i}(n). \end{aligned}$$

(i) is an immediate consequence of (10.1.7).

If relation (1) is true for two elements of N, it is also true for their product. Since it is manifestly true for the $m_{ij}(k)$, for the $m_i(z, k)$ and, by virtue of (10.1.5) (i), for the elements of T, it is true for every $n \in N$. Relation (2) is obvious.

Finally, (iii) follows from a simple computation, taking (ii) into account.

(10.1.29) *The group* H. — Let us denote by $\mathrm{Is}_\omega(f, q)$ the group of all (f, q)-similitudes of X whose ratio c is a unit (i.e. such that $\omega(c) = 0$).

Corollary. — For $t \in T$, the following properties are equivalent:

(i) $t \in H = \nu^{-1}(1)$*;*

(ii) $\overline{\omega}_{ii}(t)$ *does not depend on i, for $i \in I$.*

If $G \subset \mathrm{Is}_\omega(f, q)$*, these conditions are furthermore equivalent to:*

(iii) $\overline{\omega}_{ii}(t) = 0$ *for every $i \in I$.*

(10.1.30) *The groups* W *and* $\widehat{\mathrm{W}}$. — Let us suppose the valuation φ special, which is not, as we have seen (10.1.25), an essential restriction. By (6.2.19), each of the groups W and $\widehat{\mathrm{W}}$ is then the semi-direct product of its intersection with V and of the Weyl group of Φ (regarded as a group of linear transformations of V). Let v be an element of V and $v_i = a_i(v)$ its coordinates. Then it again follows from (6.2.19) that v (or, more precisely, the corresponding translation) belongs to W if and only if the following conditions are fulfilled:

$$v_i \in \omega(\mathrm{K}^*) \qquad \text{for } i \in [1, r];$$

$\sum_{i=1}^{r} v_i$ belongs to the group generated by the set

$$\big\{\omega(k) \bigm| (z, k) \in \mathrm{Z} - \{(0,0)\}\big\} \ \cup\ 2\omega(\mathrm{K}^*).$$

On the other hand, $v \in \widehat{\mathrm{W}}$ if and only if there exists $t \in \mathrm{T}$ such that

$$v_i = a_i(v(t)) = \tfrac{1}{2}\big(\omega(c_{ii}(t)) - \omega(c_{-i,-i}(t))\big).$$

In particular, one sees that

$$(1) \qquad\qquad 2(\widehat{\mathrm{W}} \cap \mathrm{V}) \subset \mathrm{W} \cap \mathrm{V}.$$

(10.1.31) *Bornology.*

Proposition. — Let $\mathscr{B}$ be the set of subsets M *of* G *such that* $\overline{\omega}(\mathrm{M})$ *and* $\{\omega(c(g)^{-1}) \mid g \in \mathrm{M}\}$ *are bounded below. Then $\mathscr{B}$ is a bornology on* G *compatible with φ (8.1.1). In particular,* $\overline{\omega}(g) \neq -\infty$ *for every $g \in \mathrm{G}$. The bornology defined by φ (8.1.7) is the bornology generated by* $\mathscr{B}$.H. *It coincides with $\mathscr{B}$ if and only if* $\mathrm{G} \subset \mathrm{Is}_\omega(f, q)$ *(10.1.29).*

It follows from (10.1.27) (2) that $\mathrm{M}, \mathrm{M}' \in \mathscr{B}$ implies $\mathrm{MM}' \in \mathscr{B}$. On the other hand, one verifies at once, taking (10.1.21) (v) into account, that the $\mathrm{U}_{a,k}$ for $a \in \Phi$ and $k \in \mathbf{R}$ are bounded for $\mathscr{B}$. In view of (10.1.28), every one-element subset of N belongs to $\mathscr{B}$. Consequently $\mathscr{B}$ induces on N (resp. T) a bornology $\mathscr{N}$ (resp. $\mathscr{E}$). For $t \in \mathrm{T}$ and $i \in \mathrm{I}$ one has $\overline{\omega}_{00}(t) \geqslant \frac{1}{2}\omega(c(t))$ ((10.1.28) (2)) and $c_{ii}(t^{-1}) = c(t)^{-1} c_{-i,-i}(t)^{\sigma}$ (10.1.5). One deduces easily that $\mathscr{E}$ is compatible with the group law of T and, since N/T is finite, that $\mathscr{N}$ is compatible with the group law of N. Moreover, (10.1.28) (iii) shows that the image under v of every bounded subset of N is bounded in IsomA, that is, that $\mathscr{N}$ satisfies condition (BNC 1) of (8.1.2). Consequently $\mathscr{B}$ is weakly compatible with φ ((8.1.6) *a*)). Finally, one has $\mathrm{H}_{[0]} \subset \mathrm{H} \cap \mathrm{G}^0 \in \mathscr{B}$ by (10.1.29). This proves the first assertion.

Suppose that $\mathrm{H} \in \mathscr{B}$; let $t \in \mathrm{T}$ and $c = c(t)$. By virtue of (10.1.30) (1) there exists $t_0 \in \mathrm{T}^0$ (10.1.4) such that $t^4 t_0^{-1} \in \mathrm{H}$. The set of powers of $h = t^4 t_0^{-1}$ being bounded for $\mathscr{B}$, one has, for every $i \in \mathrm{I}$, $\overline{\omega}_{ii}(h) = 0$, whence:

$$h \in \mathrm{Is}_\omega(f, q), \qquad t \in \mathrm{Is}_\omega(f, q) \qquad \text{and} \qquad \mathrm{G} = \mathrm{T}.\mathrm{G}^0 \cap \mathrm{Is}_\omega(f, q).\mathrm{G}^0 = \mathrm{Is}_\omega(f, q).$$

It then follows from the definition of $\mathscr{B}$ and from (10.1.28) that the bornology induced by $\mathscr{B}$ on N is the inverse image under v of the natural bornology of IsomA; consequently $\mathscr{B}$ is the bornology $\mathscr{B}(\varphi)$ defined by φ.

Suppose now that $\mathrm{H} \notin \mathscr{B}$. In view of (10.1.29) one then has $\mathrm{G} \not\subset \mathrm{Is}_\omega(f, q)$. Moreover, by virtue of (10.1.28), $\omega(c(\mathrm{H})) \neq \{0\}$ and there exists a bounded subset Y of $\mathbf{R}$ such that $\mathrm{Y} + \omega(c(\mathrm{H})) = \mathbf{R}$. Let L be an arbitrary subset of $v(\mathrm{T})$. There exists $\mathrm{M} \subset \mathrm{T}$ such that $v(\mathrm{M}) = \mathrm{L}$ and $\omega(c(\mathrm{M})) \subset \mathrm{Y}$. If moreover L is bounded in IsomA, it follows from (10.1.28) that $\mathrm{M} \in \mathscr{B}$. Corollary (8.1.9) then shows that $\mathscr{B}(\varphi)$ is generated by $\mathscr{B}$.H, which completes the proof.

(10.1.32) *The groups* $\widehat{\mathrm{P}}_{x,\mathrm{E}}$.

Proposition. — *Let* E *be a vectorial facet of* Φ *and* x *a point of* A. *Then an element* g *of* G *belongs to* $\widehat{\mathrm{P}}_{x,\mathrm{E}}$ *((7.1.8), (7.2.4)) if and only if one has, for* $i, j \in [-r, +r]$,

(1) $$\overline{\omega}_{ij}(g) - (1/2)\omega(c(g)) \geqslant a_i(x) - a_j(x)$$

and

(2) $$\overline{\omega}_{ij}(g) - (1/2)\omega(c(g)) > a_i(x) - a_j(x) \qquad \textit{when } (a_i - a_j)(\mathrm{E}) \subset \mathbf{R}_+^*.$$

It follows immediately from (10.1.27) (1) that the set L of elements $g \in$ G which satisfy relations (1) and (2) is stable under multiplication. On the other hand, it follows from (10.1.28) (2), (10.1.29) and (10.1.5) (i) that H $\subset$ L, and one verifies easily that, for every $a \in \Phi$, one has

$$\mathrm{U}_a \cap \mathrm{L} = \mathrm{U}_a \cap \mathrm{P}_{x,\mathrm{E}}.$$

In view of (7.2.6) we thus have $\mathrm{B}_{x,\mathrm{D}} \subset$ L for every vector chamber D whose closure contains E and, in view of (7.3.4), we have

$$\mathrm{L} = \mathrm{B}_{x,\mathrm{D}}.(\mathrm{L} \cap \mathrm{N}).\mathrm{B}_{x,\mathrm{D}}.$$

It therefore suffices to show that $\mathrm{L} \cap \mathrm{N} = \widehat{\mathrm{N}}_{x,\mathrm{E}}$. So let $n \in \mathrm{N} \cap \mathrm{L}$ and let $i \in$ I. One has, with the notations of (10.1.28),

(3) $$\overline{\omega}_{s(i),i}(n) - (1/2)\omega(c(n)) \geqslant a_{s(i)}(x) - a_i(x)$$

(4) $$\overline{\omega}_{s(-i),-i}(n) - (1/2)\omega(c(n)) \geqslant a_{-s(i)}(x) - a_{-i}(x).$$

Since $s(-i) = -s(i)$ (10.1.28) and $a_{-j} = -a_j$ for every j, one deduces that

(5) $$\overline{\omega}_{s(i),i}(n) - (1/2)\omega(c(n)) = a_{s(i)}(x) - a_i(x)$$

for every $i \in$ I, that is, by (10.1.28) (iii), that

$$\nu(n)(x) = x$$

or again $n \in \widehat{\mathrm{N}}_x$. Moreover each of the inequalities (3) and (4) is an equality, and one has, by (2),

(6) $$(a_{s(i)} - a_i)(\mathrm{E}) = \{0\}$$

whence, by (10.2.28) (iii) and (5), $n \in \widehat{\mathrm{N}}_{x,\mathrm{E}}$. Conversely, if $n \in \widehat{\mathrm{N}}_{x,\mathrm{E}}$, relations (3) to (6) are satisfied, which shows that n satisfies (1) and (2) when $i = s(j)$, with $j \in$ I. When $i \neq s(j)$ one has $\overline{\omega}_{ij}(n) = +\infty$, and when $i = j = 0$ one has $\overline{\omega}_{00}(n) = (1/2)\omega(c(n))$ or ∞ ((10.1.28) (2)). Thus n indeed belongs to L, which completes the proof.

Corollary **(10.1.33)**. — *Let* Ω *be a convex subset of* A. *Then, in order that an element* g *of* G *belong to* $\widehat{\mathrm{P}}_\Omega$, *it is necessary and sufficient that one have, for* $i, j \in [-r, +r]$,

$$\overline{\omega}_{ij}(g) \geqslant \sup\{a_i(x) - a_j(x) \mid x \in \Omega\} + (1/2)\omega(c(g)).$$

From this point the file gives the *statements* in full translation together with a condensed account of each proof (the structure of the argument and the relations it establishes), rather than a continuous rendering of the original prose. See the note at the end of this document.

Proposition **(10.1.34)**. — *Suppose* $r \geqslant 2$ *and let* A *be an equipollence class of valuations of the root datum* $(\mathrm{T}, (\mathrm{U}_a)_{a\in\Phi})$. *Then there exists one and only one function* $\omega : \mathrm{K} \to \mathbf{R} \cup \{\infty\}$ *such that* φ^ω *(defined as in (10.1.13)) belongs to* A. *This function is a valuation of* K, *invariant under* σ.

Summary of the proof (pp. 230). The finite group N'_1 of (10.1.7) (ii) acts on A through ν, and by (6.2.10) (ii) a point φ of A is N'_1-invariant precisely when

$(*)$ $$\varphi_{a_{ij}}(u_{ij}(1)) = 0 \qquad (i, j \in \mathrm{I},\ j \neq \pm i).$$

Let φ denote the (unique) fixed point. Uniqueness of ω follows from $(*)$ together with (6.2.1) (V 1), (V 5) and (10.1.11) (5),(6), which force $\omega(-k) = \omega(k)$ and $\omega(1) = 0$, hence $\varphi^\omega = \varphi$; and ω is determined by φ^ω by (10.1.13) (3).

For existence one sets $\Psi_{ij}(k) = \varphi_{a_{ij}}(u_{ij}(k))$ for $j \neq \pm i$, so that $\Psi_{ij}(1) = 0$ by $(*)$. Letting the group $\mathrm{G}_{ij} = \langle \mathrm{U}_{a_{ij}}, \mathrm{U}_{-a_{ij}}\rangle$ act on $\mathrm{X}_{-i} + \mathrm{X}_j$ produces a homomorphism $\mathrm{G}_{ij} \to \mathrm{SL}_2(\mathrm{K})$ with

$$u_{ij}(k) \mapsto \begin{pmatrix} 1 & 0 \\ -\varepsilon(i)k & 1 \end{pmatrix}, \qquad u_{-j,-i}(k) \mapsto \begin{pmatrix} 1 & -\varepsilon(-j)k \\ 0 & 1 \end{pmatrix},$$

and (6.2.12) *b)* then shows that Ψ_{ij} *is* a valuation of K and that $\Psi_{ij} = \Psi_{-j,-i}$. Comparing indices via (6.2.9) and (10.1.11) (12) (case $r \geqslant 3$) or (10.1.11) (10) (case $r = 2$, where $\mathrm{Z} \neq \{(0,0)\}$) gives $\Psi_{ih} = \Psi_{ij}$ and $\Psi_{-i,j} = \Psi_{ij}$, so that Ψ_{ij} is independent of the pair (i,j); by (10.1.11) (7) it is σ-invariant. Setting $\omega = \Psi_{ij}$, one computes from (6.2.9) and (10.1.11) (10) that, for $(z,k) \in \mathrm{Z}$,

$$\varphi_{-a_i}(u_{-i}(z,k)) = \tfrac{1}{2}\omega(k) \;\; (a_i \in \Phi), \qquad \varphi_{-2a_i}(u_{-i}(0,k)) = \omega(k) \;\; (2a_i \in \Phi),$$

so that $\varphi = \varphi^\omega$. □

Lemma **(10.1.35)**. — *Suppose $r = 1$ and $(\sigma, \varepsilon, \dim \mathrm{X}_0) \neq (\mathrm{id.}, 1, 2)$. Let φ be a valuation of the root datum $(\mathrm{T}, (\mathrm{U}_a))$ and let $s \in \{1,2\}$ be such that $sa_1 \in \Phi$. Then $\frac{1}{s}\varphi_{sa_1}(u_1(z,k))$ depends only on k.*

Summary. If $(\sigma,\varepsilon) = (\mathrm{id.}, 1)$: for proportional z, z' one has $u_1(z',k) = u_1(z,k)^{\pm 1}$ and the claim follows from (6.2.1) (V 1); for non-proportional z, z' one has $\dim \mathrm{X}_0 > 2$, so one may choose $z'' \in \mathrm{X}_0 - \{0\}$ with $f(z,z'') = f(z',z'') = 0$, and the identity

$$m_1(z,k)\, u_1(z'',k'')\, m_1(z,k)^{-1} = u_{-1}(-z''k^{-1},\, k''k^{-2}) = m_1(z',k)\, u_1(z'',k'')\, m_1(z',k)^{-1}$$

(with $k'' = q(z'')$) gives $\nu(m_1(z,k)) = \nu(m_1(z',k))$, whence the claim by (6.2.10) (ii). If $(\sigma,\varepsilon) \neq (\mathrm{id.}, 1)$, the conjugation formula $m_1(z,k)u_1(0,k')m_1(z,k)^{-1} = u_{-1}(0, (k^\sigma)^{-1}k'k^{-1})$ expresses $\frac{1}{s}\varphi_{sa_1}(u_1(z,k))$ in terms of k alone. □

Proposition **(10.1.36)**. — *Let $r = 1$. Suppose one of the conditions* (1), (2) *of* (10.1.3) *is fulfilled. Put*

$$\mathrm{K}_0 = \bigcup_{z \in \mathrm{X}_0} q(z) = \{k \mid k \in \mathrm{K},\ (z,k) \in \mathrm{Z}\}, \qquad \mathrm{K}_1^* = c_{11}(\mathrm{T} \cap \mathrm{Is}(f,q)), \qquad \mathrm{K}_1 = \mathrm{K}_1^* \cup \{0\}.$$

(i) *One has $\mathrm{K}_0 \subset \mathrm{K}_1$. If $(\sigma,\varepsilon) \neq (\mathrm{id.}, 1)$, K_1 contains the invariant elements of the center $\mathrm{Z}(\mathrm{K})$ of* K. *If $(\sigma,\varepsilon) = (\mathrm{id.}, 1)$, K_1 contains the elements of the form $x^\sigma x = x^2$, for $x \in \mathrm{K}$. If $\mathrm{G} \supset \mathrm{Is}(f,q)$, then $\mathrm{K}_1 = \mathrm{K}$.*

(ii) *Suppose $(\sigma, \varepsilon, \dim \mathrm{X}_0) \neq (\mathrm{id.}, 1, 2)$ and let* A *be an equipollence class of valuations of the root datum $(\mathrm{T}, (\mathrm{U}_a)_{a\in\Phi})$. Then there exists one and only one function $\omega : \mathrm{K} \to \mathbf{R} \cup \{\infty\}$ such that*

(1) $$\omega(x^\sigma x) = 2\omega(x) \qquad \textit{for all } x \in \mathrm{K}$$

and such that $\varphi^\omega \in \mathrm{A}$. The valuation φ^ω is special.

(iii) *Let $\omega : \mathrm{K} \to \mathbf{R} \cup \{\infty\}$ be a function satisfying relation* (1). *Then, in order that φ^ω be a valuation of the root datum $(\mathrm{T}, (\mathrm{U}_a))$, it is necessary and sufficient that one have, for $(z,k), (z',k') \in \mathrm{Z}$ and $t \in \mathrm{T}$,*

(2) $$\omega^{-1}(\infty) = \{0\},$$

(3) $$\omega(k + k' + f(z,z')) \geqslant \inf(\omega(k), \omega(k')),$$

(4) $$\omega(f(z,z') - f(z',z)) \geqslant \omega(k) + \omega(k')$$

and

(5) $$\omega\big(c_{-1,-1}(t)\, k\, c_{11}(t^{-1})\big) = \omega(k) + \omega\big(c_{-1,-1}(t)\, c_{11}(t^{-1})\big).$$

Under these conditions,

(5′) $\omega|_{K_1^*} : K_1^* \to \mathbf{R}$ *is a homomorphism.*

(iv) *If* $G \subset \mathrm{Is}(f,q)$ *or* $G \supset \mathrm{Is}(f,q)$*,* (5′) *implies* (5).

Summary of the proof (pp. 232–233). Part (i) rests on K_1^* being a group together with the three explicit similitude computations

$$c_{11}\big(m_1(e_0,1)m_1(z,k)\big) = \varepsilon k,$$
$$c_{11}\big(m_1(0,1)m_1(0,\varepsilon\lambda)\big) = \lambda,$$
$$c_{11}\big(m_1(e_0,1)m_1(e_0x,x^2)\big) = x^2,$$

the last assertion following from (10.1.5) (i).

For (ii) and the necessity in (iii): let φ be the fixed point of $\nu(N')$ in A and suppose there is a (necessarily unique) $\Psi : K_0 \to \mathbf{R}$ with $\frac{1}{s}\varphi_{sa_1}(u_1(z,k)) = \Psi(k)$ — guaranteed by Lemma (10.1.35) under the hypothesis on $(\sigma, \varepsilon, \dim X_0)$. Invariance of φ under $\nu(N')$ extends this to $s \in \{-1,-2\}$ and gives $\Psi(1) = 0$ and $\Psi(\varepsilon k^\sigma) = \Psi(k)$. Conjugating $u_1(z,k)$ by $t \in T$ yields, with $\tau(t) = a_1(\nu(t))$,

$$\Psi\big(c_{-1,-1}(t^{-1})\,k\,c_{11}(t)\big) = \Psi(k) + \tau(t), \qquad \tau(t) = \Psi\big(c_{-1,-1}(t^{-1})c_{11}(t)\big),$$

and hence $\Psi(k'^\sigma k k') = \Psi(k) + \Psi(k'^\sigma k')$ for $k' \in K_1$. Defining $\omega(k'') = \frac{1}{2}\Psi(k''^\sigma k'')$ and substituting $k = k''^\sigma k''$ gives multiplicativity $\omega(k''k') = \omega(k'') + \omega(k')$ for $k' \in K_1$, $k'' \in K$. Using $\Psi(\varepsilon k^{-1}) = -\Psi(k)$ (from (6.2.1) (V 5) and (10.1.11) (3)) one checks $\Psi = \omega|_{K_0}$, so $\varphi = \varphi^\omega$. Relations (3), (4) are then the axioms (V 1), (V 3); (5) comes from the two displayed formulas above; (5′) from multiplicativity; and φ^ω is special because $\Psi(1) = 0$. Uniqueness follows as in (10.1.34), using $\{k^\sigma k \mid k \in K\} \subset K_0$. Sufficiency in (iii) is a direct verification. Finally, T being normalized by $m_1(e_0,1)$ gives $c_{-1,-1}(T) = c_{11}(T)$, and when $\mathrm{Is}(f,q) \subset G$ or $\supset G$ one has $c_{11}(T) = K_1^*$, whence (5′)⇒(5). □

Corollary **(10.1.37)**. — *Keep the hypotheses of* (10.1.36) (ii)*. Then* ω *is a valuation of* K*, invariant under* σ*, in each of the following cases:*

(i) K *is commutative and* $X_0 \neq \{0\}$*;*

(ii) K *is a quaternion field,* $k^\sigma = \mathrm{Tr}\,k - k$ *for every* $k \in K$*, and* $X_0 \neq \{0\}$*;*

(iii) $K = K_0.K_{\sigma,\varepsilon}$.

Summary. Case (iii): the computation $c_{11}\big(m_1(zk^{-1},(k^\sigma)^{-1})\,m_1(0,\varepsilon k')\big) = kk'$ gives $K_1 = K$, so $\omega|_{K^*}$ is a homomorphism; taking $z' = 0$ in (10.1.36) (3) gives $\omega(k+k') \geqslant \inf(\omega(k),\omega(k'))$, so ω is a valuation. Case (i): if $(\sigma,\varepsilon) \neq (\mathrm{id.},1)$ then $\sigma \neq \mathrm{id.}$ (else f would be alternating and $X_0 = \{0\}$), so $\varepsilon = -1$ and $K_{\sigma,\varepsilon} = \{t \in K \mid t^\sigma = t\}$; since q is not zero on X_0, $K_0 \not\subset K_{\sigma,\varepsilon}$, whence $K = K_0.K_{\sigma,\varepsilon}$ and case (iii) applies. If $(\sigma,\varepsilon) = (\mathrm{id.},1)$ one applies (1) and (3) with $z = e_0x$, $z' = e_0y$. Case (ii) reduces to $\dim X_0 = 1$ and then to case (i) via the classical isomorphism between a quaternionic antihermitian group in three variables and a hermitian group in four variables over a commutative field. □

Remarks **(10.1.38)**. —

a) The conclusion of (10.1.36) (ii) is in general *false* when $(\sigma,\varepsilon,\dim X_0) = (\mathrm{id.},1,2)$. Indeed, take $G = G^0$ (10.1.4) with f nondegenerate and let L be the quadratic extension of K splitting f (i.e. over which $q|_{X_0}$ acquires nontrivial zeros). There is then an isomorphism of G onto $\mathrm{PSL}_2(L)$ carrying $(T,(U_a))$ onto the "natural" root datum of $\mathrm{PSL}_2(L)$. The equipollence classes A therefore correspond bijectively and naturally to the non-improper valuations ω_1 of L (by (6.2.12) *b)*), and A satisfies the conclusion of (10.1.36) (ii) only when ω_1 is invariant under the automorphism of order 2 of L fixing K.

b) Keep the hypotheses of (10.1.36) (ii), with K commutative and $X_0 = \{0\}$. If $\sigma = \mathrm{id}$. then $\varepsilon \neq 1$ by the conventions of (10.1.1), G^0 with its root datum is isomorphic to $SL_2(K)$, and (6.2.12) *b)* shows that ω is a valuation of K. If on the other hand $\sigma \neq \mathrm{id}$., then G^0 with its root datum is isomorphic to $SL_2(L)$, L being the field of invariants of σ; the restriction ω_1 of ω to L is a valuation of L which may be chosen arbitrarily ((6.2.12) *b)*), and ω is a valuation of K only if ω_1 extends in a unique way to a valuation of K ((10.1.36) (1)).

The authors add that they know no example with $X_0 \neq \{0\}$ in which ω fails to be a valuation of K.

10.2. The groups $GL_n(K)$ and $SL_n(K)$. —

(10.2.1) *Setting.* Let K be a field, not necessarily commutative, and let X be a right vector space of finite dimension $\geqslant 2$ over K, equipped with a basis $(e_i)_{1\leqslant i\leqslant r+1}$. For an endomorphism g of X, write $(g_{ij})_{1\leqslant i,j\leqslant r+1}$ for its matrix in this basis. For $1 \leqslant i, j \leqslant r+1$, $i \neq j$, let ζ_{ij} send $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(K)$ to the endomorphism $\zeta_{ij}(m)$ of X given by

$$\zeta_{ij}(m).e_i = e_i a + e_j c, \qquad \zeta_{ij}(m).e_j = e_i b + e_j d, \qquad \zeta_{ij}(m).e_h = e_h \;\; (h \neq i, j).$$

For $k \in K$ one puts

$$u_{ij}(k) = \zeta_{ij}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \zeta_{ji}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \qquad m_{ij}(k) = \zeta_{ij}\begin{pmatrix} 0 & k \\ -k^{-1} & 0 \end{pmatrix} = m_{ji}(-k^{-1}) \;\; (k \in K^*).$$

In $\mathbf{R}^{r+1}$ with its usual euclidean norm and canonical basis $(a_i)_{1\leqslant i\leqslant r+1}$, put $a_{ij} = a_j - a_i$; the set Φ of the a_{ij} is an irreducible root system of type A_r in the space V^* they span ([5], ch. VI, p. 251). For $a_{ij} \in \Phi$ set

$$U_{a_{ij}} = \{u_{ij}(k) \mid k \in K\}, \qquad M^0_{a_{ij}} = \{m_{ij}(k) \mid k \in K^*\}.$$

The map $k \mapsto u_{ij}(k)$ is an injective homomorphism $K \to \mathrm{End}(X)$, and $U_{a_{ij}}$ is the set of elements of $\mathrm{End}(X)$ of the form $1 + n$ with $n(e_h) = 0$ for $h \neq j$ and $n(e_j) \in e_i K$.

Proposition **(10.2.2)**. — *Let* G *be a subgroup of* GL(X) *containing the subgroups* $U_{a_{ij}}$ *for* $1 \leqslant i, j \leqslant r+1$, $i \neq j$, *and let* T *be the subgroup formed of the elements of* G *whose matrix relative to the basis* (e_i) *is diagonal. Put* $M_{a_{ij}} = T.M^0_{a_{ij}}$. *Then* $(T, (U_{a_{ij}}, M_{a_{ij}})_{a_{ij}\in\Phi})$ *is a generating root datum of type* Φ *in* G.

Summary. (DR 1) and (DR 3) are clear. (DR 2) follows from the commutator formulas

$$(1) \qquad (u_{ij}(x), u_{h\ell}(y)) = \begin{cases} 1 & \text{if } i \neq \ell \text{ and } h \neq j, \\ u_{i\ell}(xy) & \text{if } i \neq \ell \text{ and } h = j, \\ u_{hj}(-xy) & \text{if } i = \ell \text{ and } h \neq j, \end{cases}$$

(the case $i = \ell$, $h = j$ being automatic). (DR 4) follows from (6.1.3) *a)*. Since $m_{ij}(k)$ exchanges the lines $e_i K$, $e_j K$ and fixes the others, $m_{ij}(k) U_{a_{h\ell}} m_{ij}(k)^{-1} = U_{a_{\tau(h)\tau(\ell)}}$ with $\tau = (i\,j)$, matching $r_{a_{ij}}(a_{h\ell}) = a_{\tau(h)\tau(\ell)}$; together with T-invariance of the $U_{a_{h\ell}}$ this gives (DR 5). Taking Φ^+ to be the a_{ij} with $i < j$, the groups $U^\pm$ consist of unipotent upper/lower triangular matrices, whence (DR 6). Consequently G^0 is normal in GL(X) and equals SL(X), the group generated by all transvections (= kernel of the Dieudonné determinant) [21]; since $GL(X) = \widetilde{T}.SL(X)$ the datum is generating. In particular G may be *any* subgroup of GL(X) containing SL(X). □

(10.2.3) Let ω be a non-improper valuation of K. For $1 \leqslant i, j \leqslant r+1$, $i \neq j$, and $k \in K$, put

$$\varphi^\omega_{a_{ij}}(u_{ij}(k)) = \omega(k).$$

Proposition. — *The family* $\varphi^\omega = (\varphi^\omega_a)_{a\in\Phi}$ *is a special valuation of the root datum* $(T, (U_a)_{a\in\Phi})$.

Summary. (V 0), (V 1), (V 4) are immediate; (V 3) follows from the formulas (1) of (10.2.2); (V 5) and (V 2) for $m \in \mathrm{M}^0_a$ follow from (6.2.3) *a)* via the homomorphisms $\zeta_{ij} : \mathrm{SL}_2(\mathrm{K}) \to \mathrm{G}$. For (V 2) with $t \in \mathrm{T}$ one uses

$$t\, u_{ij}(k)\, t^{-1} = u_{ij}(t_{ii}\, k\, t_{jj}^{-1}), \tag{1}$$

which gives $\varphi^{\omega}_{a_{ij}}(tut^{-1}) = \varphi^{\omega}_{a_{ij}}(u) + \omega(t_{ii}) - \omega(t_{jj})$, together with $\varphi^{\omega}_{a_{ij}}(u_{ij}(1)) = 0$. □

(10.2.4) One proves as in (10.1.24) that the valuation φ^{ω} is *extendable*. Moreover $\Gamma_{a_{ij}} = \Gamma'_{a_{ij}} = \omega(\mathrm{K}^*)$ for every $a_{ij} \in \Phi$, and when ω is discrete the échelonnage $\mathscr{E}$ associated with φ^{ω} is of type A_r.

Proposition **(10.2.5)**. —

(i) *The group* N *is the subgroup of* G *formed of the elements whose matrix* (n_{ij}) *is monomial, and there is a surjective homomorphism* s *from* N *onto the symmetric group* $\mathfrak{S}_{r+1}$ *such that the only nonzero coefficients of the matrix of an element* $n \in \mathrm{N}$ *are the coefficients* $n_{s(n)(i),i}$.

(ii) *Let* A *be the affine space under the dual* V *of* V^* *of the valuations equipollent to* φ^{ω}, *and let* ν *be the homomorphism from* N *into the affine group of* A *defined in* (6.2.10). *Identify* A *and* V *by taking* φ^{ω} *as origin. Let* $n \in \mathrm{N}$ *and* $\sigma = s(n)$. *For* $x \in \mathrm{A}$ *and* $1 \leqslant i, j \leqslant r+1$ *one has*

$$(a_{\sigma(i)\sigma(j)})(\nu(n).x) = (a_{ij})(x) + \omega(n_{\sigma(j),j}) - \omega(n_{\sigma(i),i}).$$

(iii) *The subgroup* $\mathrm{H} = \operatorname{Ker} \nu$ *of* N *is the subgroup of the* $t \in \mathrm{T}$ *such that* $\omega(t_{ii})$ *is independent of* i *for* $1 \leqslant i \leqslant r+1$.

Summary. N is generated by T and the $m_{ij}(k)$, and $s(m_{ij}(k))$ is the transposition of i and j; (ii) is a direct computation from (10.2.3) (1), and (iii) is an immediate consequence. □

(10.2.6) Let $\mathbf{W}$ be the affine Weyl group of Φ acting on A (identified with V via φ^{ω}), and $\widetilde{\mathbf{W}}$ its normalizer in the affine group of A (cf. (1.3.18)). Using (6.2.19) one checks that the translation group $\mathrm{W} \cap \mathrm{V}$ contained in $\mathrm{W} = \nu(\mathrm{N} \cap \mathrm{G}^0)$ is $(\mathbf{W} \cap \mathrm{V}) \otimes \omega(\mathrm{K}^*)$; identifying V and V^* by the scalar product induced from $\mathbf{R}^{r+1}$, it is the restriction to V^* of the group of translations $\sum_i \gamma_i a_i$ with $\gamma_i \in \omega(\mathrm{K}^*)$ and $\sum_i \gamma_i = 0$.

If $\mathrm{G} = \mathrm{GL}(\mathrm{X})$, then by (10.2.5) (ii) the translation group of $\widehat{\mathrm{W}} = \nu(\mathrm{N})$ is identified with $(\widetilde{\mathbf{W}} \cap \mathrm{V}) \otimes \omega(\mathrm{K}^*)$, i.e. with the restrictions to V^* of the translations $\sum_i \big(\gamma_i - (r+1)^{-1} \sum_j \gamma_j\big) a_i$ with $\gamma_i \in \omega(\mathrm{K}^*)$. One deduces that for any G with $\mathrm{SL}(\mathrm{X}) \subset \mathrm{G} \subset \mathrm{GL}(\mathrm{X})$,

$$(r+1)(\widehat{\mathrm{W}} \cap \mathrm{V}) \subset \mathrm{W} \cap \mathrm{V}. \tag{1}$$

(10.2.7) Let C be the commutator group of K^* and $\det : \mathrm{GL}(\mathrm{X}) \to \mathrm{K}^*/\mathrm{C}$ the Dieudonné determinant [21]. Since $\omega(\mathrm{C}) = \{0\}$, the homomorphism $\omega : \mathrm{K}^* \to \mathbf{R}$ induces $\omega' : \mathrm{K}^*/\mathrm{C} \to \mathbf{R}$; one sets $\delta = \omega' \circ \det$.

Proposition. — Let $\mathscr{B}$ *be the set of subsets* M *of* G *such that the set of the* $\omega(g_{ij})$ *for* $g \in \mathrm{M}$, $1 \leqslant i, j \leqslant r+1$, *and the set of the* $-\delta(g)$ *for* $g \in \mathrm{M}$ *are both bounded below. Then* $\mathscr{B}$ *is a bornology compatible with* φ^{ω}. *The bornology defined by* φ^{ω} *is generated by* $\mathscr{B}$.H. *It coincides with* $\mathscr{B}$ *if and only if* $\mathrm{G} \subset \operatorname{Ker} \delta$.

Summary. Stability under products and boundedness of the $\mathrm{U}_{a,k}$ are clear (the Dieudonné determinant of a triangular matrix is the image of the product of its diagonal coefficients). For $\mathrm{M} \in \mathscr{B}$ with $\mathrm{M} \subset \mathrm{T}$, setting $p = \inf\{\omega(t_{ii})\}$ and $q = \sup\{\delta(t)\}$ over M, the relations $-\delta(t^{-1}) = \sum_i \omega(t_{ii}) \geqslant (r+1)p$ and $\omega((t^{-1})_{ii}) = -\omega(t_{ii}) \geqslant -q + rp$ give $\mathrm{M}^{-1} \in \mathscr{B}$ and $(a_i - a_j)(\nu(t)) = \omega(t_{ii}) - \omega(t_{jj}) \leqslant q - (r+1)p$ by (10.2.5) (ii). Hence $\mathscr{B}$ induces on T, and so on N, a bornology compatible with the group law for which $\nu(\mathrm{M})$ is bounded in IsomA; by (8.1.6) *a)* this makes $\mathscr{B}$ weakly compatible with φ^{ω}. Finally

$$\mathrm{H}_{[0]} \subset \mathrm{H} \cap \mathrm{SL}(\mathrm{X}) \subset \mathrm{H} \cap \operatorname{Ker}\delta = \{t \in \mathrm{T} \mid \omega(t_{ii}) = 0,\ 1 \leqslant i \leqslant r+1\} \in \mathscr{B}.$$

The proof is finished as in (10.1.31), replacing $\mathrm{Is}(f, q)$ by SL(X) and $\omega \circ c$ by δ. □

Proposition **(10.2.8)**. — *Let* E *be a vectorial facet of* Φ *and* x *a point of* A. *An element* $g \in \mathrm{G}$ *belongs to* $\widehat{\mathrm{P}}_{x,\mathrm{E}}$ (7.2.4) *if and only if one has, for* $1 \leqslant i, j \leqslant r+1$*:*

(1) $$\omega(g_{ij}) + a_j(x) - a_i(x) \ \geqslant\ (r+1)^{-1}\delta(g)$$

and

(2) $$\omega(g_{ij}) + a_j(x) - a_i(x) \ >\ (r+1)^{-1}\delta(g) \qquad \text{when } (a_i - a_j)(\mathrm{E}) \subset \mathbf{R}_+^*.$$

Summary. As in (10.1.32) it suffices to show $\mathrm{N} \cap \mathrm{L} = \widehat{\mathrm{N}}_{x,\mathrm{E}}$, where L is the set of g satisfying (1) and (2). For $n \in \mathrm{N} \cap \mathrm{L}$ with $\sigma = s(n)$, summing (1)/(2) over $i = \sigma(j)$ gives $\sum_j \omega(n_{\sigma(j),j}) \geqslant \delta(n)$, which is an *equality*; hence every one of those inequalities is an equality. By (10.2.5) (ii) this gives $\nu(n).x = x$, and one must have $(a_{\sigma(j)} - a_j)(\mathrm{E}) \not\subset \mathbf{R}_+^*$ for every j, whence $(a_{\sigma(j)} - a_j)(\mathrm{E}) = \{0\}$ and $n \in \widehat{\mathrm{N}}_{x,\mathrm{E}}$. Conversely, for $n \in \widehat{\mathrm{N}}_{x,\mathrm{E}}$ the numbers $\omega(n_{\sigma(j),j}) + a_j(x) - a_{\sigma(j)}(x)$ are all equal, giving (1); and for $i \neq \sigma(j)$ one has $\omega(n_{ij}) = +\infty$. □

Corollary **(10.2.9)**. — *Let* Ω *be a convex subset of* A. *In order that an element* $g \in \mathrm{G}$ *belong to* $\widehat{\mathrm{P}}_{\Omega}$ *it is necessary and sufficient that*

$$\omega(g_{ij}) \ \geqslant\ \sup\{a_i(x) - a_j(x) \mid x \in \Omega\} + (r+1)^{-1}\delta(g) \qquad (1 \leqslant i, j \leqslant r+1).$$

Proposition **(10.2.10)**. — *Let* ψ *be a valuation of the root datum* $(\mathrm{T}, (\mathrm{U}_a)_{a\in\Phi})$ *of* G. *There exists one and only one valuation* ω *of the field* K *such that* ψ *is equipollent to* φ^{ω}.

The proof is entirely analogous to, and simpler than, that of (10.1.34).

Note added in proof

The following is a condensed account of the note on pp. 238–239, not a sentence-by-sentence translation.

(a) GL_{r+1} **and the Goldman–Iwahori space of norms.** When K is a complete, locally compact valued field, the building of $\mathrm{SL}_{r+1}(\mathrm{K})$ for the valuation φ^{ω} of (10.2.3) is closely related to the space of norms of O. Goldman and N. Iwahori [*Acta Math.* **109** (1963), 137–177]. In general, with the notation of 10.2 and no extra hypothesis on K, call an *additive norm* γ on X (i.e. the logarithm of the inverse of a norm in the usual sense) *decomposable* if there is a basis $(u_i)_{1\leqslant i\leqslant r+1}$ of X and reals c_i with

(1) $$\gamma\Big(\sum_{i=1}^{r+1} u_i x_i\Big) = \inf_i\{\omega(x_i) - c_i\} \qquad (x_i \in \mathrm{K}),$$

(every norm has this property if K is maximally complete), and call two norms *homothetic* if they differ by a constant. Then there is exactly one G-invariant bijection from the building $\mathscr{I}$ of G (for φ^{ω}) onto the set of homothety classes of additive norms on X carrying the class of the norm γ of (1) to the point of $\mathscr{I}$ representing $\varphi^{\omega} + v$, where $v \in \mathrm{V}$ is given by $a_i(v) = c_i$. In particular, for K locally compact, $\mathscr{I}$ is the quotient of the Goldman–Iwahori space $\mathscr{N}(\mathrm{X})$ by homothety. The topology on $\mathscr{I}$ from the metric of (7.4.20) (or (2.5.4)) is the quotient of the Goldman–Iwahori topology, although their metric is unrelated to the one used here. The authors note that it is sometimes useful to attach to a reductive group over a local field a building which is the direct product of a building of the present chapter with a euclidean affine space; for GL_{r+1} this modified building is the space of norms itself, rather than its quotient by homothety.

(b) The other classical groups (after a suggestion of A. Weil). Resume the notation and conventions of (10.1.1) and (10.1.14), assume conditions (i)–(iv) of (10.1.15), and let $\mathscr{J}$ be the set of additive norms α on X satisfying

$$\omega(q(x)) \geqslant 2\alpha(x) \qquad (x \in \mathrm{X}), \tag{2}$$

$$\omega(f(x, y)) \geqslant \alpha(x) + \alpha(y) \qquad (x, y \in \mathrm{X}), \tag{3}$$

where for $\bar{t} \in \mathrm{K}/\mathrm{K}_{\sigma,\varepsilon}$ one puts $\omega(\bar{t}) = \sup\{\omega(t) \mid t \in \bar{t}\}$. Then there is exactly one G-invariant injection ι of the building $\mathscr{I}$ of G (for φ^ω) into $\mathscr{J}$ such that, for $v \in \mathrm{V}$ given by $a_i(v) = c_i$, $\iota(\varphi^\omega + v)$ is the norm

$$\gamma\Big(\sum_{i=1}^{r} e_i x_i + x_0\Big) = \inf\Big\{\omega(x_i) - c_i,\ \tfrac{1}{2}\omega(q(x_0))\Big\}. \tag{4}$$

If moreover X is finite-dimensional and K, $\mathrm{K}_{\sigma,\varepsilon}$ are maximally complete for ω, $\omega|_{\mathrm{K}_{\sigma,\varepsilon}}$, then $\iota(\mathscr{I})$ is exactly the set of maximal elements of $\mathscr{J}$, and every element of $\mathscr{J}$ is minorised by an element of $\iota(\mathscr{I})$.

Sketch of the argument. Call subspaces Y, Y' of X *orthogonal for* α if $\alpha(y + y') = \inf\{\alpha(y), \alpha(y')\}$ for $y \in \mathrm{Y}$, $y' \in \mathrm{Y}'$. Put $\mathrm{X}_+ = \sum_{i=1}^{r} \mathrm{X}_i$ and choose α-orthogonal supplements X'_- of $\mathrm{X}_0 + \mathrm{X}_+$ and X'' of X_+ in X. Setting $\alpha^*(x_+) = \sup\{f(x_+, x') - \alpha(x') \mid x' \in \mathrm{X}'_-\}$, the norm $x_+ + x'' \mapsto \inf\{\alpha^*(x_+), \alpha(x'')\}$ majorises α and lies in $\mathscr{J}$, so one may assume $\alpha^* = \alpha|_{\mathrm{X}_+}$. Take an α-orthogonal basis $(u_i)_{1\leqslant i\leqslant r}$ of X_+ and the dual basis (u'_{-i}) of X'_- with $f(u_i, u'_{-j}) = \delta_{ij}$, and for each i pick $\lambda_i \in q(u'_{-i})$ with $\omega(\lambda_i) = \omega(q(u'_{-i}))$ (possible by the hypothesis on $\mathrm{K}_{\sigma,\varepsilon}$). Then $u_{-i} = u'_{-i} - \varepsilon u_i \lambda_i$ spans a space $\mathrm{X}_- = \sum_i u_{-i}\mathrm{K}$ also orthogonal to $\mathrm{X}_+ + \mathrm{X}_0$; choosing it as X'_- one may, after transforming α and the data by a suitable element of G, assume $e_i = u_i$ for $i \in \mathrm{I}$. Writing (2) for $x = x_+ + x_- + x_0$ and (3) for that x against X_+ and X_- shows that α minorises the norm γ of (4) with $c_i = -\omega(e_i)$, and this γ is easily seen to be a maximal element of $\mathscr{J}$.

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Index of notations

Index of conditions

Terminological index

Headwords are given in English and re-alphabetized accordingly; the French original is added in parentheses where the correspondence is not transparent. Section references are those of the original index.

Appendix: French–English glossary

The renderings of the French technical vocabulary used in this translation.

échelonnage	échelonnage (retained)
épinglage	pinning
équipollent, équivalent	equipollent, equivalent
équipollentes	equipollent
adhérence	closure
alcôve	alcove
appartement	apartment
applications structurales	structural maps
base de filtre	filter base
borne inférieure / supérieure	infimum / supremum
bornologie	bornology
cône simplicial	simplicial cone
centre de gravité	barycenter
chambre	chamber
chambre vectorielle	vector chamber
cloison	panel
close	closed
complexe polysimplicial	polysimplicial complex
corps	field (not assumed commutative unless stated)
corps gauche	skew field
corps local	local field
corps résiduel	residue field
corps résiduel	residue field
corps valué	valued field
décomposition réduite	reduced decomposition
déplacements	displacements
déployé / quasi-déployé	split / quasi-split
descente	descent
donnée radicielle	root datum
donnée radicielle génératrice	generating root datum
donnée radicielle valuée	valued root datum
donnée radicielle (valuée)	(valued) root datum
donnée radicielle génératrice	generating root datum
double système de Tits	double Tits system
déployé	split
ensemble affine	affine set
enveloppe convexe	convex hull
espace affine euclidien	Euclidean affine space
facette	facet
facette / facette vectorielle	facet / vector facet
facette vectorielle	vector facet
famille filtrante croissante	increasing directed family
fixateur	fixer (pointwise stabilizer)
fortement équivalents	strongly equivalent
fortement orthogonales	strongly orthogonal
galerie	gallery
germe de quartier	sector germ
graphe de Coxeter	Coxeter graph
groupe des commutateurs	commutator group
homomorphisme B-adapté	B-adapted homomorphism

immeuble	building
mitoyennes (chambres)	adjacent (chambers)
morphisme chambré	chambered morphism
mur	wall
mur, bord	wall, boundary
ordre grignotant	nibbling order
par récurrence sur	by induction on
partie close	closed subset
parties closes	closed subsets
poids radiciels	radical weights
point extrémal	extreme point
point interne	internal point
point spécial	special point
polysimplexe	polysimplex
prolongement / prolongeable	extension / extendable
pronilpotent / prounipotent	pronilpotent / prounipotent
quartier	sector
quitte à remplacer	after replacing, if necessary
réseau	lattice
rétraction	retraction
racine affine	affine root
racine vectorielle	vector root
racines non divisibles / non multipliables	non-divisible / non-multipliable roots
radical prounipotent	prounipotent radical
sans bégaiement	non-stuttering
saturé	saturated
similitude	similarity
sommet (d'un quartier)	apex (of a sector); vertex (of a sector)
sous-groupe borné	bounded subgroup
sous-groupe distingué	normal subgroup
sous-groupe distingué	normal subgroup
sous-groupe parabolique	parabolic subgroup
sous-groupe parahorique	parahoric subgroup
stabilisateur	stabilizer
système de racines inverse	inverse (dual) root system
système de Tits	Tits system (= BN-pair)
série centrale descendante	descending central series
tendue (galerie)	taut (gallery)
valuation	valuation
valuation dense	dense valuation
valuation impropre	improper valuation
valuation prolongée	extended valuation
équipollent	equipollent (parallel and equally directed)

F. Bruhat and J. Tits