Weyl calculus, Wigner transform and beyond

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1 Introduction

In quantum mechanics, we are always interested in the probability density, or more precisely, the wave function $\psi(x)$. Then we may visualize $\rho(x) = |\psi(x)|^2$ to get the density function. On the other hand, the momentum is also an essential quantity for physicists. The well known relation

$$\phi(p) = \frac{1}{\sqrt{\hbar}} \int_{\mathbb{R}^d} e^{-ixp/\hbar} \psi(x) dx,$$

gives the relation between the density of x and the density of p. But it's somehow hard for us to reveal the nature of p immediately by the density function of x. Hence it's better to have a function that both reveals density of x as well as p. The Wigner function, introduced by Wigner in 1932, is just the thing we want.[**?**] This function contains all the information we need, and can reveal expectation values for physical observables just by integrations.

The information in Wigner function indicates that it contains information for solution of Schrödinger equations. Some basic calculation tells us that, the Schrödinger

$$i\hbar \frac{\partial \psi}{\partial t}(x,t) = \hat{H}\psi(x,t)$$

is hard to solve where $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$ is the Hamiltonian. Explicit solution only exists for a few specific potential functions. In this article, we may consider $\hbar = \epsilon$ and $m = 1$, which we call the *semiclassical Schrödinger equation*. Namely,

$$i\epsilon \frac{\partial \psi^\epsilon}{\partial t} = -\frac{\epsilon^2}{2}\Delta\psi^\epsilon + V(x)\psi^\epsilon, \quad \psi^\epsilon(0,x) = \psi^\epsilon_{\text{in}}(x)$$

To solve this equation, we may consider a more generalized problem. If we are handling problem like $F(X,D)u = f$, where X, D are operators and F is functions from a field that "acts on" operators. Can we just use elementary algebra to derive $u = F^{-1}(X,D)f$? In many time this cannot be done. But in the semiclassical Schrödinger case, this can be approximately done! The thing we need to do is to set up a "functional calculus" to do the right job.

A functional calculus is a method to assign an operator $f(A)$ to each function on phase space $f \in \Xi$.[2] Roughly speaking, functional calculus is a structure that gives the homomorphism such that $fg(A) = f(A)g(A)$. A well known result is the so called "holomorphic functional calculus", which assign the map A and holomorphic f to $f(A)$ such that the homomorphism is the ordinary product. But for multiple numbers of operators, things are quite different. Since we are dealing with operators that may act on each other, commutation should be taken into mind. There's much to be changed in the new algebra. The homomorphism may now be replaced by the "homomorphism with an error term", which, in the case we will describe below, is that $fg(A_1, A_2, \cdots, A_n) - f(A_1, A_2, \cdots, A_n)g(A_1, A_2, \cdots, A_n)$ should be small(not necessarily zero).

Weyl calculus is just the right "functional calculus" to deal with operators involved in quantum mechanics. This calculus is "unbiased" on both element and satisfy all the thing a "good calculus" may contain. With this calculus, we can define the Wigner transform easily, which is exactly the original function(or Weyl symbol) that generate the projection operator.

We also consider something on the WKB method. WKB method is a good approach to solving PDE which has high frequency term. But it suffer a great obstacle that its solution are all locally asymptotic but globally explode. So Wigner transform come into place. The Schrödinger equation (or to say more geneally, the linear dispersive model) can be solved(or more accurately approximated) by using the WKB initial data on the Wigner semiclassical limit.

Quantization problems can also be extended. The goal of quantization is to relate a Hilbert space with a "classical mechanics", or the Lagrangian manifold. We may now extend our quantization to a specific structure called the symplectic manifold. With this quantization, we may also define a "Wigner transform", which has not been done before.

The structure of this article is as follows. Firstly, we review some basic ideas and notations in Hamiltonian mechanics and quantum mechanics. Secondly, we derive the Weyl calculus for functions from polynomials to arbitrary function on phase spaces. Thirdly, we present a useful tool in the computation for quantum mechanics, which is called the WKB analysis, and use Wigner transform to overcome obstacle aroused in WKB approximation. Lastly, we will give some note on the quantization on symplectic manifold.

2 Hamiltonian analysis and Quantum mechanics

2.1 Hamiltonian analysis

In classical theory, we may dealing with the phase space $\Xi = T^*\mathbb{R}^d \sim \mathbb{R}^{2d}$. A point in this space is denoted by (p, x), representing the momentum and the location. Observables defined Ξ is exactly functions from Ξ to $\mathbb{R}$.

Hamiltonian $H(p, x)$ is of extreme importance in the classical theory of mechanics. This functional determine the time evolution through motion, that is, we have the Hamiltonian's equations of motion

$$\dot{x} = \nabla_p H(x, p), \quad \dot{p} = \nabla_x H(x, p).$$

To simplify the assertion, we may introduce the Poisson bracket of observables, defined by

$$\{A, B\} := (\nabla_p A)(\nabla_x B) - (\nabla_x A)(\nabla_p B).$$

Here, ∇_p is the gradient with respect to p, and the product is the dot product between the vectors. It's easy to see that this is an asymmetric operator. Hence Hamiltonian equation of motion is now

$$\dot{x} = \{H, x\}, \quad \dot{p} = \{H, p\}.$$

By simple calculation ,we also have the familiar identity

$$\{x_i, x_j\} = \{p_i, p_j\} = o \text{ for any } i, j \quad \{x_i, p_j\} = -\delta_{ij}$$

The most important thing is that any observable A is governed by

$$\dot{A} = \{H, A\}$$

2.2 Quantum mechanics

In quantum mechanics the system is represented by a wave function $\psi \in L^2(\mathbb{R}^d)$, and we also consider the self-adjoint bounded linear functionals on $\mathcal{H}$(denoted by $\hat{A} \in \mathcal{B}(\mathcal{H})$ and called observables). As we may know that $|\psi|^2$ is a probability density function, we may have the expectation function for the observables, denoted by

$$\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}\psi\rangle = \int_{\mathbb{R}^d} \bar{\psi}(x)(\hat{A}\psi)(x)dx.$$

This time, the Hamiltonian $\hat{H} \in \mathcal{B}(\mathcal{H})$ in the system is the operator satisfying the following equation for any observables A

$$\dot{\hat{A}} = -\frac{i}{\epsilon}[\hat{H}, \hat{A}] = -\frac{i}{\epsilon}(\hat{H}\hat{A} - \hat{A}\hat{H}).$$

In the quantum mechanics sense, ϵ is often $\hbar = h/2\pi$, when h is the Planck constant. This time we are dealing with the semiclassical case, so the ϵ is not given its meaning in quantum mechanics, but rather a constant in $(0, 1]$.

Previously, we stated that Schrödinger equation is given by

$$i\epsilon\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t).$$

The purpose is to solve the equation with this fashion"

$$\psi(x,t) = e^{-\frac{i}{\epsilon}\hat{H}t}\psi(x,0).$$

As if it is multiplying a constant. As ψ evolves, we may see $A(t)$ as the evolution of A. Namely,

$$\langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi(0)|\hat{A}(t)|\psi(0)\rangle$$

This gives the observation that $\hat{A}(t) = e^{\frac{i}{\epsilon}\hat{H}t}\hat{A}e^{-\frac{i}{\epsilon}\hat{H}t}$ And it's easy to see that $\hat{A}(t)$ is an observable, that is ,it satisfies $\dot{\hat{A}} = -\frac{i}{\epsilon}[\hat{H}, \hat{A}]$.

2.3 Strong quantization

From now on, we may see the similarity between the classical case and the quantum case. In the quantum mechanics, there's also a canonical basis. The "location operator" $\hat{x}$, which is multiply by x, and the $\hat{p}$, which is $\frac{\epsilon}{i}\nabla$. The canonical relation is then

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \text{ for all } i, j \quad [\hat{x}_i, \hat{p}_j] = i\epsilon\delta_{ij}$$

Naturally, we may wish to have a transform that maps observables in classical sense to observables in the quantum sense. Since the operators may not commute in the quantum mechanics, we may wish that xp and px are mapped to $\frac{1}{2}(\hat{p}\hat{x} + \hat{x}\hat{p})$, which is called the *Weyl ordering*. So we have the following definition:

Definition 2.1 (strong quantization)**.** *A strong quantization is a mapping from functions on $f \in \Xi$ to operators on $Op(f) \in \mathcal{H}$, such that the following assertion holds:*

1. *$Op(x_j) = \hat{x}_j, Op(p_j) = \hat{p}_j, Op(c) = cI$, where I is the identity operator and c is a constant.*

2. *$Op(af + bg) = aOp(f) + bOp(g)$, where a, b are constants.*

3. *$Op(fg) = \frac{1}{2}(Op(f)Op(g) + Op(g)Op(f))$, which is called the Jordan product and represents unbiasedness.*

4. *$[Op(f), Op(g)] = -i\epsilon\{f, g\}$*

Unfortunately, this "perfect" quantization does not exists, as was proved by Groenewold. So we are looking for quantization "in the weak sense". To be more accurate, 1 and 2 holds when 3,4 holds only when $\epsilon \to 0$. The Weyl calculus is this kind of quantization.

3 Weyl calculus

The following contents comes from [2], with a number of extension.

3.1 Weyl Calculus on polynomials

To set up the most rudimentary form of the calculus, we may consider the polynomials functions of the operators A and B. When the algebra is direct substitution, namely, for polynomial

$$F(a, b) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} a^i b^j,$$

A way to define $F(A, B)$ is done by direct substitution, which is called the *Kohn-Nirenberg calculus*. That is exactly

$$F(A, B) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} A^i B^j$$

But this calculus put A before B, and interchanging A and B gives a different calculus. What we would like is a calculus that does not change when order of A, B is changed. So the Weyl calculus of the polynomial is defined by

$$F(A, B) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} (A^i B^j)_W.$$

Here $(A^i B^j)_W$ is defined as the average of all $\binom{i+j}{i}$ possible ordering of $A^i B^j$. As an example, $(A^2 B)_W = \frac{1}{3}(AAB + ABA + BAA)$ and $(A^2 B^2) = \frac{1}{6}(AABB + ABAB + ABBA + BABA + BBAA + BAAB)$.

As a simple corollary of the definition, we may note that the Weyl quantization of $F(a, b) = (ra + sb)^n$ is exactly $F(A, B) = (rA + sB)^n$ no matter A, B are commutative or not. But for Kohn-Nirenberg calculus, $(A + B)^2 \neq A^2 + 2AB + B^2$, so the same relation does not hold.

So for the linear change of operators $A' = cA + dB + e, B' = fA + gB + h$, and polynomial that $F(a, b) = F'(ca + db + e, fa + gb + h)$, we may get the similar equation written as

$$F'(A', B')_W = F(A, B)_W.$$

This calculus can be also extended to arbitrary formal infinite series. For instance, the series

$$F(a,b)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}c_{i,j}a^ib^j\mapsto F(A,B)_W=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}c_{i,j}(A^iB^j)_W.$$

So we may get the identity

$$(\exp(rA+sB+t))_W=\exp(rA+sB+t)$$

the latter is the formal series summation of the operator.

3.2 Moyal product for polynomials

We may see that in the Weyl calculus, $(FG)(A,B)_W$ and $F(A,B)_WG(A,B)_W$ are different when A,B are not commutative. In particular, we are interested in the model of A,B satisfying the *Heisenberg commutator relationship* $[A,B]=i\epsilon$.(here identity I is omitted). So by the same identity, we know that $AB=(AB)_W-\frac{1}{2}[A,B]=(AB-\frac{1}{2}i\epsilon)_W$. So we would like to find the expression for all polynomials. Here's the definition

Definition 3.1. *The Moyal product of polynomial F and G, denoted by $F\star_\epsilon G$, is the unique polynomial such that*

$$(F\star_\epsilon G)(A,B)_W=F(A,B)_WG(A,B)_W$$

If the classical limit $\epsilon\to 0$ holds, we may obtain the relation $F_WG_W=(FG)_W$. So the following is the attempt to compute the explicit expression for Moyal product of two polynomials.

We first observe that $[sA+tB,s'A+t'B]=i\epsilon\omega((s,t),(s',t'))$ where ω is the anti symmetric form

$$\omega((s,t),(s',t')):=st'-ts'.$$

To get a more precise formulation of the relations, we have the Baker-Campbell-Hausdorff formula

Theorem 3.2 (Baker-Campbell-Hausdorff formula)**.** *For any s,t,s',t' and operators A,B satisfying $[A,B]=i\epsilon$, formally we have*

$$\exp(sA+tB)\exp(s'A+t'B)=\exp(sA+tB+s'A+t'B)\exp(\frac{i\epsilon}{2}\omega((s,t),(s',t')))\quad(1)$$

The proof is divided into two parts. First we prove the Hadamard lemma as follows

Lemma 3.3 (Hadamard)**.** $\exp(X)Y\exp(-X)=\exp(\operatorname{ad}(X))Y$ *where* $\operatorname{ad}(X)(Y):=[X,Y]$.

Proof. Note that $\frac{d}{dt}\exp(tX)=X\exp(tX)=\exp(tX)X$. We have the equation

$$\frac{d}{dt}\exp(tX)Y\exp(-tX)=\operatorname{ad}(X)\exp(tX)Y\exp(-tX).$$

As a result for solution of ODE, we conclude that $\exp(tX)Y\exp(-tX)=\exp(t\operatorname{ad}(X))Y$. □

Here we may prove the Theorem 3.2:

Proof. Secondly, we may observe that plugging in $X = sA + tB$ and $Y = s'A + t'B$. Note that $[sA+tB, [sA+tB, s'A+t'B]] = 0$ since the latter is only constant multiplied by identity. Thus we observe that

$$\exp(sA + tB)(s'A + t'B)\exp(-sA - tB) = s'A + t'B + i\epsilon\omega\{(s,t),(s',t')\}.$$

With this identity, we consider the expression

$$C(\lambda) := \exp(\lambda(sA + tB))\exp(\lambda(s'A + t'B))\exp(\frac{\lambda^2}{2}i\epsilon\omega\{(s,t),(s',t')\}).$$

Taking the formal derivative, we see that

$$\begin{aligned} C'(\lambda) =&(sA + tB)\exp(\lambda(sA + tB))\exp(\lambda(s'A + t'B))\exp(\frac{\lambda^2}{2}i\epsilon\omega\{(s,t),(s',t')\}) \\ &+ \exp(\lambda(sA + tB))(s'A + t'B)\exp(\lambda(s'A + t'B))\exp(\frac{\lambda^2}{2}i\epsilon\omega\{(s,t),(s',t')\}) \\ &- \exp(\lambda(sA + tB))\exp(\lambda(s'A + t'B))(\lambda i\epsilon\omega\{(s,t),(s',t')\})\exp(\frac{\lambda^2}{2}i\epsilon\omega\{(s,t),(s',t')\}) \end{aligned}$$

From $s'A + t'B - \lambda i\epsilon\omega\{(s,t),(s',t')\} = \exp(-\lambda(sA + tB))(s'A + t'B)\exp(\lambda(sA + tB))$. So plugging in we get

$$C'(\lambda) = (sA + tB + s'A + t'B)C(\lambda).$$

This equation yields

$$C(\lambda) = \exp(\lambda(sA + tB + s'A + t'B)).$$

Take $\lambda = 1$, we have, for any a, b as function, that

$$\exp(sa + tb) \star_\epsilon \exp(s'a + t'b) = \exp(sa + tb + s'a + t'b)\exp(\frac{1}{2}i\epsilon\omega((s,t),(s',t'))).$$

□

What we need to find is $(sa + tb)^n \star_\epsilon (s'a + t'b)^m$, we consider coefficients before the indeterminate a, b, and find that degree of s plus degree of t should equal n and degree of s' added by degree of t' should equal m. So consider the formal expansion $(\sum(sa+tb)^i/i!)(\sum(s'a+t'b)^j/j!)(\sum(\frac{1}{2}i\epsilon\omega((s,t),(s',t')))^k/k!)$,we should take $i+k = n$ with $j + k = m$, which is $\sum \frac{1}{(n-j)!(m-j)!j!}(sa+tb)^{n-j}(s'a+t'b)^{m-j}(\frac{1}{2}i\epsilon\omega((s,t),(s',t')))^j$. So by multiplying $n!$ and $m!$ as a bounce back of $\exp(sa + tb)$ and $\exp(s'a + t'b)$, we know the equation below

$$(sa+tb)^n \star_\epsilon (s'a+t'b)^m = \sum_{j=0}^{\infty} j!\binom{n}{j}\binom{m}{j}(sa+tb)^{n-j}(s'a+t'b)^{m-j}(\frac{1}{2}i\epsilon\omega((s,t),(s',t')))^j$$

To simplify this result, note that $\nabla^j(sa+tb)^n = j!\binom{n}{j}(s,t)^{\otimes j}(sa+tb)^{n-j}$ and $\omega((s,t),(s',t'))^j = \omega^{\otimes j}((s,t)^{\otimes j},(s',t')^{\otimes j})$, where ${}^{\otimes j}$ denotes the tensor product. Thus we have

$$(sa + tb)^n \star_\epsilon (s'a + t'b)^m = \sum_{j=0}^{\infty} \frac{1}{j!}(\frac{i\epsilon}{2})^j \omega^{\otimes j}(\nabla^j(sa + tb)^n, \nabla^j(s'a + t'b)^m).$$

Then by bilinearity, we have the following theorem

Theorem 3.4 (Explicit product for Moyal product)**.** *The Moyal product for arbitrary F, G is*

$$F \star_\epsilon G = \sum_{j=0}^{\infty} \frac{1}{j!}(\frac{i\epsilon}{2})^j \omega^{\otimes j}(\nabla^j F, \nabla^j G).$$

Observing the first four orders, we have

$$F \star_\epsilon G = FG + \frac{i\epsilon}{2}\omega(\nabla F, \nabla G) - \frac{\epsilon^2}{8}\omega^{\otimes 2}(\nabla^2 F, \nabla^2 G) - \frac{i\epsilon^3}{48}\omega^{\otimes 3}(\nabla^3 F, \nabla^3 G) + \cdots$$

Note that $\omega(\nabla F, \nabla G)$ is exactly the Poisson bracket! Applying simple algebra, we may deduce that $\epsilon \to 0$ implies $\frac{1}{i\epsilon}(F \star_\epsilon G - G \star_\epsilon F) = 0$.

3.3 Weyl quantization on smooth functions

If we would like to extend the above definition to arbitrary smooth function, the Fourier inversion theorem plays an important role. We note that

$$f(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi} f(y) dy d\xi.$$

Thus we would like to define

$$F(A, B)_W = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(s, t)\text{“}\delta(s - A)\delta(t - B)\text{”} ds dt.$$

To be more explicit, this formula can be written as

$$F(A, B)_W = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(s, t) e^{i(\xi(t-B)+y(s-A))} d\xi dy ds dt \tag{2}$$

We may see an example to illustrate this idea:

Example 3.5. *In this definition, $(px)_W = \hat{x}\hat{p} + \frac{\epsilon}{2i}$, which coincides with previous assertion that $f(x, p)_W = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ with $f(x, p) = xp$.*

Proof. From the Baker-Campbell-Hausdorff formula(Theorem 3.2) we know that $\exp(-i(\xi\hat{p} + y\hat{x})) = \exp(-iy\hat{x})\exp(-i\xi\hat{p})\exp(\frac{i}{2}\epsilon\xi y)$. So we substitute $s' = s + \frac{\epsilon\xi}{2}$ in the integration, and we get

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (s' - \epsilon\frac{\xi}{2}) t e^{iy(s'-\hat{x})} e^{i\xi(t-\hat{p})} d\xi dy ds' dt \\
&= \hat{x}\hat{p} - \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} t\epsilon\frac{\xi}{2} e^{i\xi(t-\hat{p})} d\xi dt = \hat{x}\hat{p} - \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} t\frac{\epsilon}{2i}\frac{\partial}{\partial t} e^{i\xi(t-\hat{p})} d\xi dt \\
&= \hat{x}\hat{p} - \int_{\mathbb{R}^d} t\frac{\epsilon}{2i}\delta'(t - \hat{p}) dt = \hat{x}\hat{p} + \frac{\epsilon}{2i}.
\end{aligned}$$

□

Furthermore, the equation 2 is a bit tedious for computation. We may simplify it to the following theorem

Theorem 3.6 (Weyl Quantization formula)**.** *The Weyl quantization for function on $\hat{x}$ and $\hat{p}$(denoted by $F(\hat{x}, \hat{p})_W$), where $\hat{x} = x$ and $\hat{p} = -i\epsilon\nabla$ and the given test function $f(x)$, is that*

$$F(\hat{x}, \hat{p})_W f(z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(\frac{y+z}{2}, \xi)\exp(-i(y - z)\xi) dy d\xi$$

Proof. First, we note two equations: $\exp(iy\hat{x})f(x) = \exp(iyx)f(x)$ and $\exp(i\xi\hat{p})f(x) = f(x+\epsilon\xi)$. So we consider the integration by definition

$$\frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(s,t)\exp(\frac{i}{2}\epsilon\xi y)\exp(i(ys+\xi t))\exp(-iyz)f(z-\epsilon\xi)dyd\xi dsdt$$

Substitute $\eta = z - \epsilon\xi$, the formula becomes

$$\frac{1}{(2\pi)^{2d}\epsilon^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(s,t)\exp(i(ys-(\eta-z)t/\epsilon))\exp(-iy(z+\eta)/2)f(\eta)dyd\eta dsdt$$

Note that

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp(iy(s-(z+\eta)/2))dy = \delta(s-(z+\eta)/2).$$

So the upper formula simplifies to

$$\frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\frac{z+\eta}{2},t)\exp(-i(\eta-z)t/\epsilon)d\eta dt$$

which is exactly the result we want. $\square$

Note that in the Kohn-Nirenberg calculus, $\frac{x+y}{2}$ is replaced by x. And in particular, the adjoint operator $F(\hat{x},\hat{p})^* = \bar{F}(\hat{x},\hat{p})$.

To proceed into the Wigner transform, we must give some definition

Definition 3.7 (Weyl symbol)**.** *The Weyl symbol for $F(\hat{x},\hat{p})_W$ is exactly $F(x,p)$. Denoted by* $\mathrm{Symb}[F_W] = F$

The Weyl quantization can be represented by an integral kernel $K_F(x,y)$, that is

$$(F(\hat{x},\hat{p})_W f(x) = \int_{\mathbb{R}^d} K_F(x,y)f(y)dy.$$

We can get the simple identity (To add the ϵ to satisfy coefficient for $\hat{p}$)

$$K_F(x,y) = \frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d} F(\frac{x+y}{2},p)e^{\frac{i}{\epsilon}p(x-y)}dp.$$

If we substitute $X = \frac{x+y}{2}$ and $\eta = x - y$, then

$$K_F(X+\eta/2, X-\eta/2) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} F(X,p)e^{ipz}dp$$

So we get an easy representation for inversion of Weyl quantization, that is,

$$\mathrm{Symb}[F_W] = F(X,p) = \int_{\mathbb{R}^d} K_F(X+\eta/2, X-\eta/2)e^{-\frac{i}{\epsilon}pz}dp$$

3.4 The Wigner transform

We consider $\hat{P}_\psi$ which projects the density function f onto state ψ, that is $(\hat{P}_\psi f)(x) = \psi\langle\psi, f\rangle = \psi\int_{\mathbb{R}^d}\bar{\psi}f dy$. So we have the definition for the Wigner transform

Definition 3.8 (Wigner transform)**.** *The Wigner transform for function ψ, denoted by $W[\psi](x,p)$, is exactly*

$$W[\psi](x,p) = \mathrm{Symb}[\hat{P}_\psi](x,p) = \int_{\mathbb{R}^d}\psi(x+\frac{z}{2})\overline{\psi(x-\frac{z}{2})}e^{-\frac{i}{\epsilon}pz}dz$$

The latter equation holds because the kernel for the projection is exactly $K_P = \psi(x)\overline{\psi(y)}$.

As we have said in the introduction, Wigner transform provides a function that contains all the information. Sometimes the function(called the Wigner function) is called a "quasi-density" function since it's marginal divided by $(2\pi\epsilon)^d$. That is to say, the following are true:

$$\frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d} W[\psi](p,x)dp = |\psi(x)|^2$$

and

$$\frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d} W[\psi](p,x)dx = |g(p)|^2,$$

where $g(p)$ is the normalized Fourier transform for ψ, which we may see as the wave function for p.

In fact, the Wigner transform can be used to compute the expectation for Weyl quantization, which is the result of the following theorem

Theorem 3.9 (Expectation for operator)**.** *Let ψ be a wave function, the following holds*

$$\langle\psi, F(\hat{x},\hat{p})_W\psi\rangle = \frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W[\psi](x,p)F(x,p)dxdp$$

where W is the Wigner transform and $F(x,p)$ is the Weyl symbol for $F(\hat{x},\hat{p})_W$.

Proof. The proof is by simple calculation as below

$$\begin{aligned}
&\frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W[\psi](x,p)F(x,p)dxdp \\
&= \frac{1}{(2\pi\epsilon)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x+\frac{z}{2})\overline{\psi(x-\frac{z}{2})}e^{-\frac{i}{\epsilon}zp}\int_{\mathbb{R}^d}K_F(x+\frac{z'}{2},x-\frac{z'}{2})e^{-\frac{i}{\epsilon}z'p}dz'dpdx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x+\frac{z}{2})\overline{\psi(x-\frac{z}{2})}K_F(x-\frac{z}{2},x+\frac{z}{2})dxdz \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\overline{\psi(y)}K_F(y,y')\psi(y')dydy' = \int_{\mathbb{R}^d}\overline{\psi(y)}(F_W\psi)(y)dy
\end{aligned}$$

To see what Wigner transform has done, we have the example for harmonic oscillator. ☐

Example 3.10 (Harmonic Oscillator)**.** *For a harmonic operator, the Hamiltonian is given by*

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

The lowest three states, are respectively

$$\psi_0(x) = \frac{1}{\sqrt[4]{\pi a^2}}e^{-x^2/(2a^2)} \quad \psi_1(x) = \frac{\sqrt{2}}{\sqrt[4]{\pi a^2}}\frac{x}{a}e^{-x^2/(2a^2)}$$

and

$$\psi_2(x) = \frac{1}{\sqrt[4]{\pi a^2}}\frac{2}{\sqrt{3}}((x/a)^2-1)e^{-x^2/(2a^2)}.$$

where $a = \epsilon/(m\omega)$. The Wigner functions corresponding to them are respectively

$$W_0(x,p) = 2\exp(-a^2p^2/\epsilon^2 - x^2/a^2) \quad W_1(x,p) = 2(-1+2(ap/\epsilon)^2+2(x/a)^2)\exp(-a^2p^2/\epsilon^2-x^2/a^2)$$

as well as

$$W_2(x,p) = \frac{2}{3}(1-12(x/a)^2+4(x/a)^4+4(ap/\epsilon)^4-4(ap/\epsilon)^2+8(px/\epsilon)^2)\exp(-a^2p^2/\epsilon^2-x^2/a^2)$$

So the graph for the Wigner transforms are respectively shown in the next page. As we can see, the Wigner function can be negative. But the one with all positive value can also display quantum phenomena, and the one for state 2, with $a = \epsilon = 1$*:*

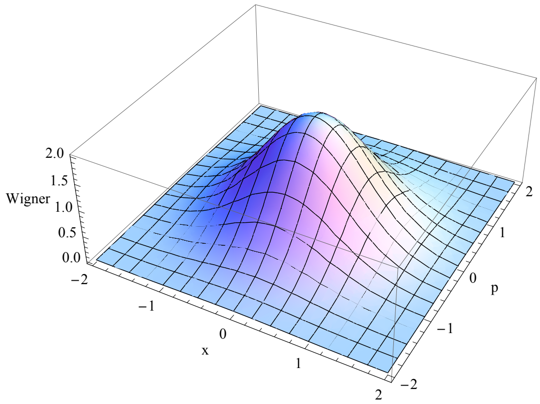

Figure 1: Plot of Wigner function for state 0

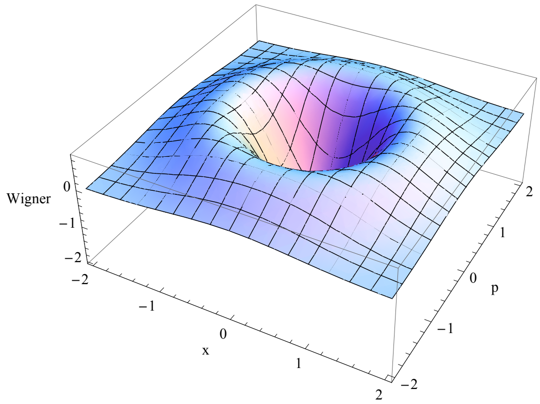

Figure 2: State 1

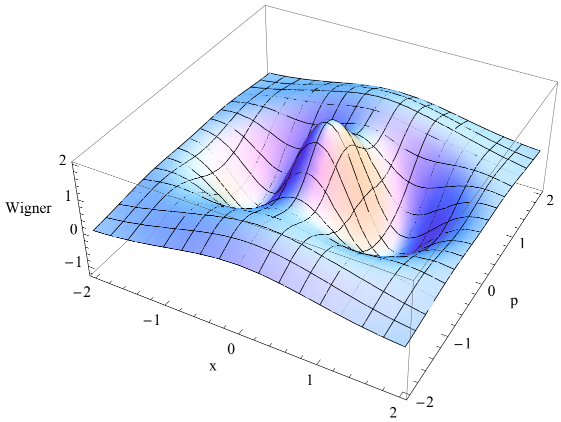

Figure 3: State 2

In general $W[f] \geq 0$ if and only if $f(x) = 0$ or $f(x)$ is a Gaussian function, as we can see in the first case. Finally, we gave a remark for the Wigner functions. There's a kind of function called the *Husimi* function, which, is the convolution of Wigner function and the Gaussian. When the Gaussian has large enough variance, the Husimi function will have positive value a.e.

3.5 Moyal product for smooth functions

We have already obtained the Moyal product for polynomials, yet we haven't said anything about the Moyal product for arbitrary function F and G. The mathematical formulation for analytic function is similar, since we have already got the Moyal product for formal series, and we can just expand to get the desired result. But for Moyal product of smooth but not analytic functions, we must change a method to tackle. The way we used is to "test" the function on a wave, called the "Gaussian wave packet" defined below.

Definition 3.11. *The Gaussian wave packets are the function*

$$g_{x_0,p_0}(x) = e^{-(x-x_0)^2/\epsilon} e^{ipx/\epsilon}$$

It seems that the series expansion around (p_0, x_0) of Weyl symbol F is a good approximation of operator acting on the packet. We state the theorem without proof below.

Theorem 3.12 (Testing principle). *For and $k \geq 0$, we have*

$$F(\hat{x},\hat{p})_W g_{x_0,p_0} = \sum_{i+j\leq k} \frac{\partial_x^i \partial_p^j F(x_0,p_0)}{i!j!}((\hat{x}-x_0)^i(\hat{p}-p_0)^j)_W g_{x_0,p_0} + O_{k,L^2}(\epsilon^{k-O(1)})$$

where $O_{k,L^2}(\epsilon^{k-O(1)})$ is a quantity with L^2 norm about $O_k(\epsilon^{k-O(1)})$.

The proof for this theorem used the symmetry principle for the Weyl quantization:

Theorem 3.13 (Symmetry). *Let the function $F_{x_0,p_0}(x,p) = F(x-x_0, p-p_0)$. Then*

$$F_{x_0,p_0}(\hat{x},\hat{p})_W = \mu_{p_0}\tau_{x_0} F(\hat{x},\hat{p})_W \tau_{x0}^{-1}\mu_{p_0}^{-1}.$$

Where $\tau_{x_0} f = f(x-x_0)$ and $\mu_{p_0} f = e^{ip_0 x/\epsilon} f$.

With this principle, we may move (x_0, p_0) to $(0,0)$, and consider the Taylor expansion at $(0,0)$. We may show that by subtracting the terms with order less than k, we may get error term with the order shown.

By approximation by polynomials, we may see that the Moyal product for smooth function also satisfies the similar equality

$$F \star_\epsilon G = \sum_{j=0}^{k} \frac{1}{j!}(\frac{ih}{2})^j \omega^{\otimes j}(\nabla^j F, \nabla^j G) + \epsilon^{k-O(1)} E_{k,\epsilon}.$$

The details we refer to [4].

We may now establish the relation between the Wigner transform and the Moyal product. Since Wigner transform is the symbol of $P_\psi : \phi \mapsto \psi\langle\phi,\psi\rangle$. So we have the relation

$$F(\hat{x},\hat{p})_W P_\psi \bar{F}(\hat{x},\hat{p})_W = P_{F(\hat{x},\hat{p})_W f}$$

Hence we may see that

$$W[F(\hat{x},\hat{p})_W f] = F \star_\epsilon W[f] \star_e \bar{F}.$$

4 WKB Analysis with Wigner transform

4.1 Motivation for WKB Method

What is the motivation for WKB Method? I think it aims to describe the high frequency part, which is usually bizarre for many equations. With the WKB analysis, we may able to state the description with mathematical rigor. We may first consider the simple ODE example[8]

$$\epsilon \frac{d^2 y}{dx^2} + y = 0, \quad 0 < \epsilon \ll 1$$

The solution for the differential equation is exactly $y(x) = \exp(\pm \frac{ix}{\epsilon})$. With simple calculus, we ay find that $y' = \pm i\epsilon \exp(\pm \frac{ix}{\epsilon})$ and $y'' = -\epsilon \exp(\pm \frac{ix}{\epsilon})$. It seems that the oscillation is at a scale of about ϵ. This is the key motivation for the WKB approximation, since we will see later that the function are approximated by high frequency function that leaves lower order of ϵ. To see the power of this method, we may look at an example:

Example 4.1 (Approximation for general case)**.** *We consider the function*

$$\epsilon^2 \frac{dy^2}{dx^2} + F(x)y = 0, \quad 0 < \epsilon \ll 1$$

We assume the solution has the form

$$y(x) = A(x, \epsilon) \exp(\frac{iu(x)}{\epsilon}).$$

Then by taking derivative, we know

$$y'' = -\frac{(u')^2}{\epsilon^2} A + \frac{i}{\epsilon}(Au'' + 2A'u') + A''$$

Then we insert into the original equation, and approximate $A(x, \epsilon)$ by $A(x, \epsilon) = A_0(x) + \epsilon A_1(x) + \cdots$. At order $O(1)$, $F(x) = (u')^2$. At order $O(\epsilon)$, we have $A_0 u'' + 2A_0 u' = 0$, thus $A_0 = C(F(x))^{-1/4}$. Finally, we get

$$y(x) \sim (F(x))^{-1/4} \exp(\pm \frac{i}{\epsilon} \int_{x_0}^{x} \sqrt{F(s)} ds)$$

One of the most used application of WKB method occur in Schrödinger equation, which we will discuss later on.

4.2 WKB analysis on generalized linear dispersive model

We may like to solve a system more general than the Schrödinger equation, so a generalized linear dispersive model is then involved. This model is a initial value problem for ψDO(pseudo differential operator). We may give the clear definition now:

Definition 4.2. *A generalized linear dispersive model for an anti-adjoint scalar pseudo-differential operator is the equation*

$$\epsilon \partial_t \psi^\epsilon + iH(\hat{x}, \hat{p})_W \psi^\epsilon = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$

with the WKB initial data

$$\psi^\epsilon = \sqrt{n_I(x)} e^{iS_I(x)/\epsilon}, \quad x \in \mathbb{R}^d$$

where H_W is the Weyl quantization of a function $H(x, p)$. $n_I \geq 0$1.e. and $S_I \in W^{1,1}_{loc}$.

Also, we may denote $n^\epsilon(x, t) = \|\psi(x, t)\|^2$ to be the energy. On some conditions the energy is conservative. Here we gave some examples of the generalized linear dispersive model:

Example 4.3. *1. The Schrödinger equation*

$$\epsilon\frac{\partial\psi^\epsilon}{\partial t} - i\frac{\epsilon^2}{2}\nabla\psi^\epsilon + iV(x)\psi^\epsilon = 0$$

Here we know that $H(x,p) = |p^2|/2 + V(x)$.

2. 1-d Airy equation

$$\epsilon\frac{\partial\psi^\epsilon}{\partial t} - \frac{\epsilon^3}{3}\psi^\epsilon_{xxx} = 0$$

Here $H(x,p) = p^3/3$.

3. Spinless Bethe-Salpeter equation

$$\epsilon\frac{\partial\psi^\epsilon}{\partial t} - i\left(\sqrt{-\frac{\epsilon^2}{2}\delta + 1} + V(x)\right)\psi^\epsilon = 0.$$

These examples are all hard to solve by exact form, so we refer to the WKB method. Namely, we may assume

$$\psi^\epsilon(x,t) = A^\epsilon(x,t)\exp(\frac{i}{\epsilon}S(x,t)), \quad x\in\mathbb{R}^d, t\in\mathbb{R}$$

with the similar approximation

$$A^\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots .$$

We consider the case of a polynomial Weyl symbol with C^∞ coefficients.

$$H(\hat{x},\hat{p})_W\phi(x) = \sum_{|k|=0}^{m}\epsilon^{|k|}D_y^k(a_k(\frac{x+y}{2})\phi(y))|_{y=x}$$

. As a result, we may get the following approximation

$$H(\hat{x},\hat{p})_W(A^\epsilon e^{iS/\epsilon}) = e^{iS/\epsilon}\sum_{|k|=0}^{m}(i\epsilon)^{|k|}R_k[A^\epsilon]$$

. Where

$$R_0[A^\epsilon] = H(x,\nabla_x S)A^\epsilon,$$

$$R_1[A^\epsilon] = \sum_{j=1}^{d}\frac{\partial H(x,\nabla_x S)}{\partial p_j}\frac{\partial A^\epsilon}{\partial x_j} + \frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 S}{\partial x_j\partial x_k}\frac{\partial^2 H(x,\nabla_x S)}{\partial p_j\partial p_k}A^\epsilon + \frac{1}{2}\sum_{k=1}^{d}\frac{\partial^2 H(x,\nabla_x S)}{\partial y_k\partial p_j}A^\epsilon$$

So by comparing coefficient, we may get the equation of the following:

$$\begin{aligned}\partial_t n + \nabla\cdot(n\nabla_p H(x,\nabla_x S)) = 0, \quad x\in\mathbb{R}^d, t\in\mathbb{R}\\ \partial_t S + H(x,\nabla_x S) = 0\end{aligned}$$

where $n = A^2$, with initial data $n(x,0) = n_I, S(x,0) = S_I$. Here are some examples of the equations we listed before.

Example 4.4. *1. Schrödinger equation*

$$\begin{aligned}\partial_t n + \nabla\cdot(n\nabla_x S) = 0, \quad x\in\mathbb{R}^d, t\in\mathbb{R}\\ \partial_t S + \frac{|\nabla_x S|^2}{2} + V(x) = 0\end{aligned}$$

2. 1-d Airy Equation

$$\partial_t n + \nabla \cdot (n(\nabla_x S)^2) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$
$$\partial_t S + \frac{(\nabla_x S)^3}{3} = 0$$

3. Spinless Bethe-salpeter equation

$$\partial_t n + \nabla \cdot (\frac{n\nabla_x S}{\sqrt{\frac{|\nabla_x S|^2}{2} + 1}}) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$
$$\partial_t S + \sqrt{\frac{(\nabla_x S)^2}{2} + 1} + V(x) = 0$$

The useful idea from WKB analysis is that, if the smooth solution exists locally, then the WKB approximation will have minor difference with the exact solution locally, with error term the order $O(\epsilon)$. The theorem states as below

Theorem 4.5 (Error theorem)**.** *Let ψ^ϵ solves the equation before, and H_W is the form as before. $n_I \in \mathcal{S}(\mathbb{R}^d)$ and S_I is bounded, with all the derivatives up to sufficient order. Let $t_{c_1} < 0 < t_{c_2}$ be such that smooth solution of (n, S) exists on $\mathbb{R}^d \times (t_{C_1}, t_{C_2})$, and define*

$$\psi^\epsilon_{wkb}(x,t) = \sqrt{n(x,t)}\exp(\frac{i}{\epsilon}S(x,t))$$

Then for any finite time-interval $[t_1, t_2] \subset (t_{c_1}, t_{c_2})$, we have

$$\sup_{t_1<t<t_2} \|\psi^\epsilon(t) - \psi^\epsilon_{wkb}(t)\|_2 \le O(\epsilon).$$

But the problem with WKB analysis is that there's only unique solution locally. Also, the phase is discontinuous at caustics, namely, the points of intersections of rays(the integral curve given by method of characteristic). So the global asymptotic description is not available by using only WKB method.

4.3 Wigner Transform as a recipe for WKB Analysis

First, we may "Wignerized" the Schrödinger equation to solve the Wigner function, that is,

$$\partial_t w^\epsilon + p \cdot \nabla_x w^\epsilon - \theta^\epsilon[V]w^\epsilon = 0.$$

There $\theta^\epsilon[V]$ is the ψDO

$$\theta^\epsilon[V]W^\epsilon(x,\xi,t) := \frac{i}{(2\pi)^d\epsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[V(x+\epsilon\frac{z}{2}) - V(x-\epsilon\frac{z}{2})] \cdot w^\epsilon(x,p,t)e^{iz(p-p')}dxdp.$$

When $V(x) = 0$, the equation becomes the free transport equation

$$\frac{\partial w^\epsilon}{\partial t} + p \cdot \nabla_x w^\epsilon = 0$$

As a typical example. the harmonic oscillator, where $V(x) = |x^2|/2$, is

$$\frac{\partial w^\epsilon}{\partial t} + p \cdot \nabla_x w^\epsilon - x \cdot \nabla_p w^\epsilon = 0$$

So with the help of Wigner function, we may proceed to get the so called "classical limit". That is

Theorem 4.6. *If f lines in bounded subset of $L^2(\mathbb{R}^d)$, then $w^\epsilon[f]$ is bounded as $\epsilon \to 0$. Moreover*

Thus by sequential compactness there's a subsequence and a distribution function on $w^0 \in S'(\mathbb{R}^d \times \mathbb{R}^d)$ such that $w^{\epsilon_k}[f^{e_k}] \to w^0$. It can also be shown that Husimi function w^ϵ_H are limit point of Wigner function. And then w^0 is non negative, and thus define a positive Borel measure. This is called the classical limit for w^ϵ. With the aid of this theorem ,we may now proceed to a limit for previous calculation, that

$$\langle \psi^\epsilon, F(\hat{x},\hat{p})_W \psi^\epsilon \rangle \overset{\epsilon\to 0}{\to} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,p) w(t,x,p)\,dxdp.$$

But what's the relation with the WKB analysis? Well, we may now proceed to look at $\partial_t w^\epsilon$. As we may find in the definition of Wigner transform, that the equation becomes the classical *Liouville equation*

$$\partial_t w = -\{H, w\}$$

as $\epsilon \to 0$. Thus we may compute the equation now, that is $\psi_I = \sqrt{n_I} e^{iS_I/\epsilon}$. Then

$$W^\epsilon[\psi^\epsilon_I](x,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\sqrt{n_I(x-\epsilon\frac{z}{2})n_I(x+\epsilon\frac{z}{2})}\exp[\frac{i}{\epsilon}(S_I(x-\epsilon\frac{z}{2})-S_I(x+\epsilon\frac{z}{2}))+izp]dz.$$

Then we have

$$w^\epsilon[\psi^\epsilon_I] \overset{\epsilon\to 0}{\to} w_I = n_I(x)\delta(p - \nabla_x S_I(x))$$

And the density

$$n^\epsilon(x,t) \overset{\epsilon\to 0}{n}(x,t) = \int_{\mathbb{R}^d} w(x,dp,t)$$

All the proof can be found in [7]

5 A taste of geometric quantization

Since the time limitation we cannot state much in this topic. All the theorems are stated without proof, and we may get more result in the WKB analysis on manifold.

5.1 Motivations and definitions

The former analysis and PDE are all defined and displayed on $\mathbb{R}^d$(as the configuration space, usually denoted by $(q_1,\cdots,q_d)$) and $T^*\mathbb{R}^d \cong \mathbb{R}^d$(the cotangent bundle, which is $p_1,\cdots,p_d$). But how about the case when the configuration space is not a Euclidean space, but a manifold? How about the quantization now? The equivalent formulation of this problem is that, if there is a "classical theory" on this manifold, how can we relate a "quantum theory"(i.e. associate a Hilbert space to function on phase space) on this manifold? To answer these questions, we must look at a kind of interesting manifold, called the symplectic manifold(sometimes a more general kind of manifold, namely, the Poisson manifold is used, but we focus on symplectic case now, since it's more elegant.). These manifold admit a structure of "classical theory".

What are q and p for classical theory now? Since we know the definition of a manifold, which is a structure that locally "looks like" $\mathbb{R}^d$. Thus the coordinate for the manifold N is denoted by $(q_1, q_2,\cdots,q_d)$. When there is a neighborhood of x,denoted by U, that the tangent bundle of U is locally $TU \cong U \times \mathbb{R}^d$. Thus we denote the coordinate system for this tangent bundle $(q_1, \cdots, q_d)$. Note that $q_i \in T^*U$. This is the analogous definition for momentum.

We next look at the definition for symplectic manifold, which is more general than $M = T^*N$:

Definition 5.1 (Symplectic manifold). *(X, ω) is a symplectic manifold if*

1. *X is a manifold.*
2. *ω is a closed $2-$form.*

What is the ω now? Actually, it represents the symplectic form ω we have mentioned earlier. Let look at the 1-form $\theta = \sum p_k dx_k$ in $M = T^*N$. Then $\omega = \sum dp_k \wedge dx_k$ is a symplectic form. Then this induces an element in the tangent bundle. That is , $c = \sum \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial x_k}$. Representing them in local coordinate(in manifold M), we have

$$\omega = \sum_{i<j} \omega_{ij} x_i \wedge x_j \quad c = \sum_{i<j} c_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Thus for any function defined on $C^\infty(M)$(we may now call it physical observables!), we have the Poisson bracket

$$\{F, G\} = c_{ij} \partial_i F \partial_j G$$

We omit the summation by the Einstein notation. Then we have the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$$

because this is equivalent to $d\omega = 0$ and the Schouten bracket

$$[c, c]^{ijk} = \sum_{\text{cyc } ijk} \sum_m c^{im} \partial_m c^{jk}.$$

The former summation is over the cyclic permutation of i, j, k. The Lagrangian submanifold N of M is a submanifold of dimension d such that ω vanishes on TN.

The classical(Hamiltonian) picture of the manifold is the function H such that the states does not depend on time, and for any functions F on $M \times \mathbb{R}$ we have

$$\dot{F} = \{H, F\}.$$

F is an arbitrary observable, and we apply to p_k and q_k gives

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \dot{p_k} = -\frac{\partial H}{\partial q_k}.$$

As we usually see, the Hamiltonian is of the usual sense. Let's see an example to illustrate this idea.

Example 5.2. *$N = S^2$ given by the $x_1^2 + x_2^2 + x_3^2 = r^2$, M is the cotangent bundle of N. So we may identify M with the subset of $\mathbb{R}^6$ such that*

$$\hat{x}_1^2 + x_2^2 + x_3^2 = r^2, \quad x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

We may also take $q_1 = x_1$ and $q_2 = x_2$. The quadratic form $T = (m/2)(y_1^2 + y_2^2 + y_3^2)$ gives the kinetic energy of system. We also have the form $p_1 = m(y_1 x_3 - x_1 y_3)/x_3$ and $p_2 = m(y_2 x_3 - x_2 y_3)/x_3$. In coordinates we have the formula

$$T = \frac{1}{2m}(p_1^2 + p_2^2 - \frac{(p_1 q_1 + p_2 q_2)^2}{r^2})$$

The potential energy $V = -mgx_3 = \pm mg\sqrt{r^2 - q_1^2 - q_2^2}$. So the Hamiltonian is $H = T + V$.

For M that is not any tangent bundle of manifold, we may also see it as the phase space! In general, for a symplectic manifold with some particular property, it may be quantized properly. We will see this in the next subsection.

5.2 Koopman-VanHove-Segal Representation and Souriau-Konstant Prequantization

Definition 5.3 (Geometric Quantization)**.** *Let (M, ω) be a symplectic manifold, and $\mathcal{P}(M, \omega)$ is the Poisson algebra on M(that is $C^\infty(M)$ with Poisson bracket $\{\cdot\}$). Then a quantization is mapping from the Poisson algebra to operator on a Hilbert(or pre-Hilbert) space that satisfies the following assertion*

1. $Op(1) = 1$
2. $Op(\{F, G\}) = [F, G]_\epsilon = \frac{i}{\epsilon}(Op(F)Op(G) - Op(g)Op(F))$
3. $Op(F^*) = (Op(F))^*$
4. *For complete $F_1, F_2, \cdots, F_m$ the $Op(F_1), \cdots, Op(F_n)$ s complete.*

A prequantization is a mapping that satisfies the first three conditions.

As we may note here, the Op is different from the strong quantization we said earlier. For the case $M = T^*N$ and $\omega = d\theta$, a quantization exists, which is called the Koopman-VanHove-Segal representation.

Theorem 5.4. *The associated operator has the form*

$$Op(F) = F + \frac{\epsilon}{i}c(F) - \theta(c(F))$$

where $c(F)$ is the Hamiltonian vector field with generating function F, with the local representation $c(F) = \omega_{ij}\partial_i f \partial_j$.

All these conditions are fufiled. Since for 2, we have $[C(F), c(G)] = c(\{F, G\})$ an.If $c(F)$ is complete, then $Op(F)$ should be self adjoint. But it is not a quantization because in $M = T^*\mathbb{R}$, $Op(q) = q + \frac{\epsilon}{i}\frac{\partial}{\partial p}, Op(p) = -\frac{\epsilon}{i}\frac{\partial}{\partial q}$. This is not a complete set.

The point is that ω should be exact for the KVS formula. So by Poincare's lemma, closed forms are always locally exact. If it's possible to do the gluing job, we are able to get the global version of the quantization. The theorem of Souriau-Konstant Prequantization follows.

If the complex line bundle over domain $U_\alpha \subset M$ has a non vanishing section s_α, then thespace of sections can be identified with space of $C^\infty(U_\alpha)$ by bijection $\phi \leftrightarrow \phi \cdot s_\alpha$. Under this representation, we have

$$\nabla_\xi \phi = \xi\phi - \frac{i}{\epsilon}\theta_\alpha(\xi)\phi$$

where θ is defined via equation

$$\nabla_\xi s_\alpha = -\frac{i}{\epsilon}\theta_\alpha(\xi) \cdot s_\alpha.$$

So we may have the definition comparing the previous KVS formula that

$$Op(F) = F + \frac{\epsilon}{i}\nabla_{c(F)}.$$

Condition 1 and 3 are justified, but when we refer to condition 2,we have the curvature form

$$\Omega(\xi, \eta) = \frac{1}{2\pi i}([\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]})$$

So $\{F, G\} = h\Omega(c(F), c(G))$ holds. Thus the following condition should be satisfied.

Theorem 5.5 (Souriau-Konstant formula condition)**.** *Souriau-Konstant formula is a prequantization if and only if curvature form Ω coincides with $h^{-1}\omega$.*

This prequantization formulation gives us much light on the operators on Hilbert space of $(C^\infty(M))$ now.

Remark 5.6 (Polarization)**.** *The Hilbert space associated to prequantization is too big to handle, so we should cut down the dimension a bit by using a technique called polarization. This is always found in the Khäler manifold. To see the detail, see[6]*

6 Conclusion and further extensions

In this article, we have set up the Weyl calculus, and apply it to the applications of Wigner transforms. Also, we investigated the WKB analysis that occur in computation of PDE in quantum mechanics. Finally, we indulge into a more general form of the quantization, called the geometric quantization. There's a bit more to be explored, but due to limitation of time, our journey stops here. We can proceed to resolve the Schrödinger equation on the geometric quantization framework. WKB analysis can also be analogously introduced.

With more consideration, we can even generalize the similar fashion to graph theory! In fact, graphs have many similarity with the manifolds. We can define the differential form, the Laplacian, the derivative on the graph. A generalization for a "symplectic graph"(a discrete version of symplectic manifold) maybe a bit interesting for us to continue our research.

Another generalization I'm thinking about apart from change of configuration or phase space is that if the quantization can be extended to the non linear operators of various kinds. Since the nonlinear PDEs are the major interest in the field of PDE. Such equations as KdV equations or Navier-Stokes are important for physicists. So is there a scheme to associate a function (or maybe a more general thing, a distribution or something else) to a nonlinear operator? Or more weakly, can we also find a phase version of other nonlinear equations (similar to Wigner transform) that can give us asymptotic form of the solution? This is not a rigorous argument, but a wild thought. Hopefully it may exist.

7 Acknowledgement

This article is a report for the topic course IV taught by professor Jin. The article is an extension of the second and third chapter of article [1] plus a bunch of details and extensions. The author specially thanks professor Jin for his excellent talk in the Topic course IV. Without his introduction, the author would never know this beautiful approach in mathematical physics.

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