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The Geometry of Ample Line Bundle

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丰富线丛的几何学

摘　　要

本论文主要研究的是凯勒流形上各种正与半正线丛的消减定理。最重要的定理即小平消减定理，它说的是在 $q > 0$ 且 L 丰富时，$H^q(X, K_X \otimes L) = 0$。而这个定理有几个推广方向。

第一个重要的推广是小平 -中野 -秋月消减定理，这个定理将原来的典范线丛变为微分形式的丛。第二个重要的推广则是川又 -Viehweg 消减定理，它将小平消减的条件弱化为大而数字有效的线丛，而之后的推广则是结合这两种的特点，本文将在这方面进行详细的讨论。

与此同时，代数消减定理的发现给我们带来新的研究方向，我们可以将代数的论述转换为几何的论述，并利用几何方法证明代数定理，我们将讨论几何证明的塞尔消减定理以及 Grauert-Riemenschneider 消减定理。最后，我们将讨论在高维的正丛，埃尔米特流形或者超凯勒流形上的消减定理。

本文中使用了大量分析方法，主要的是对于曲率正定性刻画以及微分算子的计算。

关键词： 丰富线丛, 消减定理, 拟正线丛, 数字有效, 小平消减定理

The Geometry of Ample Line Bundle

ABSTRACT

This thesis is mainly about the vanishing theorem correponding to positive and semipositive line bundles on Kähler manifold. The most important one is Kodaira vanishing theorem, which says that when $q > 0$ and L ample, the cohomology group $H^q(X, K_X \otimes L) = 0$. There're several ways to generalize this theorem.

The first one is Kodaira-Nakano-Akizuki vanishing theorem. This theorem replace the canonical bundle by bundle of differential forms. The second one is Kawamata-Viehweg vanishing theorem. It weaken the condition to big and nef bundle. Other generalizations combine the two ones. We will discuss throughly on this subject.

Meanwhile, algbraic vanishing theorem brings new way of research. We can translate algebraic claim to geometric argument and prove the algebraic vanishing theorems. We will prove by geometry Serre's vanishing theorem and Grauert-Riemenschneider vanishing theorem. At last, we will consider vanishing theorem on higher rank bundle, Hermitian manifold and hyper-Kähler manifold.

To fulfill our purpose, analytic tools are used. The most profound one in this aritcle is the characterization of curvature and operators on Kähler manifold.

KEY WORDS: Ample line bundle, Kodaira Vanishing Theorem, Semipositive line bundle, Nef bundle

目　录

第一章 Introduction

Cohomology vanishing theorem for holomorphic vector bundle on compact Kähler manifold occupies an important role in the theory of several complex variable and complex geometry. These analytic methods can be used to study algebraic objects like linear series and projective schemes. The most celebrated theorem in this subject is the Kodaira vanishing theorem. This is the most fundamental, but also the most tempting theorem in modern algebraic geometry. The statement of this theorem is a vanishing of cohomology group

$$H^q(X, K_X \otimes L) = 0, \quad \forall q > 0.$$

The proof of this theorem involves transcendental technique in the computation of cohomology group. This proof is done by Hodge theory in complex geometry. Hodge theory converts the hard computation of cohomology group of bundles to relative easier computation of harmonic differential (p, q) forms. Also, some calculations of Lefschetz and dual Lefschetz operators in Kähler manifold are required to obtain the vanishing of harmonic forms. The most direct generalization of Kodaira's vanishing theorem comes from the process of finding the appropriate computation result. This generalization is called Kodaira-Nakano-Akizuki vanishing theorem, which generalize the tensor product of canonical line bundle to the differential forms. And the power of this proof could be extended to the so called L^2 vanishing theorem. This theorem involves the Bochner technique and gave a sufficient condition for vanishing of cohomology group.

There're various applications of the Kodaira vanishing theorem. The most profound one is the application in algebraic geometry, where the Kähler manifold is replaced by smooth projective k-scheme. Here k is a filed of characteristic 0. The tool for this geometry to algebra transition is the Serre's GAGA principle and Lefschetz principle. The actual route for this transition is

$$\{X \text{ compact Kähler manifold}, L \text{ positive}\} \overset{\text{GAGA}}{\Leftrightarrow} \{X \text{ smooth projective } \mathbb{C}\text{-scheme}, L \text{ ample}\}$$
$$\overset{\text{Lefschetz}}{\Leftrightarrow} \{X \text{ smooth projective } k\text{-scheme}, L \text{ ample}\}$$

Classically, people are interested in the going right part, which states a purely algebraic theorem. But the bad thing is, the nature of this proof is transcendental. Thus Deligne, Illusie and Raynaud gave a purely algebraic proof of Kodaira vanishing theorem by the degeneration of Hodge to de Rham spectral sequence. Their proof are so elegant that it becomes the foundation for the research of algebraic side of Hodge theory.

In this article, we will consider the converse part. The algebraic theorems could also be translated into geometric flavor. These algebraic notations include the ampleness, numerical effectiveness(nef), globally generation, bigness, and semiampleness. These notions comes from the study of sheaf theory, but finally can be restated in geometric term. More precisely, these notions could be formed by the observation of curvature associated to Chern connection. These descriptions of algebraic notions could be used to reprove the algebraic vanishing theorems by analysis.

A first translation is done in Serre's vanishing theorem, which states that for any vector bundle, the tensor product of it and a "sufficiently positive" line bundle should have vanishing cohomology group. This theorem was proved by Serre in his fundamental paper "Faisceaux algébriques cohérents" on the coherent algebraic sheaf. His proof consisted of purely homological algebraic tools. But we could reprove it using Kodaira vanishing and the notion of positivity.

The second translation concerned the Kawamata-Viehweg vanishing theorem. This theorem was developed independently by Kawamata and Viehweg in 1982. It generalize Kodaira's vanishing theorem in that it uses big and nef line bundle instead of ample line bundle. This generalization is highly nontrivial since we modify the positivity in the previous proof to semipositivity. The previous geometric proof could not be used since we cannot derive vanishing of harmonic form purely with previous technique. An aid for this dilemma is to prove Nadel's vanishing theorem about principle ideal sheaf and modify this ideal sheaf to algebraic notions. But a geometric lemma could be used to prove the Kawamata-Viehweg vanishing theorem directly. A side product of this vanishing theorem is the Bogomolov vanishing theorem, which generalize Kodaira-Nakano-Akizuki vanishing to semipositivity.

The third translation is about the Grauert-Riemenschneider vanishing theorem. This theorem says that the higher direct image vanishes when our map is finite generic. This kind of theorem is called generic vanishing theorem. We could use the Kawamata-Viehweg vanishing theorem to the group of cohomology about the higher image sheaf and apply the Leray spectral sequence. Then the Grauert-Riemenschneider vanishing theorem boils down to the proof of semipositivity of curvature tensor.

Apart from these translations, Kodaira vanishing theorem has its higher rank counterpart. The search for a canonical generalization to higher rank is difficult. Currently, two kinds of positivity of vector bundle are in hand. One is called Griffiths positivity, and the other is called Hartshorne ampleness. Griffiths positivity generalize the positivity from geometric part, which use the positivity of curvature tensor. Hartshorne ampleness deals with the algebraic part, which involves the projectivation of vector bundles. The fundamental problem in this theory is called Griffiths' conjecture, which asked if these two sides are equivalent.

Another generalization is the translation of theorem from Kähler to Hermitian manifold. We are deprived of the Kähler structure on manifold, thus the vanishing cohomology is only true for some particular dimension. Also, this generalization could be stronger to be the hyper-Kähler manifold, which says that the vanishing theorem is true for even lower dimension. But since the limitation of time, we are unable to provide much interesting result in the higher rank, hermitian and hyper-Kähler case.

The organization of this thesis is as follows. Preliminaries of complex geometry are described and defined in Chapter 2. We introduce the complex manifold in 2.1, and the cohomology group of sheaves in 2.2. In section 2.3, we introduce our main character, the curvature tensor or in other word, the first Chern class. In section 2.4 we proved the Hodge theorem in real Riemannian manifold. In Chapter 3, we are mainly interested in the vanishing theorems in Kähler manifold. In section 3.1 we provide the formula and identities for canonical operators on compact Kähler manifold. We then states and proved the L^2 vanishing theorem and its relative like Girbau vanishing theorem. We also used our geometric proof to relate Serre's vanishing theorem to curvature. And then we use the similar technique to prove the Andreotti-Grauert vanishing theorem. Section 3.3 is related to the semipositivity in vanishing theorem. The main result in this section is the proof of Kawamata-Viehweg vanishing theorem, and we used similar technique to prove Koll'ar injectivity theorem and Bogomolov-Sommese vanishing. An application is the Grauert-Riemenschneider vanishing theorem. Chapter 4 is devoted to several generalizations. Due to time limitation, we are unable to present the most exciting progress in this chapter. We only state without proof of some main results. Finally algebraic vanishing theorem are provided to get a flavor.

第二章 Complex Geometry

2.1 Complex Manifold

Complex manifold is an analog of ordinary complex plane $\mathbb{C}$. It is made up of several complex open sets glued together by holomorphic functions. In this section, we will introduce the basic notions of complex manifold, assuming knowledge of several variable holomorphic functions. Some little preliminary on differentiable manifold is required.

Definition 2.1.1 (Complex manifold)**.** A $n-$dimensional *complex manifold* is a topological space admitting an open cover $\{U_i\}$ and maps $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ such that the map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$.

There are several notations needed to be stated. First, we must define the holomorphic functions on complex manifold.

Definition 2.1.2 (Holomorphic function)**.** A *holomorphic function* on a complex manifold X is a function $f : X \to \mathbb{C}$ such that $f \circ \varphi_\alpha$ is holomorphic on $\varphi_\alpha(U_\alpha)$.

With manifold, we may define the tangent space, which are intrinsic property that determine the topology of manifold. Tangent spaces define the directional derivative of functions on manifold. We let

Definition 2.1.3 (Tangent Bundle)**.** Let X be an $n-$dimensional complex manifold. Locally, we have trivialization $(z_1, z_2, \ldots, z_n)$. A *holomorphic tangent space* T_zX

defined at z is the vector space generated by $\frac{\partial}{\partial z_i}$. Similarly, we may define *anti-holomorphic tangent space*. The transition map thus defines the transition map of tangent space by obvious Lebniz law. That is, we identify the two sets $U_i \cap U_j \times \mathbb{C}^n \subset U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n \subset U_j \times \mathbb{C}^n$ by the map

$$(u, v) \mapsto (u, \phi_{ij*}(v)).$$

Here ϕ_{ij*} is a Jacobian matrix with coefficient $\frac{\partial \phi_{ij*}(u)}{\partial z_l}$, and $\phi_{ij} = \phi_j \circ \phi_i^{-1}$.

The tangent bundle is actually a special case of vector bundle. Vector bundle associate a vector space to every point on manifold. But this kind of association is continuous with respect to the differentiable structures. But in the beginning, we will not consider the vector bundle. We delay the discussion to later sections.

Complex geometry has more characteristics than real geometry, and the most important one is the multiplication by i. In the complex manifold settings, we consider this transformation by an "almost complex structure". That is

Definition 2.1.4 (Almost Complex Structure)**.** On the tangent bundle of a differentiable manifold X, an *almost complex structure* is an endomorphism I of TX such that $I^2 = -1$.

A complex manifold surely has a almost complex structure, as the endomorphism I is the multiplication of i in $\mathbb{C}^n$. An interesting problem was aroused: When a differentiable manifold with almost complex structure can actually be realized by a complex manifold? This problem was solved by Newlander and Nirenberg, which is called Newlander-Nirenberg theorem. But we will not discuss it here.

We will also introduce the notion of vector bundle. Vector bundles attach a vector space to the manifold.

Definition 2.1.5 (Vector bundle)**.** A *complex vector bundle* of rank m over a complex manifold X is a topological space E, a mapping $\pi : E \to X$ and an open covering $\{U_i\}$ of X such that we have the two conditions

1. There is a trivialization map $\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^m$.
2. Transition map $\tau_j \circ \tau_i^{-1}$ is $\mathbb{C}$-linear on $\pi^{-1}((U_i \cap U_j) \times \mathbb{C}^m)$.

As we can see that the tangent bundle is a special kind of vector bundle. Moreover, we can also define the *holomorphic vector bundle*. That is, the transition function τ_i are biholomorphic maps. In addition, we can define several operations on vector bundles. These operations include direct sum, tensor product and dual. The operations of bundle are easy in the sense that we only consider the operation to their relative vector spaces.

The rest of this section is devoted to examples of complex manifolds. These manifolds are typical examples that could be found in complex geometry textbook. But we may give the calculation of the curvature and other invariants later.

Affine Space. The most simple example is the complex affine space $\mathbb{C}^n$. The most simple metric is

$$\omega = i \sum_j dz^j \wedge d\bar{z}^j$$

But as we will see later, we can endow different kind of metric on it.

Projective Space. This is the key object in our study of positivity. A projective space $\mathbb{P}^n$ is the collection of all lines passing the origin. We can see it as $(\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*$, by seeing $(z_0, \ldots, z_n)$ and $(\lambda z_0, \ldots, \lambda z_n)$ as the same point. We may use the canonical covering

$$U_i = \{(z_0, \ldots, z_n) | z_i \neq 0\}, i = 0, \ldots n$$

to cover the whole projective space. The chart map φ_i are simply

$$\varphi_i : U_i \to \mathbb{C}^n, (z_0, \dots, z_n) \mapsto \left(\frac{z_0}{z_i}, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i}\right),$$

here $\hat{x}$ means we remove this one. The transition map is obviously biholomorphic on $U_i \cap U_j$ for all i, j.

Projective hypersurface. Let f be a homogeneous polynomial with variables $z_0, z_1, \dots, z_n$. Here we consider the induced map of f from $\mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}$. Suppose that f is regular at 0, so that the variety

$$X = V(f) := \{(z_1, \dots, z_n) \in \mathbb{P}^n | f(z_1, \dots, z_n) = 0\}$$

is exactly a manifold. By implicit function theorem and let $V(f) \cap U_i$ be a covering, we see that the manifold is actually complex manifold.

Complex Tori. This manifold is a quotient $T = \mathbb{C}^n/\Gamma$, where Γ is a discrete group of $2n$. This is a Kähler manifold. The Kähler metric is

$$\omega = i \sum_{j,k} h_{jk} dz^j \wedge d\bar{z}^k$$

Hopf Manifold. This is the first manifold that is seen to be non-Kähler . We also take some quotient of space. We also have the space $\mathbb{C}^n \backslash \{0\}$, and this time the group $\Gamma = \{\gamma^n | n \in \mathbb{Z}\} \cong \mathbb{Z}$, where $\gamma \in (0, 1)$. The manifold $X = (\mathbb{C}^n \backslash \{0\})/\Gamma$ is called Hopf manifold. This manifold is diffeomorphic to $S^{2n-1} \times S^1$.

Kodaira Surface. As we have just mentioned the complex tori $T = \mathbb{C}^n/\Gamma$. The

Kodaira surface S is obtained by a similar manner by regarding $\mathbb{C}^2$ as the group

$$\begin{pmatrix} 1 & x & u & t \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Iwasawa Manifold. The Iwasawa manifold is quotient of complex Lie group $G = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in GL(3, \mathbb{C})$ by a lattice. We use the lattice $\Gamma = G \cap GL(3, \mathbb{Z} + i\mathbb{Z})$, acting by the multiplication. Thus the Iwasawa manifold $X = G/\Gamma$ is a complex manifold of dimension 3.

Grassmannian Manifold. Let V be complex vector space of dimension $n+1$, the Grassmannian manifold $\mathrm{Gr}_k(V)$ for $k \leq n+1$ is the set of $k-$dimensional subspace of V.

Blow-up Blow-up is a method to create a new manifold out of an old one. And blowing up is a useful technique to remove singularity. We will only mention the name in this section since the full definition require much prerequisite.

2.2 Cohomology and Sheaves

Cohomology is a great invariant that used to characterize the topological structure of manifolds. Cohomology, in short, is a group that is quotient of a kernel by an image. The most fundamental cohomology in topology is simplicial cohomology, where the group is map from simplicial complex to $\mathbb{Z}$ or $\mathbb{R}$. In differential manifold theory, the easiest cohomology is de Rham cohomology, which is the kernel of differential d

modulo the image of the similar operator. In this section, we will see the definition of cohomology and its relative in sheaf theory. In order to see the true definition, we must first define sheaves:

Definition 2.2.1 (Pre-sheaf and sheaf)**.** A *pre-sheaf* $\mathcal{F}$ on a topological space M is a method to assign Abelian group to open sets with a "restriction structure". Namely, for each U open, and $V \subset U$ open. We have a restriction morphism

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V).$$

And the restriction of restriction is the same, that is

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(W).$$

We will also denote $\rho_{UV}(\sigma)$ by $\sigma|_V$ for abbreviation. A *sheaf* is a presheaf such that the natural map

$$\prod_V \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is onto the set

$$\left\{(\sigma_V)_{V\in\mathcal{V}}|\sigma_{V|V\cap W} = \sigma_{W|V\cap W}, V, W \in \mathcal{V}.\right\}$$

Here $\mathcal{V}$ is an open covering of open set U.

The creation of sheaf leads to the systematic research of local behaviors and their relation with global reaction. A complex manifold X has two fundamental sheaves. One is the sheaf of holomorphic function on X, called the structure sheaf $\mathcal{F}_X$and the other is the wedge product of cotangent bundle K_X called the canonical sheaf. The properties of these two sheaves are useful in the study of topology, and the gluing of sheaves can also be seen as another way of defining manifold, or the more generalized object: scheme. But now, we will only consider the sheaves on complex manifold. We will introduce the category of sheaf to define its cohomology.

Definition 2.2.2 (Category)**.** A *category* $\mathcal{C}$ consists of objects(denoted by obj $\mathcal{C}$) and maps hom$(,)$. Satisfying the following restrictions.

1. The morphism(formal name of map)f has an origin A and target B. Here A, B are objects in $\mathcal{C}$. We say that the morphism f maps A to B. And all morphisms that maps A to B form a set called hom(A, B).

2. There is a combination map hom$(A, B) \times$ hom$(B, C) \rightarrow$ hom(A, C), such that the associativity holds true, that $f \circ (g \circ h) = (f \circ g) \circ h \in$ hom(A, D). $f \in$ hom$(A, B), g \in$ hom$(B, C), h \in$ hom(C, D).

3. The identity map exists and unique, that is $id_A \in$ hom(A, A) that all maps combine with this map is the same.

Examples of category include category of sets, groups, rings, topological space or even manifolds. In order to study the sheaf theory, we must introduce an additional property called the Abelian category.

Definition 2.2.3 (Abelian Category)**.** An *Abelian category* $\mathcal{C}$ is a category satisfying the following conditions

1. For every $A, B \in \mathcal{C}$, hom(A, B) is an Abelian group and the composition

$$\text{hom}(A, B) \times \text{hom}(B, C) \rightarrow \text{hom}(A, C)$$

is bilinear.

2. Coproduct exists. That is, for every object A and B, there's an object $A\coprod B$ such that there exists morphisms $i_1 : A \to A\coprod B$ and $i_2 : B \to A\coprod B$ that satisfying the universal property: for every object C and morphisms $\phi_1 : A \to C$ and $\phi_2 : B \to C$, there's a unique morphism $\psi : A\coprod B \to C$ that $\phi_1 = \psi \circ i_1$ and $\phi_2 = \psi \circ i_2$. In other word, the following diagram commutes.

$$\begin{tikzcd} & C & \\ A \arrow[ur, "\phi_1"] \arrow[r, "i_1"'] & A\coprod B \arrow[u, dashed, "\psi"] & B \arrow[ul, "\phi_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

Sometimes this is called the "direct sum". This is equivalent to the existence of product, where all the arrows are reversed.

3. Every morphism $\phi : A \to B$ admits a kernel. Kernel is an object K with a map χ such that $f \circ \chi$ is the zero map that satisfying the universal property: for every object K' and morphism $k' : K' \to A$ such that the composition $f \circ \chi'$ is zero, there is a unique morphism $\psi : K' \to K$ such that $\chi \circ \psi = \chi'$. The following diagram commutes

$$\begin{tikzcd} & A \arrow[dr, "\phi"] & \\ & K \arrow[u, "\chi"] \arrow[r, "0_{KB}"] & B \\ K \arrow[uur, "\chi'"] \arrow[ur, dashed, "\psi"] \arrow[urr, "0_{K'B}"'] & & \end{tikzcd}$$

4. Every morphism $\phi : A \to B$ has a cokernel C. This is similar to what we have

seen before, so we only give a diagram:

$$\begin{tikzcd}
& B \arrow[d, "\chi"] \arrow[ddr, "\chi'"] & \\
A \arrow[ur, "\phi"] \arrow[r, "0_{AC}"] \arrow[drr, "0_{AC'}"'] & C \arrow[dr, dashed, "\psi"] & \\
& & C'
\end{tikzcd}$$

5. The cokernel of a kernel is isomorphic to the kernel of cokernel.

The category satisfying condition 1 is called $Ab-$enriched; those satisfying 1 and 2 are called additive; those satisfying 1,2 and 3 is called pre-Abelian. But we will only consider Abelian category here. We will further define the resolution of sheaves

Definition 2.2.4 (Exactness)**.** Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves on manifold. The sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is called *exact in the middle* if we have equality of sheaves $\operatorname{im}\phi = \ker\psi$.

A chain of exact mappings is called a resolution.

Definition 2.2.5 (Resolution)**.** A resolution $\mathcal{F}^{\cdot}$ is the chain

$$\mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{\phi_0} \mathcal{F}^1 \xrightarrow{\phi_1} \cdots \mathcal{F}^i \xrightarrow{\phi_i} \mathcal{F}^{i+1} \xrightarrow{\phi_{i+1}} \cdots$$

such that

$$\mathcal{F}^{i-1} \xrightarrow{\phi_{i-1}} \mathcal{F}^i \xrightarrow{\phi_i} \mathcal{F}^{i+1}$$

is exact in the middle and $j(\mathcal{F}) = \ker\phi_0$.

Some examples of resolution includes the Čech resolution, de Rham resolution and Dolbeault resolution.

Čech resolution. Let $\mathcal{F}$ be a sheaf over complex manifold X, and let U_i be an open covering of X. Let

$$U_I = \cap_{i\in I} U_i, \quad I \subset \mathbb{N}.$$

And we let $\mathcal{F}^k = \oplus_{i\in I}\mathcal{F}_I$ and $d : \mathcal{F}^k \to \mathcal{F}^{k+1}$ by formula

$$(d\sigma)_{j_0 j_1 \ldots j_{k+1}} = \sum_{i=0}^{k} (-1)^i \sigma_{j_0 \ldots, \hat{j}_i, \ldots j_k | U \cap U_{j_0 \cdots j_{k+1}}}.$$

Then we can prove that

$$0 \longrightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{\phi_0} \mathcal{F}^1 \xrightarrow{\phi_1} \cdots \mathcal{F}^i \xrightarrow{\phi_i} \mathcal{F}^{i+1} \xrightarrow{\phi_{i+1}} \cdots$$

is a resolution called the Čech resolution.

de Rham resolution. Let $\mathcal{A}^k$ denotes the C^∞ differential forms. The exterior differential d gives morphism of sheaves. We also claim that the complex

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \mathcal{A}^n \longrightarrow 0$$

Dolbeault resolution. Let $\mathcal{A}^{0,q}(E)$ denotes the sheaf of sections of $\Omega^{0,q} \otimes E$

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \mathcal{A}^n \longrightarrow 0$$

We will proceed to see the use of resolutions. This is essential in the calculation of cohomology group. Now we say the image of ϕ is cokernel of the kernel and coimage is kernel of the cokernel. And what we need is the left exact functor

Definition 2.2.6 (Left Exactness)**.** A functor F between to Abelian category $\mathcal{C}$ and $\mathcal{C}'$ is called *left exact* if $\ker F(\phi) = F(\ker(\phi))$ for $\phi : A \to B$ between $A, B \in \mathcal{C}$.

Definition 2.2.7 (Injective)**.** An object I of Abelian category is called *injective* if for every injective morphism $A \xrightarrow{j} B$ and morphism $\phi : A \to I$, there's a morphism $\psi : B \to I$ such that $\psi \circ j = \phi$. We say the category has *sufficiently many injective objects* if every object A of $\mathcal{C}$ admits an injective morphism $j : A \to I$, where I is injective.

Now we must use the similar way to define the resolutions, that

Definition 2.2.8. A complex $\mathcal{I}^\bullet$ is called a resolution of an object A of $\mathcal{C}$ if $\operatorname{im} d^i = \ker d^{i+1}$ for $i \geq 0$ and $j : A \to \mathcal{I}^0$ is isomorphic to $\ker d^0$.

Now if we have sufficiently many injective object, we may get

Lemma 2.2.9. *If $\mathcal{C}$ has sufficiently many injective objects, then every object admits an injective resolution. That is, $I^\bullet$ are all injective objects.*

The proof of the lemma is easy in that we can define the next object by the previous image. More detailed treatment could be referred to [24]. This lemma show the existence of cohomology. We may define our cohomology, using the derived functors

Theorem 2.2.10 (Derived Functor)**.** *For every object M of $\mathcal{C}$ and left exact functor F, there's object $R^iF(M)$, $i \geq 0$in $\mathcal{C}'$ determined up to isomorphism and satisfying*

1. $R^0F(M) = F(M)$
2. *For every short exact sequence*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

we can get the long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \dots$$

3. *For every injective object I we have $R^iF(I) = 0$, $i > 0$*

This is exactly defined by injective resolution $I^\bullet$ of A that $R^iF(A) = H^i(F(I^\bullet))$ and prove the properties. Now we define the sheaf cohomology to be

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F}).$$

With this definition of cohomology, we can prove the theorems of

Thus the de Rham resolution induce the de Rham cohomology

$$H^i(X, \mathcal{O}_X(E)) = R^i\Gamma(\mathcal{O}_X) = \frac{\ker(d : \mathcal{A}^i(X, E) \to \mathcal{A}^{i+1}(X, E))}{\operatorname{im}(d : \mathcal{A}^{i-1}(X, E) \to \mathcal{A}^i(X, E))}$$

And the Dolbeault resolution induce the Dolbeault cohomology

$$H^q(X, \Omega_X^p(E)) = R^q\Gamma(\Omega_X^p(E)) = \frac{\ker(\bar\partial : \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p,q+1}(X, E))}{\operatorname{im}(\bar\partial : \mathcal{A}^{p,q-1}(X, E) \to \mathcal{A}^{p,q}(X, E))}$$

And we have the following characterization

Definition 2.2.11 (Acyclic)**.** An object M of $\mathcal{C}$ is called F-acyclic if $R^iF(M) = 0$ for $i > 0$.

Definition 2.2.12 (Flasque)**.** A sheaf is said to be flasque if for every pair of open set $V \subset U$, the restriction map $\rho_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Now we have the important theorem

Theorem 2.2.13. *Every injective sheaf is flasque*

Lemma 2.2.14. *A short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves on topological space X with $\mathcal{F}$ flasque induces exact sequence of

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0, \quad U \subset X.$$

And

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

if $\mathcal{F}, \mathcal{G}$ flasque, so is $\mathcal{H}$.

The proof of this theorem is just definition and use of homological algebra.

Theorem 2.2.15. *Flasque sheaves $\mathcal{F}$ are acyclic for functor Γ. Namely $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Finally, we define the pullback sheaf, which will be used in the Grauert-Riemanshneider vanishing theorem

Definition 2.2.16 (pull-back)**.** Let $f : M \to N$ be a continuous map, and $\mathcal{F}$ is a sheaf over N. The pullback sheaf $f^{-1}\mathcal{F}$ assign every open set $U \subset M$ the space of sections $s : U \to \cup_{x \in U} \mathcal{F}_{f(x)}$ and it can be realized locally by a section on N.

2.3 Holomorphic Vector bundle, Curvature and Chern Connection

We would like to define special structures on the manifold and the associated bundles in this section. This structure is called the ”Kähler metric”. But to see what Kähler metric is, we must first see what is metric. A metric is a way to assign bundles a notion of distance. A metric on a manifold measures the “length”. More precisely, we associate a inner product on the tangent space. To understand more clearly, we may consider the standard type of length calculation aroused in calculus. If we parameterize a curve $y(x)$ where $x \in [a, b]$ and compute the length of the curve. The standard computation says that

$$l(y) = \int_a^b \sqrt{1 + y'(x)^2} dx.$$

This formula is actually a assignation of metric to the tangent space of curve. For a point $(x_0, y(x_0))$ on the $1-$dimensional manifold, we use the metric $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \left(1 + \left(\frac{dy}{dx}\right)^2\right)$. The differentiation actually comes from the formula $l(y) = \int_a^b \sqrt{g(y'(x), y'(x))} dx$. Similarly, we define the Hermitian metric on manifold. This metric is a way to associate a Hilbert space structure to the vector bundles.

Definition 2.3.1 (Hermitian Metric)**.** An *Hermitian metric* on holomorphic vector bundle E is an Hermite scalar product h_x which depends differentiably with respect to x. Which means, on a local trivialization $U \times \mathbb{C}^n$, the map $(h_{ij}) : U \to GL_n(\mathbb{C})$ is differentiable.

Also, we know a “Meta-theorem” for vector bundle, which says that any construction on vector space could be translated to construction on vector bundle. For example, we can take the direct sum and the tensor product of vector space, then the corresponding operation is available on vector bundle. Other operations available are the dual and exterior product. Finally, we could take the determinant, which is the r-th exterior product of E. Thus the Hermitian metric is a section of $\Gamma(E \otimes \bar{E})^*$ and $h_p(\eta, \bar{\zeta}) = \overline{h_p(\zeta, \bar{\eta})}$ which is positive definite ($h_p(\zeta, \bar{\zeta}) = 0$ for non zero ζ). Now this metric could induce a metric on the bundle of differentiable forms.

Definition 2.3.2 (Induced Metric on differential form)**.** The induced metric on differentiable 1-form is

$$(h^{ij}) = (h_{ij})^{-1}, h^{ij} = h(dz^i, d\bar{z}^j)).$$

And the exterior product

$$(z_1 \wedge \ldots \wedge z_k, w_1 \wedge \ldots \wedge w_k) = \det((v_i, w_j)).$$

The Hermitian metric is useful in the structure of vector bundle. The good thing is ,we can associate every vector bundle this structure

Theorem 2.3.3. *Every complex vector bundle admits a Hermitian metric*

The proof is chosen by taking the partition of unity. We could also associate another computation method: the operator of taking derivative. This operation is called “connection” in that it connects the nearby tangent spaces and gives the result of infinitesimal variant of the original tangent vector. To specify it in more mathematical term, we can see the following definition:

Definition 2.3.4 (Connection)**.** A *Connection* on a vector bundle E is a homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ that.

$$\nabla(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s).$$

for any differentiable function f and section of E.

The set of all connection can be seen as a space of 1-form over $E \otimes E^*$. That is

Theorem 2.3.5. *The space of all connection on E is $\mathcal{A}^1(X, E \otimes E^*)$.*

Proof of the theorem and following lemma could be referred to [11]. Now we may define the fundamental connection that is often used in complex geometry. This connection is called Chern connection. But first, we may define several properties that are essential in defining the Chern connection.

Definition 2.3.6 (Hermitian Connection)**.** A connection on a complex vector bundle is said to be *Hermitian* (or compatible with Hermitian structure h) if for any section $s_1, s_2 \in E$,

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2).$$

Here $h(\alpha \otimes s, t) = \alpha h(s, t)$ for 1-form α.

We can prove the following theorem

Theorem 2.3.7. *The set of all Hermitian connection is the space*

$$\mathcal{A}^1(X, End(E, h)).$$

by $End(E, h)$ we denote the $a \in End(E)$ such that

$$h(a(s_1), s_2) + h(s_1, a(s_2)) = 0.$$

Now we define the Chern connection, that is ,the connection is compatible is holomorphic structure.

Definition 2.3.8 (Chern connection)**.** The *Chern connection* on the holomorphic vector bundle is a connection that

1. is compatible with Hermitian structure, and
2. is compatible with holomorphic structure, i.e. $\nabla^{0,1}$ the antiholomorphic part is equivalent to $\bar{\partial}$.

Chern connection is uniquely determined when we already have an Hermitian structure. For a holomorphic line bundle L, locally Chern connection is

$$\nabla = d + \partial \log H.$$

Chern-Weil theory then exists as a analytic characterization of topology. The Chern-Weil theory says that the curvature form could be translated to representative of de Rham cohomology group. In this article, we are only concerned with the first Chern class. But we shall see what curvature is before we say something about Chern class. The curvature of a connection depends how much a connection commutes. Since we all know that $d^2 = 0$ on the tangent bundle. But in general $\nabla^2 \neq 0$. Thus we should consider how much it does not commute, the analytic invariant to see this property is the curvature. We could define $\nabla = d + A$, then

Definition 2.3.9. The *curvature tensor* for connection ∇ is

$$\Theta = dA + A \wedge A \in \mathcal{A}^2(X, End(E)).$$

Locally, the curvature tensor for Chern connection is

$$\Theta_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}$$

Now several useful equalities are given for curvature on operation of vector bundles

Proposition 2.3.10. *Let E_1, E_2 be two holomorphic vector bundle with connection ∇_1 and ∇_2 and curvature form Θ_1, Θ_2. Then*

1. *The direct sum $E_1 \oplus E_2$ has the curvature form*
$$\Theta_1 \oplus \Theta_2.$$
2. *The tensor product $E_1 \otimes E_2$ has curvature*
$$\Theta_1 \otimes 1 + 1 \otimes \Theta_2$$
3. *The dual ∇^* on E_1^* induce*
$$\Theta_1^* = -\Theta_1^t$$

The proposition is done by direct brute force checking. Since we also have the Bianchi identity that

$$(\nabla\Theta)(s) = \nabla(\Theta(s)) - \Theta(\nabla(s)) = \nabla^3(s) - \nabla^3(s) = 0.$$

Thus

$$\bar{\partial}\Theta = (\nabla\Theta)^{1,2} = 0.$$

We know that $[\Theta] \in H^1(X, \Omega_X \otimes End(E))$. Now we should multiply a constant $\frac{i}{2\pi}$ to make it an integral class. This $\left[\frac{i}{2\pi}\Theta\right]$ is called the *first Chern class*. A first definition of our most important subject, positive line bundle, is shown in the following definition

Definition 2.3.11. A line bundle is said to be positive if and only if there's an Hermitian structure on it such that $\frac{i}{2\pi}\Theta > 0$ as a $(1,1)$ form.

As a particular example, we see that on a standard coordinate on $\mathbb{P}^n$, the holomorphic line bundle $\mathcal{O}(1)$ is endowed with the metric

$$\frac{i}{2\pi}\Theta = \omega_{FS}.$$

Here ω_{FS} is the canonical Fubini-Study metric. This will motivate our Kodaira vanishing theorem. We should see how to analyze the topological invariant with analytic tool known as Hodge theory in the next section.

2.4 Hodge Theory in Real Settings

Let M be a n dimensional C^∞ manifold, for exterior algebra $\wedge^k T_p^* M$, we have the natural isomorphism $\Lambda^k T_p^* M \cong \Lambda^{n-k} T_p^* M$. This can induces a linear isomorphism between $\mathcal{A}^k(M) \cong \mathcal{A}^{n-k}(M)$(vectors spaces of k-forms and $(n-k)$-forms). The isomorphism can be seen as follows. given an oriented orthonormal basis $\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n$, we can get a linear map:

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^*$$

by letting

$$*(\theta_{i_1} \wedge \theta_{i_2} \wedge \ldots \theta_{i_k}) = \theta_{i_{k+1}} \wedge \ldots \wedge \theta_{i_n}$$

where $(i_1, i_2, \cdots, i_n)$ is an even permutation of $(1, 2, \cdots, n)$. It's easy to check that this is well defined. This definition can be extended to $\mathcal{A}^k$, where for an orthonormal basis, the only difference is to multiply a function f. That is, if $x_1, x_2, \ldots, x_n$ is a local frame, then we may take the Gram-Schmidt orthogonal algorithm to get a frame that is normal. In particular, $*1 = \theta_1 \wedge \ldots \wedge \theta_n$, and this is called the volume form denoted by v_M. By change of variable, we can easily check that $v_M = \sqrt{\det(g_{ij})}$,where g_{ij} is the Riemannian metric. Also, we define the inner product $(\cdot, \cdot)$ between alternating forms. That is, $\omega = \alpha_1 \wedge \alpha_2 \wedge \cdots \alpha_n$ and $\eta = \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$. $\langle \omega, \eta \rangle = \det((\alpha_i, \beta_j))$, and the inner product between 1-form are defined canonically.

Thus we have the following theorems that give the properties of Hodge star operator

Lemma 2.4.1. *For $f, g \in C^\infty(M)$ and $\omega, \eta \in \mathcal{A}^k(M)$, one has*

1. $*(f\omega + g\eta) = f(*\omega) + g(*\eta)$.
2. $** \omega = (-1)^{k(n-k)}\omega$.
3. $\omega \wedge *\eta = \eta \wedge *\omega = \langle \omega, \eta \rangle v_M$.
4. $*(\omega \wedge *\eta) = *(\eta \wedge *\omega) = \langle \omega, \eta \rangle$.
5. $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$

Proof. (1),(2) are obvious, (4) is just applying $*$ to (3), so we may only prove (3). We first assume that $\omega_p = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_k$. We assume $\eta_p = \theta_{i_1} \wedge \theta_{i_2} \wedge \ldots \wedge \theta_{i_k}$, and then $*\eta_p = \text{sgn}(I, J)\theta_{j_1} \wedge \theta_{j_2} \wedge \ldots \wedge \theta_{j_{n-k}}$. Thus $\omega \wedge \eta \neq 0$ only when $\{i_1, i_2, \cdots, i_k\} = \{1, 2, \cdots, k\}$. Thus $\omega_p \wedge *\eta_p = \text{sgn}\, I\theta_1 \wedge \ldots \wedge \theta_n$. Also, we know that $\langle \omega_p, \eta_p \rangle = \text{sgn}\, I$. The proof i done. $\square$

Remark 2.4.2. *With this definition, we can get the famous divergence theorem by defining*

$$\text{div}\, X = *d * \omega_X$$

where $X \in \mathfrak{X}(M)$ and ω_X is the 1-form corresponding to isomorphism $\mathfrak{X}(M) \cong \mathcal{A}^1(M)$. The definition gives $X = \sum_i f_i \frac{\partial}{\partial x_i}$, then $\operatorname{div} X = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$. So when M is an oriented compact Riemannian manifold, and X is a vector field, we have equality

$$\int_M \operatorname{div} X v_M = \int_{\partial M} \langle X, n \rangle v_{\partial M}$$

here n is the outward unit vector field on ∂M.

With the Hodge star operator, we can introduce the inner product in $\mathcal{A}^k(M)$ by integrating the function , that is

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle v_M$$

This definition clearly satisfies all the requirement for inner product. In other word, we can write it as

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega = (*\omega, *\eta)$$

With the inner product, we may get the adjoint operator δ for the exterior differentiation $d : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$. That is, $(d\omega, \eta) = (\omega, \delta\eta)$. It's easy to check that $\delta = (-1)^{n(k+1)+1} * d *$. Thus we may give the

Definition 2.4.3 (Laplacian-Beltrami operator)**.** For Riemannian manifold M, the operator $\Delta : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$ by

$$\Delta := d\delta + \delta d$$

A form $\omega \in \mathcal{A}^*(M)$ that $\Delta\omega = 0$ is called a harmonic form.

Remark 2.4.4. *By some computation, we know that the Laplacian defined before on a*

Euclidean space $\mathbb{R}^n$, acting on the form $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ is

$$\Delta\omega = -\left(\sum_{s=1}^{n} \frac{\partial^2 f}{\partial x_s^2}\right) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

We can see that the Laplacian defined before is just the opposite sign version, we can soon see that Δ defined this way is a positive semi-definite operator. That is,

Lemma 2.4.5. *For a Laplacian, we have*

1. $*\Delta = \Delta *$. *Thus ω is harmonic implies $*\omega$ to be harmonic.*
2. Δ *is self-adjoint.*
3. $\Delta\omega = 0$ *if and only if $d\omega = 0$ and $\delta\omega = 0$.*

The proof for the lemma is straight forward, with only the adjoint argument in the definition of the Laplacian. We may get an analogue for famous Liouville theorem:

Theorem 2.4.6. *If M is a connected, oriented, compact Riemannian manifold with dimension n, then harmonic function is constant, and the harmonic n-form is constant multiple of v_M.*

The proof is also obvious that $df = 0$, and $*\omega$ is harmonic implies ω is constant multiple. The theorem this time implies that every element in the de Rham cohomology group H^k_{DR} can be represented by a harmonic form when $k = 0, n$. we may prove the assertion for other k below.

We denote $\mathbb{H}^k(M)$ to be all harmonic k-form in M. That is

$$\mathbb{H}^k(M) = \{\omega \in \mathcal{A}^k(M), \Delta\omega = 0\}.$$

Since the harmonic form is closed, we have the natural map $\mathbb{H}^k(M) \to H^k_{DR}(M)$, the de Rham cohomology group. We may prove that the canonical map is actually an isomorphism. It's easy to see that this is an injection. This is because when ω is exact, $(\omega, \omega) = (d\eta, \omega) = (\eta, \delta\omega) = 0$ since $\delta\omega = 0$. This prove that $\mathbb{H}$ is actually a finite dimensional vector space. To prove the onto-ness, we may first prove the theorem

Theorem 2.4.7 (Hodge decomposition)**.** *For a oriented compact Riemannian manifold, the space of all k-forms on M can be decomposed into three subspaces by direct sum:*

$$\mathbb{H}^k \oplus d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1} = \mathcal{A}^k(M)$$

To prove the theorem, we must first define the projection map $H : \mathcal{A}^k(M) \to \mathbb{H}^k(M)$ by choosing an orthonormal basis on $\mathbb{H}^k(M)$, this can be done since the space is a Hilbert space with finite dimension as we can see in the injection with de Rham cohomology. We choose an orthonormal basis $\beta_1, \ldots, \beta_n$, and the projection is

$$H\omega := \sum_{i=1}^{r} (\omega, \beta_i)\beta_i$$

We only gives some sketch of the proof, and the gap will be filled later

Proof. Orthogonality: $(\omega, d\eta) = (\delta\omega, \eta) = 0$, $(\omega, \delta\eta) = (\delta\omega, \eta) = 0$ and $(d\omega, \delta\eta) = (d^2\omega, \eta) = 0$. The three are obviously orthogonal. And if an element is orthogonal to the three space, then $\omega = 0$. So we may prove that $\omega - H\omega \in d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1}$.

Existence We need to prove that, for $\omega - H\omega = \omega_0 \in (\mathbb{H}^k)^\perp$, there's a unique $\eta_1 \in (\mathbb{H}^k)^\perp$ such that $\Delta\eta_1 = \omega_0$ weakly. Then $\omega - H\omega = \Delta\eta_1 = d\delta\eta_1 + \delta d\eta_1$. We may define the Green's operator $G : \mathcal{A}^k(M) \to (\mathbb{H}^k)^\perp$ by mapping ω to η_1. But to find this η_1, we may refer to the Sobolev theory on manifold. $\square$

Before stating the Sobolev theory, we may first look at the proof for the isomorphism we mentioned before, namely,

Theorem 2.4.8 (Hodge theorem)**.** *The map $\mathbb{H}^k(M) \to H^k_{DR}(M)$ is an isomorphism. Then there's an unique representative in harmonic form for the de Rham cohomology group.*

Proof. We let $\omega \in \mathcal{A}^k(M)$ to be a closed form. The decomposition gives

$$\omega = \omega_H + d\eta + \delta\theta.$$

We take the representative to be ω_H. Since $d\omega = 0$, then $d\delta\theta = 0$,and $(d\delta\theta, \theta) = \|\delta\theta\|^2 = 0$. Thus we know that the representative is unique. □

We may now turn to the "solution" for the equation $\Delta\eta = \omega_0$, and if we have a solution, we may use $\eta_1 = \eta - H\eta$ to get a solution in $(\mathbb{H}^k)^\perp$. First we introduce the Sobolev norm

Definition 2.4.9 (Sobolev norm)**.** We define the inner product $(\cdot,\cdot)_s$ for $\omega_1, \omega_2 \in \mathcal{A}^*(M)$ by

$$(\omega_1, \omega_2)_s = \sum_{i=0}^{s} (\nabla^i \omega_1, \nabla^i \omega_2)$$

and then $\|\omega_1\|_s^2 = (\omega_1, \omega_1)_s$. Here ∇ is a connection.

Thus several tools in Sobolev space can be applied, we state these results without proof here.

First, we may have a look at the "weak" solution. Define $H_s(M)$ to be the completion of space of differential form under Sobolev norm $\|\cdot\|_s$. We would like to solve $\Delta\eta = \omega$, but a theorem of Weyl stated that

Lemma 2.4.10 (Weak form of Weyl lemma)**.** *If $\eta \in H_1(M)$ and for a fixed $\omega \in \mathcal{A}^*(M)$, the equation $(\mu, \omega) = (\mu, \Delta\eta)$ holds for all $\mu \in \mathcal{A}^*(M)$, then $\eta \in \mathcal{A}^*(M)$.*

Proof of the lemma and following lemma is in [25]. With the lemma, we only need to prove the weak solution exists(i.e. the condition in Weyl's lemma). We also write $P = d + \delta$ acting on $\mathcal{A}^*(M)$. Then we have the following regularity theorem

Lemma 2.4.11 (Regularity)**.** *If $\omega \in H_0(M)$ and $P\omega \in \mathcal{A}^*(M)$, then $\omega \in \mathcal{A}^*(M)$*

Also, two inequalities are of special importance in proving the theorem. These are

Lemma 2.4.12 (Garding's inequality)**.** *There exists constant $c_1, c_2 > 0$ and for any $\eta \in \mathcal{A}^*(M)$, we have*

$$(\Delta\eta, \eta) \geq c_1\|\eta\|_1^2 - c_2\|\eta\|_0^2.$$

The following lemma is also useful in the proof, namely

Lemma 2.4.13 (Rellich's lemma)**.** *If $\{\omega_i\} \in \mathcal{A}^*(M)$ is a bounded sequence in norm $\|\cdot\|_1$, then there exists a Cauchy subsequence in norm $\|\cdot\|_0$.*

We may proceed to prove the Garding's lemma:

Lemma 2.4.14 (Garding's lemma)**.** *There's a positive constant c_0 such that for all $\eta \in (\mathbb{H}^k(M))^\perp$, we have*

$$\|\eta\|_1^2 \leq c_0(\Delta\eta, \eta).$$

Proof. Suppose not, that there's a sequence of $\{\eta_i\}$ such that $\|\eta_i\|_1 = 1$ and $(\Delta\eta_i, \eta_i) \to 0$.By Rellich lemma, we may WLOG assume that $\eta_i \to \eta$ in $H_0(M)$. Clearly $\eta \in$

$(\mathbb{H}^k(M))^\perp$. Also, $(\Delta\eta, \eta) = 0$ means that $(d\eta, d\eta) + (\delta\eta, \delta\eta) = 0$. By previous theorems we know that $\eta \in \mathbb{H}^k(M)$. Thus $\eta = 0$. But this contradicts that $\|\eta_i\|_1 = 1$ since by Garding's lemma, we should have

$$\|\eta_i\|_1^2 \leq \frac{1}{c_1}(\Delta\eta_i, \eta_i) + \frac{c_2}{c_1}\|\eta_i\|_0^2 \to 0$$

So the lemma is proved. □

This lemma actually prove the continuity we should use in the Riesz representation theorem. With this lemma, we may prove the last part of Hodge theorem with certainty. We recall the Riesz representation theorem

Theorem 2.4.15 (Riesz representation)**.** *Let ϕ be a bounded linear map on Hilbert space H, then $\phi(x) = \langle x, y\rangle$ for a fixed y.*

With this theorem, we only need to prove that for the Hilbert space with norm $\langle \omega_1, \omega_2\rangle = (\omega_1, \Delta\omega_2)$, the mapping $f(\cdot) = (\cdot, \omega_0)$ is a bounded linear map. Then by the representation theorem the result is obvious. Linearity is crystal clear, so we only need to use our lemmas to prove boundedness.

Theorem 2.4.16 (Solution for Δ)**.** *There exists a unique solution for $\Delta\eta_1 = \omega_0$ with fixed $\omega \in (\mathbb{H}^k(M))^\perp$ and $\eta \in (\mathbb{H}^k(M))^\perp$.*

Proof. Uniqueness is shown before, so for existence, we need to prove boundedness, which is because

$$|(\omega_1, \omega_0)| \leq \|\omega_1\|_0\|\omega_0\|_0 \leq \|\omega_1\|_1\|\omega_0\|_0 \leq c_0\|\omega_0\|_0\sqrt{\langle\omega_1, \omega_1\rangle}.$$

The last inequality follows from Garding's lemma. □

In general, we can consider the L^2 differential forms without the compactness setting. But this is beyond the scope of our current discussion. We must continue our study on the Hodge theory on Kähler manifold.

第三章 Vanishing Theorems on Kähler manifold

Vanishing theorems on Kähler manifold is a tool to see whether the manifold is projective. This tool may transform a difficult problem of classifying manifold to the computation of projective space. The proof of this theorem depends on the Kodaira-Akizuki identity and the Kähler identities. We will discuss the proof in full detail. We omit the use of algebraic technique since our main purpose of this paper is to investigate the power geometric method. In particular, we will show the power of curvature tensor in the definition of various algebraic properties, which constructs a bridge between the algebraic side and geometric side. We will use the Hodge theorem on Kähler manifold to reduce the problem of finding cohomology group to solving the laplacian. Then we use the identities on Kähler manifold to derive several useful techniques that are often used in the computation of harmonic forms. With these formulae, we will derive some vanishing theorem on positive and semi-positive line bundles.

3.1 Hodge Theory on Kähler manifold

In this part, we may prove the Hodge theorem for the compact complex manifold. But proof for a special manifold, namely, the complex tori, will not suffer the pain of mulling difficult Sobolev theory since it admits Fourier analysis to construct the Green's operator on the manifold. So let us begin to study the theorems. Then we will proceed to study the Hodge theory of compact Kähler manifold

Definition 3.1.1 (Complex Tori)**.** A complex torus X is the quotient $V = \mathbb{C}^g$ modulo a discrete group Λ.

In order to get the analogous result for complex torus, we may first consider the de Rham theorem on complex torus. Note that group of the differential forms on complex torus is canonically isomorphic to the differential forms on V with period Λ. These are the Λ-invariant differential forms. For simplicity, when $x_1, \ldots, x_{2g}$ are coordinate for $\mathbb{C}$, we write Λ-invariant form on torus to be

$$\omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq 2g} \omega_{i_1, \ldots, i_k} dv_{i_1} \wedge \ldots \wedge dv_{i_k}$$

In particular, when all $\omega_{i_1, \ldots, i_k}$ are constant(and the integration over a closed oriented submanifold is integer), we may denote the space spanned by all these forms $IF^k(X)_{\mathbb{Z}}$ and those which takes value in $\mathbb{C}$ by $IF^k(X)$, namely, the invariant form(the name comes from that the form will not change under any translation). We may prove that any of the forms can be represented by an invariant form, and thus the de Rham cohomology is equivalent to the space of invariant forms. We denote the differential forms on X by $\mathcal{A}^{p,q}(X)$, which has p component with dz_i and q components with $d\bar{z}_j$.

Lemma 3.1.2. *Let ω be a closed form on X and τ be any translation in X. Then $\tau^*\omega - \omega$ is exact. As a result, there's a mapping $H : \mathcal{A}^{p,q}(X) \to IF^{p+q}(X)$ which differs an exact form.*

Proof. Let $X = V/\Lambda$ and $\tau(x) = x + a$ on V. We define

$$\eta_a(x)(x_1, x_2, \cdots, x_{r-1}) = \int_0^1 \omega(x + at)(a, x_1, x_2, \cdots, x_{r-1})dt.$$

Then η_a is an $(r-1)$-form on X. In addition $d\eta_a = \tau^*\omega - \omega$ as desired. Thus, we may define H to be

$$H(\phi dv_I \wedge d\bar{v}_J) = \frac{1}{\text{vol}(X)} \left(\int_X \phi * 1 \right) \phi dv_I \wedge d\bar{v}_J.$$

This satisfies our assertion since

$$H(\omega) - \omega = \frac{1}{\text{vol}(X)} \left(\int_X d\eta_a(x) da \right) = d(\frac{1}{\text{vol}(X)} \left(\int_X \eta_a(x) da \right)).$$

The projection is done. $\square$

Since the de Rham isomorphism $H^k(X,\mathbb{Z}) \cong IF^k(X)_{\mathbb{Z}}$ by using the integral invariant forms. Using the universal coefficient theorem, we have $H^k(X,\mathbb{Z}) \otimes \mathbb{C} = H^k(X,\mathbb{C}) \cong IF^k(X)$.

Although the projection seems to be done, we have not yet defined our Laplacian $\Delta!$. This involves the ∂ and $\bar{\partial}$ computations. For the integration on torus, we may introduce the notion of Kähler metric. Any hermitian metric on V induces a translation invariant metric on X using the H we defined before. We denote Hermitian metric that induced from that of V by

$$ds^2 = \sum_{j=1}^{g} dv_j \otimes d\bar{v}_j.$$

Then the $(1,1)$-form associated to the metric $\omega = \frac{i}{2} \sum_{i=1}^{g} dv_j \otimes d\bar{v}_j$. If ω is $d-$closed, then the metric is called a Kähler metric. In particular complex tori are all Kähler . So we define the corresponding volume form by

$$dv = \frac{1}{g!} \wedge^g \omega = (-1)^{\frac{1}{2}g(g-1)} \left(\frac{1}{2}\right)^g dv_1 \wedge \ldots \wedge dv_g \wedge d\bar{v}_1 \wedge \ldots \wedge d\bar{v}_g.$$

For each form on $\mathcal{A}^{p,q}(X)$, we define the inner product to be

$$(\omega,\eta)=\sum_{|I|=p,|J|=q}\int_X \omega_{IJ}\eta_{IJ}dv,$$

where $\omega=\sum_{I,J}\omega_{IJ}dv_I\otimes d\bar{v}_J$ and the similar with η, and the summation is taken over all indices I,J with p elements in I and q elements in J. Thus $\mathcal{A}^{p,q}$ is made to be a pre-Hilbert space as before. Since we will use the Dolbeault's cohomology, we may consider the adjoint operator with respect to $\bar{\partial}$, denoted by $\bar{\partial}^*$. It's easy to verify that

$$\bar{\partial}^*(\phi dv_I\wedge d\bar{v}_J)=\sum_{j=1}^{q+1}(-1)^{p+j}\frac{\partial\phi}{\partial v_{k_j}}dv_I\wedge dv_{J-k_j}.$$

Here $J-k_j$ means the indices set that remove k_j. Thus the Laplacian operator associated to $\bar{\partial}$ is defined by

$$\Delta=\bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}.$$

Then we have the following lemma:

Lemma 3.1.3. *For the Laplacian defined before, we have*

$$\Delta(\phi dv_I\wedge d\bar{v}_J)=-\sum_{j=1}^{g}\frac{\partial^2\phi}{\partial v_j\partial\bar{v}_j}dv_I\wedge d\bar{v}_J.$$

The proof follows straightforward from the definition. We also define the form that $\Delta\omega=0$ to be the harmonic forms. Thus it's easy to see that any invariant form is harmonic. And for invariant forms, we have the Dolbeault cohomology that say something about the cohomology class of holomorphic sheaf on X. Namely the following assertion

Theorem 3.1.4 (Dolbeault Isomorphism theorem)**.** *For $\Omega^{p,q}(X)$ to be the vector bun-*

dles of holomorphic differential forms of type (p,q) *on* X*, and* $\bar{\partial} : \Gamma(\Omega^{p,q}(X)) \to \Gamma(\Omega^{p,q+1}(X))$. *Then*

$$H^q(X,\Omega^p) \cong \frac{\ker\{\bar{\partial} : \Gamma(\Omega^{p,q}(X)) \to \Gamma(\Omega^{p,q+1}(X))\}}{\bar{\partial}\Gamma(\Omega^{p,q-1}(X))}.$$

Here Ω^p *is the sheaf of holomorphic* $p-$*forms, and* $H^q(X,\Omega^p)$ *is the cohomology of sheaf.*

For the proof we refer to Griffiths-Harris[10]. This is a result based on the Dolbeault resolution.

Thus we may provide our Hodge theorem for complex tori:

Theorem 3.1.5 (Hodge theorem)**.** *For a complex tori* X*, we have the decomposition*

$$\mathcal{A}^{p,q}(X) = \mathbb{H}^{p,q}(X) \oplus \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X).$$

As a result, there's an isomorphism

$$H^n(X,\mathbb{C}) \cong \bigoplus_{j=0}^{n} H^j(X,\Omega^{n-j}).$$

We have already proved that $H^n(X,\mathbb{C}) \cong IF^n(X)$. Also we have the Dolbeault isomorphism. Also, we have proved that $IF^n \subset \ker\Delta = \mathbb{H}^{p,q}$. Thus the things we need is

1. $\mathbb{H}^{p,q} \subset IF^{p,q}$.
2. $\mathbb{H}^{p,q} \cong \frac{\ker\{\bar{\partial}:\Gamma(\Omega^{p,q}(X))\to\Gamma(\Omega^{p,q+1}(X))\}}{\bar{\partial}\Gamma(\Omega^{p,q-1}(X))}$.

So we begin our proof for the first assertion. This is because

Lemma 3.1.6 (Green's operator)**.** *There's a $\mathbb{C}$-linear operator from $\mathcal{A}^{p,q} \to \mathcal{A}^{p,q}$ that $\Delta G = G\Delta = 1 - H$ and $HG = GH = 0$.*

Proof. We may use Fourier analysis on torus to prove the proposition. This is simple since we may solve the Laplacian equation for periodic function on a one dimensional line easily by Fourier expansion. The analogous will help. Take $x_1, x_2, \ldots, x_{2g}$ be the real coordinates of the vector space V. That is, we think of $\mathbb{C}^g$ as $(x_1 + i \cdot x_{g+1}, x_2 + i \cdot x_{g+2}, \ldots, x_g + i \cdot x_{2g})$. Then we take the "orthogonal group"

$$\Lambda^{\perp} = \{y \in V \,|\, \exp(2\pi i \sum_{j=1}^{2g} \lambda_j y_j) = 1\}.$$

This is the group for the Fourier coefficient. Thus for any form ϕ that is periodic on Λ, we have the expansion

$$\phi(x) = \sum_{y \in \Lambda^{\perp}} a_y \exp(2\pi i \sum_{j=1}^{2g} x_j y_j).$$

Then the operator G is defined to be $G(\phi dv_I \wedge d\bar{v}_J) = \phi' dv_I \wedge d\bar{v}_J$, where

$$\phi'(x) = \frac{1}{\pi^2} \sum_{y \in \Lambda^{\perp} \setminus \{0\}} \frac{a_y}{\sum y_j^2} \exp(2\pi i \sum_{j=1}^{2g} x_j y_j).$$

We can simply check that $G\Delta = \Delta G = 1 - H$ since we have remove $y = 0$ term(and the equation $\frac{\partial^2}{\partial v_i \partial \bar{v}_i} = \frac{1}{4}(\partial_{x_i}^2 + \partial_{x_{i+g}}^2)$). The series converges since $\|y\| \geq 1$ except finitely many terms. The last equality is also easy since the Fourier expansion does not process constant term. $\square$

Then we may have the following corollary

Corollary 3.1.7. *$\mathbb{H}^{p,q} \subset IF^{p,q}$, then $IF^{p,q} \cong \mathbb{H}^{p,q}$.*

Proof. This follows from $\ker(1-H) \subset \ker G = IF^{p,q}$. □

To prove the orthogonal decomposition, we only note that $\operatorname{img} \Delta = \ker H$. This is because $\Delta G = 1$ on $\ker H$. And $H\Delta = \Delta H = 0$ since the derivative of constant is zero. Hence

$$\mathcal{A}^{p,q} = \operatorname{img} H \oplus \ker H = \mathbb{H}^{p,q} \oplus \operatorname{img} \Delta = \mathbb{H}^{p,q}(X) \oplus \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X)$$

By simple calculation, we know that $G\bar{\partial}\phi = \bar{\partial}G\phi$. As a result, the last part of the theorem follows

Lemma 3.1.8. *$H : \mathcal{A}^{p,q} \to IF^{p,q}$ induces an isomorphism between $H^q(X, \Omega^p)$ and $IF^{p,q}$, when the first one is determined by the Dolbeault cohomology.*

Proof. Observe that $\operatorname{img} H \subset \ker \bar{\partial}$, and on the other hand $H^{p,q} = \ker \bar{\partial} / \bar{\partial}\mathcal{A}^{p,q-1}$. Then it suffices to show that $\bar{\partial}\mathcal{A}^{p,q-1} = \ker(H : \ker \bar{\partial} \to IF^{p,q}) = \ker H \cup \ker \bar{\partial}$. If ϕ is in the intersection, we have

$$\phi = H\phi + G\Delta\phi = G\bar{\partial}\bar{\partial}^*\phi = \bar{\partial}G\bar{\partial}^*\phi.$$

The converse inclusion follows from $H\bar{\partial} = 0$. □

On a general compact complex manifold, the proof also will rely on the Sobolev space technique. We consider the Hodge theory on bundles of Kähler manifold. This theory relies more on the laplacian. We also have the Hodge decomposition for compact Hermitian manifold. That is

Theorem 3.1.9 (Hodge theorem)**.** *For a complex manifold X, we have the decomposition*

$$\mathcal{A}^{p,q}(X) = \mathbb{H}^{p,q}(X) \oplus \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X).$$

The isomorphism also holds

$$H^n(X, \mathbb{C}) \cong \bigoplus_{j=0}^{n} H^j(X, \Omega^{n-j}).$$

The proof has the similar technique with what we have seen in the real case. For a more detailed treatment one can refer to Wells[25]. Now we will prove and use some structural identities on Kähler manifold, which play essentially important roles in the proof of Kodaira vanishing and other vanishing theorems. We first define the operators on Kähler manifold

Definition 3.1.10 (Operators on Kähler manifold)**.** The *Lefschetz operator* L is defined to be $L\alpha = \omega \wedge \alpha$ when $\alpha \in \mathcal{A}^*(M)$. The *dual Lefschetz operator* Λ is defined to be $(\alpha, L\beta) = (\Lambda\alpha, \beta)$ when the inner product is induced from the Hermitian inner product.

Theorem 3.1.11 (Lefschetz Identities)**.** *X is a Kähler manifold with Kähler form ω. Now if we have the Lefschetz operator, we may define the counting operator $F|_{\mathcal{A}^k(X)} := (k-n) \cdot id$. Now the Lefschetz identities are*

$$1)[F, L] = 2L, \quad 2)[\Lambda, F] = 2\Lambda, \quad 3)[L, \Lambda] = 2F$$

This theorem is a corollary of a more generalized theorem. Also, we could prove this theorem on $\mathbb{C}^n$ and claim that this is also true on a manifold. The first two identities are trivial in that this is only a count of the order of differential form. We must observe the third identity. We could produce a more generalized theorem by computing the commutator of arbitrary $(1,1)$-form with Λ. Namely

Theorem 3.1.12 (Generalized Lefschetz Identities)**.** *For $u \in \mathcal{A}^{j,k}(X,E)$ on a Kähler manifold. Then for a $(1,1)$-form γ, note that the Kähler form ω and γ can be simultaneously diagonalize by orthonormal basis $\zeta_1, \zeta_2, \dots, \zeta_n$ in TX.*

$$\omega = i \sum_{1 \le j \le n} \zeta_j^* \wedge \bar{\zeta}_j^*, \quad \gamma = i \sum_{1 \le j \le n} \gamma_j \zeta_j^* \wedge \bar{\zeta}_j^*.$$

Then for $u = \sum_{J,K} \zeta_J^* \wedge \bar{\zeta}_K^*$, we have

$$[\gamma, \Lambda] u = \sum_{J,K} \left(\sum_{j \in J} \gamma_j + \sum_{k \in K} \gamma_k - \sum_{i=1}^{n} \gamma_i \right) u_{J,K} \zeta_J^* \wedge \bar{\zeta}_K^*.$$

Proof. This could be verified by brute force checking. □

Corollary 3.1.13. *As a corollary, we have*

$$[L^i, \Lambda]\alpha = i(k - n + i - 1) L^{i-1}(\alpha), \quad \alpha \in \mathcal{A}^k(X).$$

Proof. Note that

$$[L^i, \Lambda]\alpha = L[L^{i-1}, \Lambda]\alpha + [L, \Lambda] L^{i-1}\alpha$$

and proceed by induction. □

Theorem 3.1.14 (Kähler Identities)**.** *Suppose X is a compact Kähler manifold with a Kähler form ω. Then the identities here hold true:*

1. $[\bar{\partial}, L] = [\partial, L] = 0$ *and* $[\bar{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$.
2. $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$ *and* $[\Lambda, \bar{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\bar{\partial}^*$.

3. $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ *and* Δ *commutes with* $*, \partial, \bar\partial, \partial^*, \bar\partial^*, L$ *and* Λ.

The proof of the theorem is in [10]. Since we are doing all these operations on differential forms, we must improve it to arbitrary forms with values in any holomorphic vector bundle E. Suppose we are endowed with the Chern connection on the manifold, whose $(0,1)$ part is the same as $\bar\partial$, we may assume its $(1,0)$ part to be ∂_E. We need to prove the following more generalized identities.

Theorem 3.1.15 (Kähler Identities on bundles)**.** *Suppose X is a compact Kähler manifold and E is a holomorphic vector bundle on it. Then the identities here hold true:*

1. $[\bar\partial, L] = [\partial_E, L] = 0$ *and* $[\bar\partial^*, \Lambda] = [\partial_E^*, \Lambda] = 0$.

2. $[\bar\partial^*, L] = i\partial_E$, $[\partial_E^*, L] = -i\bar\partial$ *and* $[\Lambda, \bar\partial] = -i\partial_E^*$, $[\Lambda, \partial_E] = i\bar\partial^*$.

Theorem 3.1.16 (Kodaira-Nakano Identity)**.** *On a compact Kähler manifold X and the line bundle L, the following identity holds.*

$$\Delta_{\bar\partial} = \Delta_{\partial_E} - i[\Lambda, \Theta].$$

Proof. Note that

$$[\Lambda, \partial_E] = i\bar\partial^*.$$

Thus

$$\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^* = -i(\bar\partial\Lambda\partial_E - \bar\partial\partial_E\Lambda + \lambda\partial_E\bar\partial - \partial_E\Lambda\bar\partial).$$

Since $\bar\partial\Lambda = [\bar\partial, \Lambda] + \Lambda\bar\partial = i\partial_E^* + \Lambda\bar\partial$ and $\Lambda\bar\partial = -i\partial_E^* + \bar\partial\Lambda$. Now

$$\Delta_{\bar\partial} = -i((i\partial_E^* + \Lambda\bar\partial)\partial_E - \bar\partial\partial_E\Lambda + \Lambda\partial_E\bar\partial - \partial_E(-i\partial_E^* + \bar\partial\Lambda)) = \Delta_{\partial_E} - i[\Lambda, \Theta].$$

□

3.2 Kodaira Vanishing Theorem and L^2 Vanishing

Kodaira Vanishing theorem is actually a theorem on the relation of ampleness and positivity. This theorem was originally developed by Kunihiko Kodaira(小平邦彦) in 1956. Ampleness is a notion from algebraic geometry, where a line bundle is ample if it's "quasi-embedded" to a projective space. Moreover, ampleness could be defined by many interesting properties. The most geometric one we used here is the positivity of curvature, or more topologically the Chern class. We will used the Kähler identities computed before to see the relation of ampleness and cohomology vanishing. The proof of these theorems depends heavily on the L^2 estimates of differential forms. Thus a general L^2 Vanishing theorem is provided. The Kodaira theorem will then boils down to the computation a certain operator. More generally, the technique could be easily generalized to the (p, q) differential form instead of canonical bundle called the Kodaira-Nakano-Akizuki theorem, which was the work of Shigeo Nakano(中野茂男) and Yasuo Akizuki(秋月康夫). We will also prove the Serre's ampleness criterion by geometric approach. This theorem was originally proved by Serre by homological algebra. But here we will use the geometric argument to reprove this theorem.

Theorem 3.2.1 (Kodaira-Nakano Inequality)**.** *Suppose E is a holomorphic vector bundle on the Kähler manifold X. Then the curvature form of this manifold Θ has the following inequalities for harmonic form α*

$$1)(i\Theta\Lambda(\alpha), \alpha) \leq 0 \quad 2)(i\Lambda\Theta(\alpha), \alpha) \geq 0$$

Proof. Since Θ as an operator is by definition $\nabla^2\alpha = \Theta \wedge \alpha$. We know that

$$\Theta\alpha = \nabla^2\alpha = (\partial_E + \bar{\partial})(\partial_E + \bar{\partial})\alpha = \partial_E\bar{\partial}\alpha + \bar{\partial}\partial_E\alpha.$$

Now for a harmonic form we know that $\partial\alpha = \bar{\partial}\alpha = 0$. Now the computation follows that

$$\begin{aligned}(i\Theta\Lambda(\alpha), \alpha) &= i(\partial_E\bar{\partial}\Lambda\alpha + \bar{\partial}\partial_E\Lambda\alpha, \alpha) \\ &= i(\bar{\partial}\Lambda\alpha, \partial_E^*\alpha) = (\bar{\partial}\Lambda\alpha, -i\partial_E^*\alpha) \\ &= (\bar{\partial}\Lambda\alpha, [\Lambda, \bar{\partial}]\alpha) = -(\bar{\partial}\Lambda\alpha, \bar{\partial}\Lambda\alpha) = -|\bar{\partial}\Lambda\alpha|^2 \leq 0.\end{aligned}$$

Also the second formula holds since

$$\begin{aligned}(i\Lambda\Theta(\alpha), \alpha) &= i(\Lambda\partial_E\bar{\partial}\alpha + \Lambda\bar{\partial}\partial_E\alpha, \alpha) \\ &= (i\Lambda\bar{\partial}\partial_E\alpha, \alpha) = (\Lambda\bar{\partial}[\bar{\partial}^*, L]\alpha, \alpha) = (\bar{\partial}^* L\alpha, \bar{\partial}^* L\alpha) = |\bar{\partial}^* L\alpha|^2 \geq 0\end{aligned}$$

□

Theorem 3.2.2 (Kodaira Vanishing Theorem)**.** *Let L be a holomorphic line bundle over Kähler manifold X. Suppose in addition that L is positive, Then*

$$H^k(X, K_X \otimes L) = 0, \forall k > 0.$$

But we can prove a stronger theorem using the inequalities we have proved before. Namely, the vanishing theorem on sheaf of differential forms. And use the fact that $K_X = \Omega_X^n$.

Theorem 3.2.3 (Kodaira-Nakano-Akizuki Vanishing Theorem)**.** *Let L be a holomorphic line bundle over Kähler manifold X. Suppose in addition that L is positive, Then*

$$H^p(X, \Omega_X^q \otimes L) = 0, \forall p + q > n.$$

Proof. Since we already have that

$$(i\Theta\Lambda(\alpha), \alpha) \leq 0, \quad (i\Lambda\Theta(\alpha), \alpha) \geq 0$$

We can see that

$$(i[\Theta, \Lambda](\alpha), \alpha) \leq 0.$$

On the other hand, since our line bundle is positive, we know that there's a metric such that $\frac{i}{2\pi}\Theta = L$ as operator. Thus

$$\frac{1}{2\pi}([L, \Lambda](\alpha), \alpha) = \frac{1}{2\pi}((p+q) - n)\|\alpha\|^2 \geq 0, \quad \text{when } p + q > n.$$

It is straightforward that the harmonic form $\alpha \equiv 0$. That means, the cohomology group

$$H^p(X, \Omega_X^q \otimes L) = 0$$

when the conditions are satisfied. □

The most fundamental algebraic vanishing theorem is the Serre vanishing theorem given by Serre in his famous "Faisceaux algébriques cohérents"(simplified to be FAC). This theorem states that the positivity of a line bundle is equivalent to the vanishing of a certain cohomology group. Here we present a geometric method to prove one side. Since

Theorem 3.2.4 (Serre Vanishing Theorem)**.** *Let L be a holomorphic line bundle over Kähler manifold X. Suppose in addition that L is positive, Then for all holomorphic* *vector bundle E, there exists a $m_0 \geq 0$ such that*

$$H^q(X, E \otimes L^m) = 0, \forall m > m_0, q > 0.$$

Proof. If we choose a Chern connection on both E and L and denote the corresponding curvature form by Θ_E and $\Theta_L = \omega$ since L is ample. From previous lemmas we have seen that

$$(i[\Lambda, \Theta](\alpha), \alpha) \geq 0$$

for any $\alpha \in \mathcal{A}^{p,q}(X, E \otimes L^m)$. Also, we know that

$$\frac{i}{2\pi}\Theta = \frac{i}{2\pi}\Theta_E \otimes 1 + m(1 \otimes \omega).$$

Therefore the equality

$$\frac{i}{2\pi}([\Lambda, \Theta]\alpha, \alpha) = \frac{1}{2\pi}([\Lambda, i\Theta_E]\alpha, \alpha) + m(n - (p+q))\|\alpha\|^2.$$

Since by our Lefschetz identity $[L, \Lambda] = F = n - (p+q)$ when $\alpha \in \mathcal{A}^{p,q}(X, E \otimes L^m)$. By Cauchy-Schwartz inequality we know that

$$|([\Lambda, i\Theta_E]u, u)| \leq \|[\Lambda, i\Theta_E]\|\|\alpha\|^2,$$

where the constant $\|[\Lambda, i\Theta_E]\| \leq C$ does not depend on m. Thus there is a m such that $C + 2m\pi(n - p - q) < 0$, which proves the theorem since we know that $\alpha \equiv 0$. □

Theorem 3.2.5 (L^2 Vanishing Theorem)**.** *For a holomorphic line bundle L on X, sup-*

pose that the operator $[i\Theta_L, \Lambda]$ is positive on $\mathcal{A}^{p,q}(X, L)$. Then

$$H^q(X, \Omega_X^p \otimes E) = 0.$$

The proof is just one of the steps in the proof of Kodaira vanishing theorem. Now we may do something on the analysis of the eigenvalues of the curvature tensor at each point. These eigenvalues, by our previous generalized Lefschetz formula, are useful in determining the positivity of operator $[i\Theta_L, \Lambda]$. We then have the Girbau vanishing theorem on the eigenvalues.

Theorem 3.2.6 (Girbau Vanishing Theorem)**.** *For a holomorphic line bundle L on compact Kähler manifold X, suppose $i\Theta_L$ is semipositive and has at least $n - s + 1$ positive eigenvalues at every point $x \in X$. We have*

$$H^q(X, \Omega_X^p \otimes E) = 0, \quad p + q \geq n + s.$$

Proof. We consider a new Kähler metric

$$\omega_\epsilon = \epsilon\omega + i\Theta_L, \quad \epsilon > 0$$

Now $i\Theta_L = i\sum \gamma_j \zeta_j \wedge \bar{\zeta}_j$ is a diagonalization of the curvature form with respect to ω. Then

$$\omega_\epsilon = i\sum(\epsilon + \gamma_j)\zeta_j \wedge \bar{\zeta}_j.$$

Thus eigenvalue of $i\Theta_L$ with respect to ω_ϵ is then

$$\gamma_{j,\epsilon} = \frac{\gamma_j}{\epsilon + \gamma_j} \in [0, 1).$$

Since $j \geq s$ we have $\gamma_j \geq \gamma_s$ and $\gamma_s > 0$. Then we have

$$\gamma_{j,\epsilon} \geq 1 - \epsilon/\gamma_s.$$

Now this becomes

$$([i\Theta_L, \Lambda]\alpha, \alpha) \geq ((q-s+1)(1-\epsilon/\gamma_s) - (n-p))\,|\alpha|^2 = (p+q-n-s+1-(q-s+1)\epsilon/\gamma_s)|\alpha|^2.$$

this is true since we could choose

$$\epsilon < \frac{p+q-n-s+1}{q-s+1} \min_{x\in X} \gamma_s(x).$$

□

We could also give a generalization of the Serre vanishing theorem, which claims that the ample line bundle tensor with any bundle could have vanishing cohomology group for any $q \geq 1$. This could be generalized to any vector bundle which has semi-positive curvature form and certain number of positive eigenvalues. That is

Theorem 3.2.7 (Andreotti-Grauert Vanishing Theorem)**.** *Let L be a holomorphic line bundle over a tangent space such that $i\Theta_L$ has $n-s+1$ positive eigenvalues at every point and E be a certain holomorphic vector bundle. Now there exists an integer $m_0 \geq 0$ such that*

$$H^q(X, E \otimes L^m) = 0, \quad \forall m > m_0, q > 0$$

Proof. The proof is similar to previous proof to Serre vanishing theorem, and the only difference is that the $2\pi m(n-p-q)$ part is replaced by

$$|([\Lambda, i\Theta]\alpha, \alpha)| \leq (C + 2m\pi(p+q-n-s+1-(q-s+1)\epsilon/\gamma_s))|\alpha|^2.$$

Thus when m is large enough, this theorem is proved. □

Since the eigenvalues for the curvature tensor are quite hard to find in real settings, thus the former cohomology vanishing theorem are used as the analytic generalization for Kodaira vanishing theorem. We will see the semipositivity in the algebraic side in the next section.

3.3 Semipositive Vanishing Theorems

In this section, we will discuss the vanishing theorem in various semipositive conditions. A first theorem was due to Grauert and Riemenschneider in 1970[9]. They claim that for a semi-ample bundle a similar vanishing theorem holds. In 1976 another theorem by Girbau claim that the semi-positive bundle allow some vanishing theorem when restricted to certain condition. Some years later in 1980, a most interesting theorem called the Kawamata-Viehweg theorem appeared. This theorem was independently discovered by Kawamata[12] and Viehweg[23] as a generalization of Kodaira vanishing theorem. This generalization is a big step from positivity of Kodaira's original theorem to semipositivity. We will emphasize mostly on the Kawamata-Viehweg generalization in this chapter, but we will use a geometric proof of this theorem instead of algebraic one which appears in most books like [13]. This theorem is essential in that it has important application to the minimal model program(Mori's program).

First, we introduce some definitions. There are several types of semi positivity. The most simple generalization is the semi positivity of curvature tensor. But the algebraic application leads to several interesting generalizations

Definition 3.3.1. Let X be a projective Kähler manifold. A line bundle L over X is said to be *nef* (numerically effective) if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve $C \subset X$.

Why do we need this definition? This is because apart from the curvature description of ample line bundle, an important criterion to determine whether a bundle is ample is the Nakai-Moishezon theorem developed by Yoshikazu Nakai(中井喜和) and Boris Moishezon. This theorem is an algebraic characterization of ampleness. For the sake of simplicity we omit the proof. This proof could be referred to [10].

Theorem 3.3.2 (Nakai-Moishezon)**.** *Suppose that X is a projective manifold. A line bundle over X is ample if and only if $L^p \cdot Y > 0$ for every p-dimensional subvariety Y. That is to say $\int_Y c_1(L)^p > 0$.*

And it is easy to see that if a line bundle is nef, $\int_Y c_1(L)^p \geq 0$ for every p-dimensional subvariety Y. Thus nef is actually a semi-positive generalization of ampleness.

We are primarily concerned about the curvature condition for line bundles. As we may see in the proof of Kodaira Vanishing Theorem. Thus a restriction of curvature definition is more welcomed in actual computation. Since for definition of every subvariety, to show a bundle is nef we must use argument of contradiction. But to show a bundle is nef could also be realized by finding a good metric as said by the following theorem

Lemma 3.3.3. *A line bundle L is on the projective manifold X. Then the following are equivalent*

1. *L is nef*

2. *For any integer $k \geq 1$ and any ample line bundle A, the bundle $L^{\otimes k} \otimes A$ is ample.*

3. *For any $\epsilon > 0$, there's a Hermitian metric h_ϵ such that the curvature $i\Theta_{L,h_\epsilon} \geq -\epsilon\omega$ for given Kähler metric ω.*

Proof. (1)⇒(2) If L is nef and A is ample, then the corresponding bundle

$$c_1(L^{\otimes k} \otimes A) = kc_1(L) + c_1(A).$$

Thus the Nakai-Moishezon criterion is well satisfied.

(2)⇒(3) Since A is ample, by the theorem before we know that there's a Hermitian metric h_A on A such that $i\Theta_{A,h_A} = \omega$. Thus by choosing the metric h_B on $L^{\otimes k} \otimes A$ so that it has positive curvature. Thus we consider the metric

$$h_L = (h_A^{-1} \otimes h_B)^{1/k}.$$

Since $L^{\otimes k} = A^{-1} \otimes L^{\otimes k} \otimes A$, we have the curvature

$$i\Theta_{L,h_L} = \bar{\partial}\partial \log((h_A^{-1} \otimes h_B)^{1/k}) = \frac{1}{k}(i\Theta_B - i\Theta_A) \geq -\frac{1}{k}\omega.$$

So when k is large enough, (3) is satisfied.

(3)⇒(1) It is easy since

$$L \cdot C = \int_C \frac{i}{2\pi}\Theta_{L,h_\epsilon} \geq -\frac{\epsilon}{2\pi}\int_C \omega$$

for every curve C and every $\epsilon > 0$. Thus $L \cdot C \geq 0$. □

We may generalize the notion of nef to compact complex manifold. This is due to the differential geometric definition.

Definition 3.3.4 (Nef)**.** Let L be a holomorphic line bundle on a compact manifold with a metric ω. It is called *nef* if for every positive ϵ there is a Hermitian metric h_ϵ on L such that $i\Theta_{L,h_\epsilon} \geq -\epsilon\omega$.

We have also a cohomology characterization of ampleness. Since Serre Vanishing theorem should check all cohomology group greater than zero, which is almost impossible if we don't know the ampleness of bundle. We may reduce it to a simpler definition called the big line bundle.

Definition 3.3.5 (Kodaira Dimension and Bigness)**.** If L is a line bundle, the *Kodaira dimension* $\kappa(L)$ is defined to be the smallest constant for

$$h^0(X, mL) \leq O(m^{\kappa(L)})$$

to hold. The line bundle is called *big* if the *volume*

$$Vol(L) = \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, L\otimes k) > 0, \quad \dim X = n.$$

The reason for this definition of volume comes from the Hirzebruch-Riemann-Roch formula. When we consider the ample line bundle, we can see that $h^q(X, L^{\otimes k}) = 0$ for $k \gg 1$. Also note the famous theorem in [11]

Theorem 3.3.6 (Hirzebruch-Riemann-Roch)**.** *Let E be a holomorphic vector bundle on an $n-$dimensional compact complex manifold X. We have*

$$\chi(X, E) := \sum_{k=0}^{n} (-1)^n h^i(X, E) = \int \text{ch}(E)\text{td}(X).$$

Here ch *is the Chern character and* td *is the Todd class.*

In our ample line bundle, since we only consider the integration over (n, n)-forms, this boils down to the formula

$$\begin{aligned}\limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k}) &= \limsup_{k\to+\infty} \frac{n!}{k^n}\left(\int_X \mathrm{ch}(E)\mathrm{td}(X)\right)\\ &= \limsup_{k\to+\infty} \frac{n!}{k^n}\left(\int_X \sum_{i=0}^{n} \frac{(k\cdot c_1(L))^i}{i!}\mathrm{td}_{n-i}(X)\right)\end{aligned}$$

We can see that the volume is exactly

$$Vol(L) = \int_X c_1(L)^n.$$

This is also true when L is nef. But we need more argument to show this. We omit the detailed proof here since this may diverge our topic too far.

Apart from the Kodaira dimension, we also have another dimension that is useful in characterizing our line bundle. This dimension is called the numerical Kodaira dimension, or numerical dimension, denoted by $\nu(L)$, defined by

Definition 3.3.7 (Numerical Kodaira Dimension)**.** Let L be a nef bundle over X and the numerical Kodaira dimension $\nu(L)$ is defined to be

$$\nu(L) := \max\{k | c_1(L)^k \neq 0 \in H^{2k}(M, \mathbb{R})\}.$$

An interesting theorem states that the numerical Kodaira dimension bounds the Kodaira dimension, as

Theorem 3.3.8. *Let L be a nef bundle over a compact Kähler manifold. Then $\kappa(L) \leq \nu(L)$*

The proof of this theorem require much algebraic technique. Since we're mainly concerned with the geometric invariant $\nu(L)$, we will not talk about the Kodaira dimension $\kappa(L)$ later.

Actually, we will now prove by geometry that the Kawamata-Viehweg vanishing theorem and the Bogomolov-Sommese vanishing theorem. The proof of these theorems comes from further discussion of the Kähler identities. Recall that on vector bundle E over a Kähler manifold, we have

$$[\partial_E^*, L] = -i\bar{\partial}, \quad [\bar{\partial}^*, L] = i\partial_E$$
$$[\partial_E, \Lambda] = -i\bar{\partial}^*, \quad [\bar{\partial}, \Lambda] = i\partial_E^*$$

To get the final vanishing theorem, we may well observe the situation when a form is harmonic. And for a (n,q) form on a compact Kähler manifold of dimension n and line bundle L we have

Lemma 3.3.9. *An (n,q)-form α with value in L is harmonic if and only if*

$$\bar{\partial}_L \Lambda^q \alpha = 0, \quad \Theta_L \Lambda \alpha = 0.$$

Here Θ_L is by an abuse of notation, a operator that gives $\Theta_L \wedge$.

Some of the analogue of Nakano inequality will be used. Namely when we will generalize Nakano inequality $(i[\Theta_L, \Lambda]\alpha, \alpha) \leq 0$ to the following lemma

Lemma 3.3.10 (Exponential Nakano)**.** *If (n,q)form is harmonic, then*

$$((\Theta + \partial\bar{\partial} f)\Lambda\alpha, e^f \alpha) \leq 0.$$

Here $f \in C^{\infty}(M, \mathbb{R})$.

Apart from this theorem, some theorems on the convergence of functions are needed.

Lemma 3.3.11. *Let* X *be a compact Riemannian manifold. Suppose* $\{f_j\}$ *is a sequence of function with* $\int_M f_j = 0$*. And* $\lim_{j\to\infty} f_j = 0$ *as generalized functions. Then* $\lim_{j\to\infty} f_j = 0$ *as generalized function.*

Proof. For this we refer to the Hodge theory in real setting. We see that for any smooth function f on M, the Green operator exists, and solves the Poisson equation $\Delta\phi = \psi$ for any $\int \psi dv = 0$. Now that we choose a smooth function ψ and set $\psi_0 = \psi - \int \psi$. Since

$$(f_j, \psi) = (f_j, \psi_0) = (\Delta f_i, G\psi_0) \to 0, \quad j \to \infty$$

we see that f_j is zero as generalize function. □

Lemma 3.3.12. *Let* $\{f_j\}$ *be a sequence of functions less than or equal to zero on* M*, and they converge in the space of generalized function. Then* $\lim_{j\to\infty} \int_U e^{f_j} \neq 0$ *for any open subset* U *on* M*.*

The proof of this theorem is standard real analysis using the proof of contradiction and take logarithm.

It seems quite natural that Calabi-Yau's existence theorem[28] will play a role in the proof of this theorem. Since we are considering a Kähler form that will be a multiple of other. This theorem has the following statement

Theorem 3.3.13 (Yau)**.** *Let* X *be a Kähler manifold of dimension* n *and* ω_0 *is its Kähler* *form. If for an function $f \in C^\infty(X, \mathbb{R})$ such that*

$$\int_X e^f \omega_0^n = \int_X \omega_0^n.$$

There exists a unique Kähler metric such that

$$[\omega] = [\omega_0], \quad \omega^n = e^f \omega_0^n.$$

We must also allow the Aronszajn's theorem for unique extension, that a form will not vanish on an open set if it is a nontrivial kernel of differential operator.

Theorem 3.3.14 (Aronszajn)**.** *Let M be a C^∞ manifold and E a vector bundle over M. Let P be a second order elliptic differential operator on $C^\infty(M, E)$. Then any $\alpha \in \ker(P)$ that vanishes on an open set of M will be zero.*

Proof of this theorem is in [2]. As a simple corollary, we know that the nontrivial element in $\mathbb{H}$ will not vanish on an open set. We can use this theorem to extend Kodaira vanishing to a semi positive vanishing theorem

Theorem 3.3.15 (Grauert-Riemenschneider Vanishing I)**.** *Let L be a holomorphic line bundle over Kähler manifold X. Suppose in addition that L is semi positive($i\Theta_L \geq 0$) and is positive on at least a point. Then*

$$H^q(X, K_X \otimes L) = 0, \forall q > 0.$$

Proof. Since we have the Aronszajin's theorem, we can see from the generalized Lefschetz formula that

$$([i\Theta_L, \Lambda](\alpha), \alpha) \geq (\gamma_1 + \gamma_2 + \ldots + \gamma_q)|\alpha|^2, \quad \gamma_1 \leq \ldots \leq \gamma_n \text{ are eigenvalues of } i\Theta_L.$$

Since $i\Theta$ is positive on an open neighborhood of p, then α should be zero in this open neighborhood since $\gamma_i > 0$ for all i. Then α should be zero on all the manifold, which means the cohomology group should vanish. □

This argument could be extend to line bundle tensor the sheaf of differential forms. We could claim that when a line bundle is semi positive and is positive on at least a point, then

$$H^q(X, \Omega_X^p \otimes L) = 0, \forall p + q > n.$$

Since we have made all our preparations, we now give our most important Kawamata-Viehweg vanishing theorem

Theorem 3.3.16 (Kawamata-Viehweg Vanishing)**.** *Let L be a nef line bundle on the Kähler manifold X such that its numerical Kodaira dimension is $\nu(L)$. Then the cohomology group*

$$H^q(X, K_X \otimes L) = 0, \quad \forall q > n - \nu(L).$$

Proof. Since we are considering the group $H^q(X, K_X \otimes L)$, we will only consider the (n, q)-forms that are harmonic and $q > \nu(L)$. We will claim that these forms are exactly zero. This is proved by contradiction. Suppose these forms are nonzero, we consider the function in ϵ

$$p(\epsilon) = \frac{\int_X (\epsilon\omega + i\Theta)^n}{\int_X \omega^n}.$$

Since L is nef, there exists a metric on L such that $\epsilon + i\Theta_L + i\partial\bar{\partial}\psi_\epsilon$ is Kähler on X. By the theorem before, we let $e^{f_1}(\epsilon\omega + i\Theta)^n = \omega^n$ and apply Calabi-Yau theorem. We observe that

$$\int_X (\epsilon\omega + i\Theta + i\partial\bar{\partial}\psi_\epsilon)^n = \int_X p(\epsilon)\omega^n = \int_X e^f p(\epsilon)(\epsilon\omega + i\Theta)^n.$$

Thus there's a function ψ_ϵ such that

$$(\epsilon\omega + i\Theta + i\partial\bar{\partial}(\psi_\epsilon + \phi_\epsilon))^n = p(\epsilon)e^f(\epsilon\omega + i\Theta + i\partial\bar{\partial}\psi_\epsilon)^n = p(\epsilon)\omega^n.$$

To simplify the notation, we denote $f_\epsilon = \psi_\epsilon + \phi_\epsilon$ and $\rho_\epsilon = \epsilon\omega + i\partial\bar{\partial}(\Theta + f_\epsilon)$. Now because $\int \rho_j \wedge \omega^{n-1}$ is bounded when $\epsilon \to 0$, there's a subsequence $\lim_{j\to\infty} \epsilon_j = 0$ and ρ_{ϵ_j} has a limit as current. Then By previous lemma f_{ϵ_j} converge as current with $\int_M f_{\epsilon_j} = 0$. We use the simplified notation that $f_\epsilon = f_{\epsilon_j}$ that replace ϵ_j by ϵ. And $j \to \infty$ is replaced by $\epsilon \to 0$.

We now choose a local frame that diagonalize ρ_ϵ to $\rho_\epsilon = \sum_i \lambda_i z^i \wedge \bar{z}^i$. Hence by previous equation we know that $\prod_i \lambda_i = p(\epsilon)$. And $p(\epsilon)$ is a polynomial with degree n. Since our numerical Kodaira degree is $\nu(L)$ and $\Theta^n = 0$ when $n \geq \nu(L)$, we know that if we write

$$p(\epsilon) = \sum_{j=0}^{n} a_j \epsilon^j \Rightarrow \begin{cases} a_j > 0 & j \geq n - \nu(L) \\ a_j = 0 & j < n - \nu(L) \end{cases}$$

Thus

$$a_{n-\nu}\epsilon^{n-\nu} \leq \prod_{i=1}^{n} \lambda_i, \quad a_{n-\nu} \text{ is independent of } \epsilon.$$

Next we consider the function For the function $u_\epsilon = \frac{\langle \rho_\epsilon \Lambda\alpha, \alpha\rangle}{\langle \alpha, \alpha\rangle}$ We may see from the computation and previous lemma

$$(\rho_\epsilon \Lambda\alpha, \alpha) = ((\epsilon L + i(\Theta + \partial\bar{\partial} f_\epsilon))\Lambda\alpha, \alpha) = ((\Lambda L + q)\epsilon\alpha, \alpha) + (i(\Theta + \partial\bar{\partial} f_\epsilon))\Lambda\alpha, \alpha) < q\epsilon(\alpha, \alpha).$$

Thus $u_\epsilon < q\epsilon$. Since

$$n\int_M \sum_{\beta\in\{1,2,\ldots,n\}} \lambda_\beta \omega^n \leq \int_M \sum_{\beta\in\{1,2,\ldots,n\}} \rho_\epsilon \wedge \omega^{n-1} < \infty, \quad \epsilon \to 0.$$

We see that there's a constant satisfying the relation $\max \lambda_\beta \leq C$ on an open set $V(\epsilon)$ of M.

For ordered set $A \subset [n]$ and $|A| = q$ we write $\theta^A = \prod_{\alpha=1}^q \theta^{i\alpha}$ where $i_{\alpha+1} > i_\alpha$. Then we have

$$\rho_\epsilon \Lambda \alpha = \sum_A (\sum_{\beta\in A} \lambda_\beta)\alpha_A \theta^N \wedge \theta^A.$$

By Aronszajin's theorem, α will be not always zero when in the open set $V(\epsilon)$. Thus we may assume that there's a "greatest" form, i.e. that α_{A_0} never comes to zero when we shrink $V(\epsilon)$ and $|\alpha_A| \leq |\alpha_{A_0}|$ for all A when the norm is taken to be the norm inherited from Hermitian form h. Now that

$$\sum_{i\in A_0} \lambda_i |\alpha_{A_0}|^2 \leq u_\epsilon \langle \alpha, \alpha \rangle = q\epsilon \binom{n}{q} |\alpha_{A_0}|^2.$$

Then $\lambda_\beta \leq C\epsilon$ for large enough C that does not depend on ϵ on $V(\epsilon)$. Now that the constant

$$\prod_{\beta=1}^n \lambda_\beta \leq C^n \epsilon^q.$$

As a consequence of this inequality

$$a_{n-\nu}\epsilon^{n-\nu} \leq C^n \epsilon^q. \quad \epsilon \to 0.$$

This is not true since $n - \nu < q$. □

As a Corollary, we simply have a modest improvement of Kodaira-Nakano-Akizuki vanishing theorem, which computes the zeros cohomology of nef bundle.

Theorem 3.3.17 (Bogomolov-Sommese Vanishing)**.** *Let L be a nef line bundle on the Kähler manifold X such that its numerical Kodaira dimension is $\nu(L)$. Then the cohomology group*

$$H^0(X, \Omega^q \otimes L) = 0, \quad \forall q < \nu(L).$$

Proof. This is simply taking conjugation and note that $L^* = \bar{L}$ and take the Serre's duality. □

Apart from the easy analogy, we will also consider the local version of this theorem, called the Grauert-Riemenschneider Vanishing theorem. This theorem tells the vanishing of derived functors. But we need some preparation for the proof of this theorem, we must equip ourselves with the essential Leray's spectral sequence tool.

Lemma 3.3.18. *Let $f : X \to Y$ be a continuous map of topological space. $\mathcal{F}$ is a coherent sheaf over X. Then the spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \cong H^{p+q}(X, \mathcal{F}).$$

We can now prove this theorem of Grauert-Riemenschneider, which is a local version of Kawamata-Viehweg.

Theorem 3.3.19 (Grauert-Riemenschneider Vanishing II)**.** *Let $f : X \to Y$ be a surjective holomorphic map from two projective hypersurfaces X to Y with* $\dim X = \dim Y$ *and f^{-1} has d preimages for every point $y \in Y$. Then the derived group*

$$R^j f_* K_X = 0 \quad \forall i > 0.$$

Proof. Firstly we prove that for any ample line bundle L over Y, the cohomology group of pullback bundle is zero. we need to show that

$$H^i(X, K_X \otimes f^*L^n).$$

To show this we only need our Kawamata-Viehweg vanishing theorem, to see that

$$f^*L^n \text{ is nef and big.}$$

f^*L is actually semi-ample. *Semi-ample* line bundle are bundles that some of its tensor power is globally generated. We can see that semi-ample line bundle are nef since for some L there's an integer n and a map $\phi : X \to \mathbb{CP}^n$(generated by globally generated sections) such that $L^n = \phi^*\mathcal{O}(1)$. Hence the integration

$$\int_C c_1(L) = \frac{1}{n}\int_C c_1(L^n) = \frac{1}{n}\int_{f_*C} c_1(\mathcal{O}(1)) \geq 0.$$

This proves that semiampleness implies nef. Now we need to see that a pullback of ample line bundle is semiample. This is because we see that L^n is generated by global section when $n \gg 0$. Thus the surjection $\mathcal{O}_Y^I \twoheadrightarrow L^n$ exists and we then have the map

$$\mathcal{O}_X^I = f^*\mathcal{O}_Y^I \twoheadrightarrow (f^*L)^n.$$

This means that $(f^*L)^n$ is globally generated, and f^*L is semiample.

The cohomology group is big since the pullback metric induced the Chern class $c_1(f^*L^n) = f^*c_1(L)^n \neq 0$ since f is holomorphic finite map.

Secondly we observe that if $H^i(X, K_X \otimes f^*L^n)$ for n sufficiently large and all

$i > 0$, this result from the Leray spectral sequence that

$$H^i(X, K_X \otimes f^*L^n) = H^0(Y, R^i f_* K_X \otimes L^n).$$

If $R^i f_* K_X \neq 0$ and n sufficiently large, the right hand side will be nonzero, which is impossible. Thus we claim that

$$R^i f_* K_X = 0, \quad \forall i > 0$$

$\square$

As we may see that we have another Grauert-Riemenschneider vanishing theorem before, which is purely geometric in that it has $i\Theta \geq 0$ and $i\Theta$ is positive at some point. This could actually be proved for the pullback line bundle f^*L, where the metric is induced.

We finally introduced the Kollár's injectivity criterion from the geometric side. We see the following injective theorem by Kollár in algebra.

Theorem 3.3.20 (Kollár injectivity criterion)**.** *Let X be a compact Kähler manifold of dimension n and L a semi-positive line bundle over X. Suppose L^k allows a nonzero global section, then the homomorphism induced by multiplication*

$$s : H^q(X, K_X \otimes L^m) \to H^q(X, K_X \otimes L^{m+k})$$

is injective for any $m > 0, q \geq 0$.

Proof. for $s \in H^0(X, L^k)$ and α is harmonic in L^m, we should prove that $s \otimes \alpha$ is also harmonic as (n, q) form taking value in L^{m+k}. From lemma before we know that

$$\bar{\partial}\Lambda^q\alpha = 0, \quad \Theta\Lambda\alpha = 0.$$

It's easy to check $\bar{\partial}\Lambda(s\otimes\alpha) = 0$ and $(m+k)\Theta\Lambda(s\otimes\alpha) = 0$ since s is $(0, 0)$ form. $\square$

第四章 Miscellaneous

4.1 Higher rank, Hermitian and Hyper-Kähler

Kodaira's vanishing theorem works on positive line bundles. We would like to extend this result to bundles of higher ranks. Higher rank line bundles are actually harder to construct a "good" notion of positivity. The algebraic and geometric construction both give satisfying generalization, but proving their equivalence stands long unsolved, and is really deep in mathematics. We will present the conflict between these constructions and show the Griffiths' conjecture, which is a grand conjecture in this field. Since this conjecture is really hard to be solved by an undergraduate, we only give some evidence of this conjecture here. We then state and prove the cohomology vanishing due to Demailly.

Firstly, we may define these generalizations. For any vector bundle we consider its We consider the curvature tensor

$$i\Theta_E = i \sum_{1\le j,k\le n, 1\le \lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu.$$

We could define various generalization of ampleness here

Definition 4.1.1 (Griffiths Positivity)**.** E is said to be *Griffiths positive* if for all $\xi \in T_xX, \xi \neq 0$ and $s \in E_x, s \neq 0$, we have

$$\theta_E = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(dz_j \otimes e_\lambda^*) \otimes \overline{(dz_k \otimes e_\mu^*)} > 0$$

We say $E >_{Grif} 0$. The negativity or semipositivity are defined similarly.

Also, we could define Nakano positivity

Definition 4.1.2 (Nakano Positivity and dual)**.** E is said to be *Nakano positive* if for all $u \in TX \otimes E$, we have

$$\theta_E(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \bar{u}_{k\mu} > 0$$

We say $E >_{Nak} 0$. The negativity or semipositivity are defined similarly.

It is said to be *dual-Nakano positive* if for all $u \in TX \otimes E$ we have

$$\theta_E(u,u)^* = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\mu} \bar{u}_{k\lambda} > 0$$

Hartshorne ampleness of the bundle relates to the projectivation of vector bundle. That is, we consider the map $s : X \to E$ that map $x \in X$ to $0 \in E$. This map is called zero section. Now the complement $E \backslash s(X)$ has a natural $\mathbb{C}^*$ action. We modulo this action to get

$$\mathbb{P}(E) = (E \backslash s(X))/\mathbb{C}^*.$$

The result is called projectivation, and there's a canonical line bundle $\mathcal{O}_{\mathcal{P}(E)}(-1)$ on it. Ampleness of vector bundle will be related to ampleness of line bundle by

Definition 4.1.3. A vector bundle is called *Hartshorne ample* or simplified as *ample* if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample over $\mathbb{P}(E^*)$.

Known various kind of positivity, the fundamental problem here is the Griffiths' conjecture

Conjecture 4.1.4 (Griffiths Conjecture)**.** Is ampleness the same thing as Griffiths positivity?

Griffiths positivity can derive ampleness, but the converse remains unknown. Proving the converse theorem is the same as finding a metric that has positive curvature form, which is hard since we're handling an arbitrary manifold. Thus we consider finding some topological invariant like cohomology group to aid the proof.

There're various vanishing theorem in the positivity theory. The earliest one is described as follows

Theorem 4.1.5 (Le Potier vanishing theorem)**.** *If E is an ample line bundle of dimension r on compact complex manifold. Then*

$$H^q(X, \Omega_X^p \otimes E) = 0, \quad p + q \geq n + r.$$

And Manivel generalize this theorem to a seemly strongest version of vanishing theorem, that

Theorem 4.1.6 (Manivel vanishing theorem)**.** *For E an ample line bundle of rank r and L a nef line bundle. Then for any sequence of integer $k_1, k_2, \ldots, k_l$ and $j_1, j_2, \ldots j_m$ that*

$$H^{p,q}(X, S^{k_1}E \otimes \cdots \otimes S^{k_l}E \otimes \bigwedge^{j_1} E \otimes \cdots \otimes \wedge^{j_m} E \otimes (\det E)^{l+n-p} \otimes L) = 0$$

if $p + q > n + \sum_{i=1}^m (r - j_i)$.

This theorem is a state-of-the-art theorem in the vanishing theorem about positivity in [16]. Currently, vanishing theorems on Griffiths positive or other bundles concern the positivity of $S^k E$,$\wedge^k E$,det E, E^* and their various tensor product. For example, if E is Griffiths positive, then $E \otimes \det E$ is Nakano positive and so on. There're many theorems in this area.

On a Hermitian manifold we consider the Bismut connection instead of Chern connection.

Theorem 4.1.7. *Let X be a compact almost-Kähler manifold and with a dd^* harmonic Kähler form and a Bismut connection. Suppose that the trace of Bismut connection over the tangent bundle is a $(1,1)$ form that is semipositive(denoted by Ric to be the Ricci curvature). Then*

$$H^p(X, \mathcal{O}_X) = 0, \quad p > 0$$

On the other hand, we could also consider the Chern connection

Theorem 4.1.8. *Let X be a compact Hermitian manifold, if the Ricci-Chern curvature is nonnegative everywhere and positive at some point, then*

$$H^0(X, \Omega_X^p) = 0, \quad 1 \leq p \leq n$$

Proof of these theorems use the more generalized Kähler identities. We refer to [1] and [15].

On a hyper-Kähler manifold, the vanishing theorem could be reduced to lower rank. Hyper-Kähler manifold associate Kähler manifold the structure of quaternion. Apart from the usual complex structure I, we have two more different complex structure J, K that satisfies the relation for quaternion. Now the vanishing theorem could be reduced to

Theorem 4.1.9. *Let X be a compact hyper-Kähler manifold of real dimension $4n$, and*

L is a holomorphic line bundle over X. If in addition L is ample, then for and $p > 0$.

$$H^q(X, \Omega_X^p \otimes L) = 0, \quad \forall p > n.$$

Proof of this theorem could be seen at [26].

As we can see, generalization and application of Kodaira's vanishing theorem is everywhere. But due to limitation of time, we are unable to provide our own theorems and proof here. But further study on these areas will be proceeded at personal mathematical interest.

4.2 Algebraic Flavor of Vanishing theorems

In this section, we will prove the vanishing theorem for affine scheme and the quasi-coherent sheaf. We will see that huge difference between the algebraic style proof and the geometric proof. But we must first state a topological lemma.

Lemma 4.2.1. *If we have a compact topological space X with $\mathcal{V}$ as basis for open sets of X. Suppose we have a sheaf $\mathcal{F}$ and integer $i \geq 0$, we say that $\mathcal{F}$ has property (P_i) if for all $\alpha \in H^i(X, \mathcal{F})$ there's an open covering $U_1, \ldots, U_r$ of X by the elements of $\mathcal{V}$ such that α maps to zero in $H^i(X, \mathcal{F}_j)$, here $F_j := (u_j)_*(F|_{U_j})$, $u_j : U_j \to X$ is the inclusion map. Then*

1. *Property (P_1) holds for every sheaf $\mathcal{F}$*
2. *If $i > 1$, assume for all $U \in \mathcal{V}$, we have $H^p(U, \mathcal{F}) = 0$ for $0 < p < i$, then (P_i) holds for $\mathcal{F}$.*

Remark 4.2.2. *We may consider this topological lemma as a statement of the Poincaré* *lemma, that every closed differential 1-form is locally exact. This lemma actually gives a This actually generalize this statement.*

Proof. 1. We first embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, then the cokernel for the map is $\mathcal{C}$ induces exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{C} \to 0$$

So for given $\alpha \in H^1(X, \mathcal{F})$, we may lift it to $s \in \mathcal{C}(X)$. Since the morphism of sheaves $\mathcal{I} \to \mathcal{C}$ is surjective, X is compact, so there's an open covering $U_1, \ldots, U_r$ in elements of $\mathcal{V}$ such that $s|_{U_j}$ lift to $s_j \in I(U_j)$.

We denote $\mathcal{I}_j = (u_j)_*(\mathcal{I}|_{U_j})$, and $\bar{\mathcal{C}}_j$ the cokernel of the induced morphism $\mathcal{F}_j \to \mathcal{I}_j$, here $\mathcal{F}_j(V) = \mathcal{F}(v \cap U_j)$. Now $\bar{\mathcal{C}}_j$ can be viewed as a subsheaf $\mathcal{C}_j := (u_j)_*(\mathcal{C}|_{U_j})$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_j \arrow[r] & \mathcal{I}_j \arrow[r] & \bar{\mathcal{C}}_j \arrow[r] & 0
\end{tikzcd}$$

We now see that the image s_j of $s \in \mathcal{C}(X)$ in $\bar{\mathcal{C}}_j(X) \subset \mathcal{C}_j(X) = \mathcal{C}(U_j)$ is from some $s_j \in \mathcal{I}_j(X) = \mathcal{I}(U_j)$. Thus s_j maps to 0 in $H^1(X, \mathcal{F}_j)$ by long exact sequence. So that $\alpha \in H^1(X, \mathcal{F})$ maps to 0 in $H^1(X, \mathcal{F}_j)$ by functoriality.

2. We use induction on i. That we take an arbitrary finite open covering $U_1, \ldots, U_r$ of X by elements in $\mathcal{V}$. Now we take another U in $\mathcal{V}$. Thus

$$0 \to \mathcal{F}_j(U) \to \mathcal{I}_j(U) \to \mathcal{C}_j(U) \to 0$$

is exact since $\mathcal{F}_j(U) = \mathcal{F}(U \cap U_j)$ and $H^1((U \cap U_j), \mathcal{F}) = 0$. Similarly

$$0 \to \mathcal{F}_j(U) \to \mathcal{I}_j(U) \to \bar{\mathcal{C}}_j(U) \to 0$$

is also exact. Since for all $U \in \mathcal{V}$ holds, so $\bar{\mathcal{C}}_j \cong \mathcal{C}_j$. Now we replace the $\bar{\mathcal{C}}_j$ by $\mathcal{C}_j$, we obtain

$$\begin{tikzcd} H^i(X,\mathcal{C}) \arrow[r, "\cong"] \arrow[d] & H^i(X,\mathcal{F}) \arrow[d] \\ H^{i-1}(X,\mathcal{C}_j) \arrow[r, "\cong"] & H^i(X,\mathcal{F}_j) \end{tikzcd}$$

The horizontal is isomorphism since $\mathcal{I}$ is injective, thus flasque, and acyclic. So the cohomology is 0. Since $\mathcal{C}$ satisfies (P_1), note the diagram show that $\mathcal{F}$ satisfies (P_2) when $i = 2$. Now it suffice to show that $\mathcal{C}$ satisfies (P_{j-1}). It is exactly $H^p(U,\mathcal{C}) = 0$ for all $U \in \mathcal{V}$ and $0 < p < i-1$. But this comes exactly from the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{C} \to 0$$

and $H^p((U,\mathcal{I})$, $H^{p+1}(U,\mathcal{F}) = 0$(later holds when $p+1 < i$).

□

Thus we're equipped with enough tool(and following lemma) to prove the Serre's vanishing theorem

Lemma 4.2.3. *For quasi-coherent $\mathcal{F}, \mathcal{G}, \mathbb{H}$ and short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathbb{H} \to 0$ on affine scheme X, $0 \to \Gamma(\mathcal{F}(X)) \to \Gamma(\mathcal{G}(X)) \to \Gamma(\mathbb{H}(X))$ is also exact.*

Theorem 4.2.4 (Serre's Vanishing I)**.** *If X is an affine scheme and $\mathcal{F}$ a quasi-coherent sheaf on X, then $H^i(X,\mathcal{F})=0$ for $i>0$.*

Proof. We take $\mathcal{V}$ to be the affine open sets. Now assume $i=1$, and we pick $\alpha\in H^1(X,\mathcal{F})$. Now by first part of lemma, there exists an open covering U_j such that α maps to 0 in $H^1(X,\mathcal{F}_j)$. Now we consider the short exact sequence

$$0\to\mathcal{F}\to\prod_{j=1}^{r}\mathcal{F}_j\to\mathcal{G}\to 0$$

when $\mathcal{F}\to\prod_j\mathcal{F}_j$ is by restriction map, and $\mathcal{G}$ the cokernel. Now by previous lemma,

$$0\to\Gamma(\mathcal{F}(X))\to\Gamma(\prod_{j=1}^{r}\mathcal{F}_j(X))\to\Gamma(\mathcal{G}(X))\to 0$$

is short exact sequence, so the map $H^1(X,\mathcal{F})\to H^1(X,\prod_j\mathcal{F}_j)$ is injective, but its image is always 0, thus $\alpha=0$.

For $i>1$ we use induction. Now for all $U\subset X$ and $0<p<i$ we have $H^p(U,\mathcal{F}|_U)=0$, so by second part of the lemma, for given α we may find $U_1,\dots,U_r$ such that its image in $H^i(X,\mathcal{F}_j)$ are all zero. By inductive hypothesis to $\mathcal{G}$ we have $H^{i-1}(X,\mathcal{G})=0$ since $\mathcal{G}$ is also a quasi-coherent sheaf. Now the long exact sequence show that the map $H^i(X,\mathcal{F})\to H^i(X,\prod_j\mathcal{F}_j)$ is injective, which complete the proof. $\square$

At last, we prove the Serre's vanishing theorem. First, we give a lemma on the isomorphism of cohomology.

Corollary 4.2.5. *Let $\phi:X\to Y$ be an affine morphism and $\mathcal{F}$ a quasi-coherent sheaf* *on X, then there's canonical isomorphism*

$$H^i(Y, \phi_*\mathcal{F}) \cong H^i(X, \mathcal{F}), \forall i > 0.$$

Proof. Given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have the flasque resolution $\phi_*\mathcal{F} \to \phi_*\mathcal{I}^\bullet$, since $\phi_*\mathcal{I}^j$ is flasque followed from $\mathcal{I}^j$ flasque. It is a resolution since $R^i\phi_*\mathcal{F} = 0$ So we may compute the cohomology of $\mathcal{I}^j$ by the resolution $\phi_*\mathcal{I}^\bullet$. Now since $\Gamma(Y, \phi_*\mathcal{I}^j) = \Gamma(X, \mathcal{I}^j)$, and we take the cohomology they are exactly the same. □

Now we must also have the cohomology of projective space to get the theorem. We omit the proof since this require some computation with Čech cohomology.

Lemma 4.2.6. *Let A be a ring then*

$$H^i(\mathbb{P}^l_A, \mathcal{O}_{\mathbb{P}^l_A}(d)) = 0, \quad \forall d \in \mathbb{Z}, 1 \leq i \leq l-1.$$

Theorem 4.2.7 (Vanishing for projective scheme)**.** *Let X be a scheme over a Noetherian ring A, embedded as closed subscheme in projective space over A with canonical sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be coherent on X, then for all $i \geq 0$, $H^i(X, \mathcal{F})$ is a finite A-module. Furthermore, there's n_0 such that for all $n \geq n_0$ we have*

$$H^i(X, \mathcal{F}) = 0, \forall i \geq 1$$

Proof. We know that with the embedding $X \to \mathbb{P}^r_A$, the cohomology of $\mathcal{F}$ is the same as the cohomology of $\mathcal{F}$ viewed as a sheaf over projective space, so WLOG, we may let $X = \mathbb{P}^r_A$. When the computation for cohomology of projective space showed that $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(n)$ is true, so we let $\mathcal{F}$ be arbitrary coherent sheaf on $\mathbb{P}$ and the short exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{F} \to 0$$

where $\mathcal{O}_{\mathbb{P}}(d)$ is twisting sheaf for some positive integer d and $\mathcal{G}$ is the kernel. Using the cohomology sequence, we have the long exact sequence

$$\cdots \to H^i(\mathcal{O}_{\mathbb{P}}(d)) \to H^i(\mathcal{F}) \to H^{i+1}(\mathcal{G}) \to \cdots$$

We apply descend induction. When $i > r$, $H^i(\mathcal{F}) = 0$ by Céch cohomology, so if by induction $H^{i+1}(\mathcal{G})$ is finite over A, $H^i(\mathcal{O}_{\mathbb{P}}(d))$ implies that $H^i(\mathcal{F})$ is finite.

Thus by taking tensor product with $\mathcal{O}_{\mathbb{P}}(n)$, we have the exact sequence

$$\to H^i(\mathcal{O}_{\mathbb{P}}(d+n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{G}(n))$$

The right is 0 by inductive hypothesis for large n, thus we've proved the theorem. □

Theorem 4.2.8 (Serre's Vanishing)**.** *Let X be a scheme proper over some Noetherian A. Let $\mathcal{L}$ be an ample invertible sheaf over X. Then for any coherent sheaf $\mathcal{F}$ there's some n_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \quad \forall i \geq 1$$

Proof. Since $\mathcal{L}$ is ample, that's $\mathcal{L}^d$ is very ample for some d. And now X is projective over A, and $\mathcal{L}^d = \mathcal{O}_X(1)$. We now apply the theorem before to the tensor products

$$\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{d-1}$$

And the theorem is proved. □

To prove Kodaira's vanishing theorem from algebraic side, we simply used the

Kollár's injectivity theorem seen before. Since The map

$$H^i(X, K_X \otimes L) \hookrightarrow H^i(X, K_X \otimes L^{N+1}), \quad i > 0$$

for some N, when N is sufficiently large, we know from Serre vanishing theorem that $H^i(X, K_X \otimes L^{N+1}) = 0$, which means that the original cohomology group is zero. This proves the Kodaira vanishing theorem. In fact, we could ride further to prove Kollár's vanishing theorem, which is a variant of Grauert-Riemenschneider vanishing theorem:

Theorem 4.2.9 (Kollár's vanishing theorem)**.** *Let $f : X \to Y$ be a morphism from smooth projective variety X to projective variety Y, and let L be an ample line bundle over Y, then*

$$H^j(Y, R^i f_* K_X \otimes L) = 0.$$

for every $j > 0$ and $i \in \mathbb{N}$.

This theorem, as we can see before, is the same as the Grauert-Riemenschneider vanishing. Since there're too much algebraic lemmas needed to be proved in this theorem, we omit the proof.

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